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WEIGHTED COMPOSITION OPERATORS IN WEIGHTED BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract

We characterize the boundedness and compactness of weighted composition operators between weighted Banach spaces of analytic functions H_v^0 and H_v^∞. We estimate the essential norm of a weighted composition operator and compute it for those Banach spaces H_v^0 which are isomorphic to c_0. We also show that, when such an operator is not compact, it is an isomorphism on a subspace isomorphic to c_0 or ℓ_∞. Finally, we apply these results to study composition operators between Bloch type spaces and little Bloch type spaces.

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1. Introduction

Weighted Banach spaces have been widely studied. These spaces appear, in a natural way, in the study of the increase of analytic functions. See, for example [1, 14, 15], and references therein. In [2, 3], Bonet, Domański, Lindström and Taskinen studied composition operators between weighted Banach spaces of analytic functions. This class of operators have been considered as defined on different classes of Banach spaces of analytic functions. See, for example, the papers of Shapiro [20, 21], the survey of Jarchow [11] and the monograph of Cowen and MacCluer [7].

Weighted composition operators $C_{\varphi,\psi}$ (see Section 2 for the definition) have been studied by different authors. For example, it is well known that all isometries of the Hardy spaces $H^p(\mathbb{D})$ for $1 \leq p < \infty, p \neq 2$, are weighted composition operators (see

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[7, 10]). In this paper, we deal with the boundedness and compactness of weighted composition operators between the weighted Banach spaces of analytic functions H_v^0 and H_v^∞ (see Section 2 for the definitions). We estimate the distance of a weighted composition operator to the space of compact operators (the so-called essential norm) and compute this distance for those Banach spaces H_v^0 which are isomorphic to c_0. We also give some properties of these operators when they are not compact. Namely, we show that they act as isomorphisms on subspaces isomorphic to c_0 or ℓ_∞. This property was obtained in [6] in the case of H^∞. In particular, we have that every weakly compact weighted composition operator between the Banach spaces H_v^0 and H_v^∞ is compact.

In the final section, we apply our previous results to study composition operators between Bloch type spaces $\mathcal{B}_p$ and little Bloch type spaces $\mathcal{B}_p^0$, including the Bloch space ($p = 1$) and spaces of analytic Lipschitz functions ($0 < p < 1$) (see [5, 23, 24]). We characterize the boundedness and compactness of a composition operator C_φ from $\mathcal{B}_p$ into $\mathcal{B}_q$ (from $\mathcal{B}_p^0$ into $\mathcal{B}_q^0$), compute its essential norm and show that it is either compact or an isomorphism on a subspace isomorphic to ℓ_∞ (c_0). The results we present in this section include previous ones due to Madigan [16], Madigan and Matheson [17] and Montes-Rodríguez [18].

This paper is organized in six sections. In Section 2, we summarize preliminaries on the spaces H_v^0 and H_v^∞. Section 3 and Section 4 are devoted to the boundedness and compactness of weighted composition operators. In Section 5 we study properties of non-compact weighted composition operators. The last section deals with composition operators between Bloch type spaces.

2. The spaces H_v^0 and H_v^∞ and notations

Let us denote by $H(\mathbb{D})$ the set of analytic functions from $\mathbb{D}$ into $\mathbb{C}$. A *weight* is a function $v : \mathbb{D} \to \mathbb{R}_+$ which is radial (that is, $v(z) = v(|z|)$ for all $z \in \mathbb{D}$), nonincreasing with respect to $|z|$ and continuous. As we have commented, the aim of this paper is to study weighted composition operators $C_{\varphi,\psi}$ given by $C_{\varphi,\psi}(f) = \psi(f \circ \varphi)$, where $\varphi, \psi \in H(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, when they are considered as operators between the spaces

$$H_v^\infty = \left\{ f \in H(\mathbb{D}) : \sup_{z \in D} v(z)|f(z)| < \infty \right\} \quad \text{and}$$

$$H_v^0 = \left\{ f \in H(\mathbb{D}) : \lim_{|z| \to 1} v(z)|f(z)| = 0 \quad \text{(uniform limit)} \right\},$$

endowed with the norm $\|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)|$. If $\limsup_{|z|\to 1} v(z) > 0$, then H_v^∞ is isomorphic to H^∞ and $H_v^0 = \{0\}$. On account of this, if $\lim_{|z|\to 1} v(z) = 0$, we call v a *typical weight*.

Many results on weighted spaces of analytic functions have to be formulated in terms of the so-called *associated weights* which are defined by

$$\widetilde{v}(z) := \left(\sup\left\{|f(z)| : f \in H_v^\infty, \|f\|_v \leq 1\right\}\right)^{-1}.$$

We have that $\widetilde{v}$ is also a weight. If we take $\widetilde{v}$ instead of v, both the spaces H_v^0 and H_v^∞ and the norm $\|\cdot\|_v$ do not change. We use the following properties of the associated weights: $v(z) \leq \widetilde{v}(z)$; given $z \in \mathbb{D}$, the element δ_z of $(H_v^\infty)^*$ defined by $\delta_z(f) = f(z)$ satisfies $\|\delta_z\|_v = 1/\widetilde{v}(z)$. If v is typical, then $\widetilde{v}$ is typical and

$$\widetilde{v}(z) := \left(\sup\left\{|f(z)| : f \in H_v^0, \|f\|_v \leq 1\right\}\right)^{-1}.$$

A weight v is called *essential* if there exists a constant $C > 0$ such that

$$v(z) \leq \widetilde{v}(z) \leq Cv(z) \quad \text{for each } z \in \mathbb{D}.$$

It is worth pointing out that $\widetilde{v}_p = v_p$ whenever $0 < p < \infty$ and $v_p(z) = (1 - |z|^2)^p$. In particular, they are essential typical weights.

Let us recall (see [19]) that if v is typical, then $(H_v^0)^*$ is isometric to $L^1(\mathbb{D})/N$, where

$$N = \left\{g \in L^1(\mathbb{D}) : \int_{\mathbb{D}} g(z)f(z)v(z)dA(z) = 0, \text{ for all } f \in H_v^\infty\right\}$$

and dA is the normalized Lebesgue measure on $\mathbb{D}$. Moreover, given $f \in H_v^0$ and $[g] \in L^1(\mathbb{D})/N$ we have that the duality is given by

$$\langle [g], f\rangle := \int_{\mathbb{D}} g(z)f(z)v(z)\,dA(z).$$

We also have that $(H_v^0)^{**}$ is isometric to H_v^∞, where the inclusion map is the canonical injection from a Banach space into its bidual.

Finally, we introduce some notations and agreements. If $f \in H(\mathbb{D})$, we note $M(f, r) := \sup_{|z|=r} |f(z)|$. If f is non-zero, we have that $v(z) = M(f, r)^{-1}$ is a weight satisfying $\widetilde{v} = v$. As usual, we denote the norm of H^∞ by $\|\cdot\|_\infty$. We reserve the letters v and w for weights. When $A \subseteq \mathbb{R}_+^0$ is the empty set, we take $\sup A = 0$. We always assume that given a weighted composition operator $C_{\varphi,\psi}$, there exists a point z such that $\psi(z) \neq 0$.

3. Boundedness of weighted composition operators

In this section we characterize the boundedness of weighted composition operators defined on H_v^∞ and H_v^0. The proofs in this section are strongly inspired by [3].

PROPOSITION 3.1. *Let v and w be weights. Then the operator $C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$ is bounded if and only if* $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/\widetilde{v}(\varphi(z)) < \infty$. *Moreover, the following holds*

$$\left\| C_{\varphi,\psi} \right\| = \sup_{z\in\mathbb{D}} \frac{|\psi(z)|\, w(z)}{\widetilde{v}\,(\varphi(z))}.$$

If v is essential, then $C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$ is bounded if and only if $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/v(\varphi(z)) < \infty$.

PROOF. Suppose that $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/(\widetilde{v}(\varphi(z))) = \infty$. Then we can find a sequence (z_n) in $\mathbb{D}$ such that $|\psi(z_n)|w(z_n) > n\widetilde{v}(\varphi(z_n))$ for all n. For all n, there exists $f_n \in B_{H_v^\infty}$ such that $|f_n(\varphi(z_n))| > 1/(2\widetilde{v}(\varphi(z_n)))$. We have that

$$\left\| C_{\varphi,\psi}(f_n) \right\|_w \geq w(z_n)\, |\psi\,(z_n)|\, |f_n\,(\varphi\,(z_n))| > w(z_n)\, |\psi\,(z_n)|\, \frac{1}{2\widetilde{v}\,(\varphi\,(z_n))} > \frac{n}{2}$$

and hence the operator $C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$ is not bounded.

Conversely, if $M = \sup_{z\in\mathbb{D}} |\psi(z)|w(z)/\widetilde{v}(\varphi(z)) < \infty$, we have that $|\psi(z)|w(z) \leq M\widetilde{v}(\varphi(z))$ for all $z \in \mathbb{D}$. Thus

$$\begin{aligned} w(z)\left|C_{\varphi,\psi}(f)(z)\right| &= w(z)|\psi(z)||f\,(\varphi(z))| \\ &= \frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))}\widetilde{v}(\varphi(z))|f\,(\varphi(z))| \leq M\|f\|_v. \end{aligned}$$

So the operator $C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$ is bounded.

It remains to calculate $\|C_{\varphi,\psi}\|$. On the one hand, the foregoing argument shows that

$$\left\| C_{\varphi,\psi} \right\| \leq \sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))}.$$

On the other hand, since $(C_{\varphi,\psi})^*(\delta_z) = \psi(z)\delta_{\varphi(z)}$ for each $z \in \mathbb{D}$, we have that

$$\left\| C_{\varphi,\psi} \right\| = \left\| (C_{\varphi,\psi})^* \right\| \geq \frac{\left\| (C_{\varphi,\psi})^*\,(\delta_z) \right\|_v}{\|\delta_z\|_w} = \frac{|\psi(z)|\,\widetilde{w}(z)}{\widetilde{v}\,(\varphi(z))}$$

for all $z \in \mathbb{D}$. □

PROPOSITION 3.2. *Let v and w be typical weights. Then the operator $C_{\varphi,\psi} : H_v^0 \to H_w^0$ is bounded if and only if $\psi \in H_w^0$ and* $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/\widetilde{v}(\varphi(z)) < \infty$. *In this case, the following holds*

$$\left\| C_{\varphi,\psi} \right\| = \sup_{z\in\mathbb{D}} \frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))}.$$

If the weight v is essential, then $C_{\varphi,\psi} : H_v^0 \to H_w^0$ is bounded if and only if $\psi \in H_w^0$ and $\sup_{z\in\mathbb{D}} |\psi(z)|w(z)/v(\varphi(z)) < \infty$.

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PROOF. When $C_{\varphi,\psi} : H_v^0 \to H_w^0$ is bounded we have that $(C_{\varphi,\psi})^{**} = C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$. So, by Proposition 3.1, it is enough to show that $\psi \in H_w^0$. But this is obvious from the fact that $C_{\varphi,\psi}(1)$ must be in H_w^0.

On the other hand, it suffices to show that $C_{\varphi,\psi}(f)$ belongs to H_w^0 for each f in H_v^0. So, take $f \in H_v^0 = H_{\tilde{v}}^0$. Given $\varepsilon > 0$, there is $r_1 \in]0, 1[$ such that $\tilde{v}(z)|f(z)| < \varepsilon/M$ whenever $|z| > r_1$. Moreover, there is $r_2 \in [r_1, 1[$ such that $w(z)|\psi(z)| < \varepsilon/\sup_{|\zeta|\le r_1}|f(\zeta)|$ whenever $|z| \ge r_2$. Now, if $|\varphi(z)| > r_1$, then

$$w(z)|\psi(z)||f(\varphi(z))| = \frac{|\psi(z)|w(z)}{\tilde{v}(\varphi(z))}\tilde{v}(\varphi(z))|f(\varphi(z))| < \varepsilon.$$

And if $|\varphi(z)| \le r_1$, we have

$$w(z)|\psi(z)||f(\varphi(z))| \le w(z)|\psi(z)| \sup_{|z|\le r_1} |f(z)| < \varepsilon.$$

Thus $w(z)|C_{\varphi,\psi}(f)| < \varepsilon$ for $|z| \ge r_2$. □

In Proposition 3.1 and Proposition 3.2, when the weight v is not essential, it is necessary to work with the associated weight as it was pointed out for composition operators in [3].

Our next result characterize the functions φ and ψ such that the operator $C_{\varphi,\psi}$ is bounded for every weight v. We need the following lemma which is a slight generalization of [3, Lemma 2.5].

LEMMA 3.3. *For any two positive sequences $(r_n) \to 1$, $(R_n) \to 1$ such that $r_0 < R_0 < r_1 < R_1 < r_2 < R_2 < \cdots$ and for every positive sequence (α_n), there is a function $f \in H(\mathbb{D})$ satisfying*

$$M(f, R_n) \ge \alpha_n M(f, r_n).$$

PROOF. Take $f_0 \equiv 1$. Assume we have already found polynomials $f_1, \dots, f_{n-1}$ such that

(a) $|f_k(z)| < 1/2^k$, for $|z| \le r_k$ and $k = 1, \dots, n-1$, and
(b) $M(f_k, R_k) > M_k$ for $k = 1, \dots, n-1$, where

$$M_k := \left(M\left(\sum_{i=1}^{k-1} f_i, r_k\right) + 1\right)\alpha_k + 2\left(\sum_{i=1}^{k-1} \|f_i\|_\infty + 1\right).$$

Take the function $\tilde{f}_n(z) = A/((1+\varepsilon)R_n - z)$, where

$$A = \frac{2M_n(R_n - r_n)}{2^{2n+1}M_n + 1} \quad \text{and} \quad \varepsilon = \frac{R_n - r_n}{R_n(2^{2n+1}M_n + 1)}.$$

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If $|z| \leq r_n$, then

$$\left|\widetilde{f}_n(z)\right| \leq \frac{A}{(1+\varepsilon)\,R_n - |z|} \leq \frac{A}{(1+\varepsilon)\,R_n - r_n} = \frac{M_n}{2^{2n}M_n + 1} < \frac{1}{2^{2n}}.$$

Moreover, $\widetilde{f}_n(R_n) = A/(\varepsilon R_n) = 2M_n$. Now, by Runge's Theorem, we can approximate $\widetilde{f}_n$ on $\overline{R_n\mathbb{D}}$ by a polynomial f_n satisfying (a) and (b). We define $f = \sum_{i=1}^{\infty} f_n$. By (a), we have that f is an analytic function. Finally, by (a) and (b), we have that $M(f, R_n) \geq \alpha_n M(f, r_n)$. □

THEOREM 3.4. *Let φ, ψ be two functions in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then the following statements are equivalent*

(i) *for every weight v the operator $C_{\varphi,\psi}$ is bounded on H_v^∞;*

(ii) *for every typical weight v, the operator $C_{\varphi,\psi}$ is bounded on H_v^0;*

(iii) *the function ψ belongs to H^∞ and there is a $r \in]0, 1[$ such that $|\varphi(z)| \leq |z|$ for every $z \in \mathbb{D}$ with $|z| \geq r$.*

PROOF. (i) implies (iii). For every weight v, we have that $C_{\varphi,\psi}(1) = \psi$ belongs to H_v^∞. If $\psi \notin H^\infty$, then $v(z) = M(\psi, |z|)^{-1/2}$ is a weight satisfying $\psi \notin H_v^\infty$. So $\psi \in H^\infty$.

Assume that there is a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$ and $|\varphi(z_n)| > |z_n|$ for all n. Since $\psi \neq 0$, we take z_n such that $|\psi(z_n)| \neq 0$. Define $r_n = |z_n|$ and $R_n = |\varphi(z_n)|$. Without loss of generality we assume that $r_0 < R_0 < r_1 < R_1 < r_2 < R_2 < \cdots$. Take $\alpha_n = n/|\psi(z_n)|$. By Lemma 3.3, there exists a function $f \in H(\mathbb{D})$ such that $M(f, R_n) \geq \alpha_n M(f, r_n)$ for all n. Take $v(z) = M(f, |z|)^{-1}$ and, for each n, choose $\theta_n \in \mathbb{R}$ such that $M(f, R_n) = |f(e^{i\theta_n}\varphi(z_n))|$. It is easy to check that $C_{\rho_{\theta_n}\circ\varphi,\psi}$ is bounded on H_v^∞ and that

$$\left\| C_{\rho_{\theta_n}\circ\varphi,\psi} \right\| \leq \left\| C_{\varphi,\psi} \right\|$$

for all n, where $\rho_{\theta_n}(z) = e^{i\theta_n}z$. This implies that

$$\begin{aligned}\left\| C_{\varphi,\psi} \right\| \geq \left\| C_{\rho_{\theta_n}\circ\varphi,\psi}(f) \right\|_v &\geq |\psi(z_n)| \left| f\left(\rho_{\theta_n}(\varphi(z_n))\right) \right| \frac{1}{M(f, |z_n|)} \\ &= |\psi(z_n)| \frac{M(f, R_n)}{M(f, r_n)} \geq n,\end{aligned}$$

for all n and we have a contradiction.

(iii) implies (i). By [3, Theorem 2.4], the operator C_φ is bounded for every weight. Moreover, since $\psi \in H^\infty$, the pointwise multiplication by ψ is also bounded. So $C_{\varphi,\psi}$ is bounded.

(iii) implies (ii). Bearing in mind Proposition 3.2, since $\psi \in H^\infty$ we have to show that $C_{\varphi,\psi}$ is bounded on H_v^∞. But this is true because (iii) implies (i).

(ii) implies (iii). This implication can be obtained in a similar way as (i) implies (iii), noting that the function f of Lemma 3.3 can be taken non-bounded. □

In [3], Bonet, Domański, Lindström, and Taskinen obtained some interesting characterizations of the functions φ satisfying the following condition: there is $r \in]0, 1[$ such that $|\varphi(z)| \leq |z|$ for every $z \in \mathbb{D}$ with $|z| \geq r$.

The results of this section were obtained for composition operators by Bonet, Domański, Lindström, and Taskinen in [3].

4. Compactness of weighted composition operators

We now characterize the compactness of weighted composition operators. In fact, we estimate the essential norm of these operators. Recall that the essential norm of a continuous linear operator T is defined by

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}.$$

Since $\|T\|_e = 0$ if and only if T is compact, these estimations give us the conditions in order to get that T is compact.

Define $T_k : H_v^0 \to H_v^0$ ($T_k : H_v^\infty \to H_v^\infty$) by $T_k(f)(z) = f(kz/(k+1))$. By [3, Theorem 3.3], T_k is a compact operator and $\|T_k\| \leq 1$ and it is not difficult to check that T_k converges to the identity in the strong topology of operators on $L(H_v^0)$. In particular, we have that $\|Id - T_k\| \leq 2$. We do not know if $\|Id - T_k\|$ tends to 1. Following the arguments given in [18], we infer that when H_v^0 is isomorphic to c_0, the sequence (T_k) can be replaced by a sequence (L_k) in order to obtain that the distance to the identity tends to 1 and keeping the others properties. The existence of this sequence of operators allows us to compute the essential norm of the weighted composition operator when the space H_v^0 is isomorphic to c_0. It is worth mentioning that, in [14, 15], Lusky showed that under quite general assumptions over the weight v, H_v^0 is isomorphic to c_0.

LEMMA 4.1. *Let v be a typical weight such that H_v^0 is isomorphic to c_0. Then there exists a sequence of compact operators (L_k) such that*

$$\lim_{n\to\infty} L_k(f) = f \quad \textit{for all } f \in H_v^0 \quad \textit{and} \quad \lim_{k\to\infty} \|Id - L_k\| = 1.$$

In fact, for each k, the operator L_k is a convex combination of $\{T_n : n \geq k\}$.

PROOF. First, we prove that $(T_k)^*$ converges to the identity in the strong topology of operators on $L((H_v^0)^*)$. So, take $[g] \in L^1(\mathbb{D})/N$. We have to show that $\|(T_k)^*([g]) -$

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$[g]\|$ goes to zero. Fix $\varepsilon > 0$, $g \in [g]$. There exists r_0 such that $\int_{|z|>r_0} |g(z)|\, dA < \varepsilon/6$. So, since v is nonincreasing, if $f \in B_{H_v^0}$, we have that

$$\int_{|z|>r_0} \left| g(z) \left(f\left(\frac{k}{k+1} z\right) - f(z) \right) \right| v(z)\, dA < \varepsilon/3.$$

By [2, proof of Theorem 4],

$$\lim_{k\to\infty} \sup_{f\in B_{H_v^\infty}} \sup_{|z|\le r_0} |(Id - T_k)(f)(z)| = 0.$$

So, there exists $n \in \mathbb{N}$ such that if $k \ge n$, then

$$\sup_{|z|\le r_0} \left| f\left(\frac{k}{k+1} z\right) - f(z) \right| < \frac{\varepsilon}{3\|g\|_1 v(0)}$$

for all $f \in B_{H_v^\infty}$ and we obtain that

$$\int_{|z|\le r_0} |g(z)| \left| f\left(\frac{k}{k+1} z\right) - f(z) \right| v(z)\, dA < \varepsilon/3.$$

Thus, if $k \ge n$ we have

$$\begin{aligned}
\|(T_k)^*([g]) - [g]\| &= \sup_{f\in B_{H_v^0}} |\langle (T_k)^*([g]) - [g], f\rangle| \\
&= \sup_{f\in B_{H_v^0}} \left| \int_{z\in\mathbb{D}} g(z) \left(f\left(\frac{k}{k+1} z\right) - f(z) \right) v(z)\, dA \right| \\
&\le \sup_{f\in B_{H_v^0}} \int_{z\in\mathbb{D}} \left| g(z) \left(f\left(\frac{k}{k+1} z\right) - f(z) \right) \right| v(z)\, dA < \varepsilon.
\end{aligned}$$

The rest of the proof is similar to [18, proof of Proposition 2.3]. □

THEOREM 4.2. *Let v and w be weights. Suppose that the operator $C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$ is continuous. Then*

$$\left\| C_{\varphi,\psi} \right\|_e \le 2 \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{|\psi(z)| w(z)}{\widetilde{v}(\varphi(z))}.$$

If v is a typical weight, then we have

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{|\psi(z)| w(z)}{\widetilde{v}(\varphi(z))} \le \left\| C_{\varphi,\psi} \right\|_e.$$

Moreover, if H_v^0 is isomorphic to c_0, we have that

$$\left\| C_{\varphi,\psi} \right\|_e = \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{|\psi(z)| w(z)}{\widetilde{v}(\varphi(z))}.$$

PROOF. First, we check the upper estimate. Fixing $r \in]0,1[$, we have

$$\left\|C_{\varphi,\psi}\right\|_e \leq \left\|C_{\varphi,\psi} - C_{\varphi,\psi}T_k\right\| = \sup_{f \in B_{H_v^\infty}} \sup_{z \in \mathbf{D}} w(z)|\psi(z)||(Id - T_k)(f)(\varphi(z))|$$
$$\leq I_{r,k} + J_{r,k},$$

where

$$I_{r,k} := \sup_{f \in B_{H_v^\infty}} \sup_{|\varphi(z)| \geq r} w(z)|\psi(z)||(Id - T_k)(f)(\varphi(z))|$$

and

$$J_{r,k} := \sup_{f \in B_{H_v^\infty}} \sup_{|\varphi(z)| \leq r} w(z)\,|\psi(z)|\,|(Id - T_k)\,(f)\,(\varphi(z))|\,.$$

Now we estimate $I_{r,k}$ and $J_{r,k}$. On the one hand, we have that

$$I_{r,k} \leq \sup_{f \in B_{H_v^\infty}} \sup_{|\varphi(z)| \geq r} \frac{w(z)|\psi(z)|}{\widetilde{v}(\varphi(z))} \widetilde{v}(\varphi(z))|(Id - T_k)(f)(\varphi(z))|$$
$$\leq \sup_{|\varphi(z)| \geq r} \frac{w(z)|\psi(z)|}{\widetilde{v}(\varphi(z))} \sup_{f \in B_{H_v^\infty}} \sup_{z \in \mathbf{D}} \widetilde{v}(z)|(Id - T_k)(f)(z)| \leq 2 \sup_{|\varphi(z)| \geq r} \frac{w(z)|\psi(z)|}{\widetilde{v}(\varphi(z))}.$$

On the other hand,

$$J_{r,k} \leq \|\psi\|_w \sup_{f \in B_{H_v^\infty}} \sup_{|z| \leq r} |(Id - T_k)(f)(z)|.$$

By [2, proof of Theorem 4]

$$\lim_{k \to \infty} \sup_{f \in B_{H_v^\infty}} \sup_{|z| \leq r} |(Id - T_k)(f)(z)| = 0.$$

Consequently,

$$\left\|C_{\varphi,\psi}\right\|_e \leq \limsup_{k \to \infty} I_{r,k} + \limsup_{k \to \infty} J_{r,k} \leq 2 \sup_{|\varphi(z)| \geq r} \frac{w(z)|\psi(z)|}{\widetilde{v}(\varphi(z))}.$$

The last inequality implies that

$$\left\|C_{\varphi,\psi}\right\|_e \leq 2 \lim_{r \to 1} \sup_{|\varphi(z)| \geq r} \frac{w(z)|\psi(z)|}{\widetilde{v}(\varphi(z))}.$$

When v is a typical weight such that H_v^0 is isomorphic to c_0, the above arguments can be followed just replacing the sequence (T_k) by the sequence (L_k) that we got in Lemma 4.1.

We now prove the lower estimate of the essential norm. We proceed by contradiction. Assume we can find constants $b > c > 0$, a compact operator $K : H_v^\infty \to H_w^\infty$ and, for each n, a point $z_n \in \mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and

$$\left\| C_{\varphi,\psi} - K \right\| < c < b \leq \frac{|\psi(z_n)| w(z_n)}{\widetilde{v}(\varphi(z_n))} \quad \text{for all } n.$$

For each n, take a function $f_n \in B_{H_v^0}$ such that $|f_n(\varphi(z_n))|\widetilde{v}(\varphi(z_n)) \geq (c/b)^{1/2}$ and $\alpha(n) \in \mathbb{N}$ such that $|\varphi(z_n)|^{\alpha(n)} \geq (c/b)^{1/2}$ and $\alpha(n)$ goes to infinity. Define $g_n(z) := z^{\alpha(n)} f_n(z)$ for all n. Clearly, (g_n) is a sequence in $B_{H_v^0}$ and converges to zero uniformly on the compact subsets of $\mathbb{D}$. So, (g_n) converges to zero in the weak topology of H_v^0 and, thus, converges to zero in the weak topology of H_v^∞. Since K is compact, we infer that $\|Kg_n\|_w \to 0$. Now, we have that

$$c > \left\| C_{\varphi,\psi} - K \right\| \geq \left\| \left(C_{\varphi,\psi} - K \right) g_n \right\|_w \geq \left\| C_{\varphi,\psi} g_n \right\|_w - \|K g_n\|_w .$$

This inequality implies that

$$\begin{aligned} c &> \limsup_{n\to\infty} \left\| C_{\varphi,\psi} g_n \right\|_w \geq \limsup_{n\to\infty} w(z_n) \, |\psi(z_n)| \, |g_n(\varphi(z_n))| \\ &= \limsup_{n\to\infty} \frac{w(z_n) \, |\psi(z_n)|}{\widetilde{v}(\varphi(z_n))} \, |\varphi(z_n)|^{\alpha(n)} \, \widetilde{v}(\varphi(z_n)) \, |f_n(\varphi(z_n))| \\ &\geq \limsup_{n\to\infty} b \left(\frac{c}{b}\right)^{1/2} \left(\frac{c}{b}\right)^{1/2} = c, \end{aligned}$$

which is a contradiction. □

Applying Theorem 4.2 we obtain a characterization of weighted composition operators which are compact.

COROLLARY 4.3. *Let v and w be weights. Then the operator $C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$ is compact if and only if $\psi \in H_w^\infty$ and*

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{|\psi(z)| \, w(z)}{\widetilde{v}(\varphi(z))} = 0.$$

When v is essential, $\widetilde{v}$ can be replaced by v.

PROOF. Suppose that $C_{\varphi,\psi}$ is compact. Then it is continuous and $\psi = C_{\varphi,\psi}(1) \in H_w^\infty$. If $\lim_{r\to 1} \sup_{|\varphi(z)|>r} |\psi(z)|w(z)/\widetilde{v}(\varphi(z)) > \varepsilon > 0$, there exists a sequence (z_n) in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and $|\psi(z_n)|w(z_n) \geq \varepsilon\widetilde{v}(\varphi(z_n))$ for all n. Take, for each n, a function $f_n \in B_{H_v^\infty}$ such that $|f_n(\varphi(z_n))|\widetilde{v}(\varphi(z_n)) \geq 1/2$ and $\alpha(n) \in \mathbb{N}$ such that $|\varphi(z_n)|^{\alpha(n)} \geq 1/2$ and $\alpha(n)$ goes to infinity. Define $g_n(z) := z^{\alpha(n)} f_n(z)$ for all n.

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Clearly, (g_n) is a sequence in $B_{H_v^\infty}$ and converges to zero uniformly on the compact subsets of $\mathbb{D}$. Since $C_{\varphi,\psi}$ is compact, it is not difficult to check that $\|C_{\varphi,\psi}(g_n)\|_w \to 0$. But

$$\left\|C_{\varphi,\psi}(g_n)\right\|_w \geq |\psi(z_n)|\,|g_n(\varphi(z_n))|\,w(z_n)$$
$$\geq |\psi(z_n)|\,|\varphi(z_n)|^{\alpha(n)}\,|f_n(\varphi(z_n))|\,w(z_n) \geq \frac{1}{4}\frac{|\psi(z_n)|\,w(z_n)}{\widetilde{v}(\varphi(z_n))} \geq \frac{\varepsilon}{4},$$

which is a contradiction.

Conversely, applying Theorem 4.2, it is enough to prove that $C_{\varphi,\psi}$ is continuous providing that $\psi \in H_w^\infty$ and $\lim_{r\to 1}\sup_{|\varphi(z)|>r}|\psi(z)|w(z)/\widetilde{v}(\varphi(z)) = 0$. Let $r < 1$ be such that $\sup_{|\varphi(z)|>r}|\psi(z)|w(z)/\widetilde{v}(\varphi(z)) \leq 1$. Then $\sup_{z\in\mathbb{D}}|\psi(z)|w(z)/\widetilde{v}(\varphi(z)) \leq \max\{1, \|\psi\|_w/\widetilde{v}(r)\}$. □

The proof of Theorem 4.2 can be also adapted to show the following result.

THEOREM 4.4. *Let v and w be typical weights and suppose that the operator $C_{\varphi,\psi} : H_v^0 \to H_w^0$ is continuous. Then*

$$\lim_{r\to 1}\sup_{|z|\geq r}\frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))} \leq \left\|C_{\varphi,\psi}\right\|_e \leq 2\lim_{r\to 1}\sup_{|z|\geq r}\frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))}.$$

If v is a typical weight such that H_v^0 is isomorphic to c_0, then

$$\left\|C_{\varphi,\psi}\right\|_e = \lim_{r\to 1}\sup_{|z|\geq r}\frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))}.$$

COROLLARY 4.5. *Let v and w be typical weights. Then the operator $C_{\varphi,\psi} : H_v^0 \to H_w^0$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{|\psi(z)|w(z)}{\widetilde{v}(\varphi(z))} = 0 \quad \textit{(uniform limit)}.$$

When v is essential, $\widetilde{v}$ can be replaced by v.

Now we characterize the pairs of analytic functions such that the weighted composition operator is always compact.

COROLLARY 4.6. *Let φ, ψ be two functions in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then the following statements are equivalent*

(i) *for every weight v, the operator $C_{\varphi,\psi}$ is compact on H_v^∞;*
(ii) *for every typical weight v, the operator $C_{\varphi,\psi}$ is compact on H_v^0;*
(iii) *the function ψ belongs to H^∞ and the following statements hold*

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(a) *for every sequence* (z_n) *in* $\mathbb{D}$ *such that* $\varphi(z_n) \to b$ *with* $b \in \mathbb{T}$, *then we have* $\psi(z_n) \to 0$;

(b) *there exists* $r_0 \in]0, 1[$ *such that* $|\varphi(z)| \leq |z|$ *for every* $z \in \mathbb{D}$ *with* $|z| \geq r_0$.

PROOF. (i) implies (iii) and (ii) implies (iii). By Theorem 3.4, ψ belongs to H^∞ and satisfies (b).

Assume that (a) is false. Then, we are going to find a typical weight v such that $C_{\varphi,\psi}$ is not compact on H_v^0 and on H_v^∞.

Fix $\delta > 0$. There exist $c > 0$ and a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$, $|\varphi(z_n)| \to 1$, $|\psi(z_n)| \geq c$, $|z_n| < |z_{n+1}|^\delta$ and $|z_n| < |\varphi(z_{n+1})|$ for all n. Set $r_n = |z_n|$. We define an increasing function $u : [0, 1) \to \mathbb{R}_+$ which is equal 1 on $[0, r_1]$, $u(r_n) = 2^n$ and it is affine on each interval $[r_{n-1}, r_n]$. The weight is defined by $v(z) = 1/u(|z|)$. This weight v is essential (see the proof of [3, Theorem 3.7]). By (b) and Theorem 3.4, $C_{\varphi,\psi}$ is continuous on H_v^0 and H_v^∞. On the other hand,

$$\frac{|\psi(z_n)|v(z_n)}{v(\varphi(z_n))} \geq c\frac{u(|\varphi(z_n)|)}{u(r_n)} \geq c\frac{u(r_{n-1})}{u(r_n)} = \frac{c}{2}$$

and $C_{\varphi,\psi}$ is not compact on H_v^0. Moreover, $(C_{\varphi,\psi})^{**} = C_{\varphi,\psi} : H_v^\infty \to H_v^\infty$ is not compact.

(iii) implies (i). Note that it is enough to show that (i) holds for essential weights. So, fix an essential weight v. We apply Corollary 4.3. Since ψ belongs to H^∞, we only have to prove that

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{|\psi(z)|v(z)}{v(\varphi(z))} = 0.$$

Take a sequence (z_n) in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$. By (b), we assume that $|\varphi(z_n)| \leq |z_n|$, so $v(|\varphi(z_n)|) \geq v(|z_n|)$. This shows that $|\psi(z_n)|v(z_n)/v(\varphi(z_n)) \leq |\psi(z_n)|$ and, by (a), we have that $|\psi(z_n)|$ goes to zero.

(iii) implies (ii). Again, it is enough to show that (ii) holds for essential weights. So, fix an essential and typical weight v. We apply Corollary 4.5. We have to prove that

$$\lim_{|z|\to 1} \frac{|\psi(z)|v(z)}{v(\varphi(z))} = 0.$$

Take a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$. Without loss of generality, we assume that either $|\varphi(z_n)| \to 1$ or there exists a constant $c < 1$ such that $|\varphi(z_n)| \leq c$ for all n. If $|\varphi(z_n)| \to 1$, by (b), we have that $|\varphi(z_n)| \leq |z_n|$, so $v(|\varphi(z_n)|) \geq v(|z_n|)$. Thus $|\psi(z_n)|v(z_n)/v(\varphi(z_n)) \leq |\psi(z_n)|$. But, by (a), we have that $|\psi(z_n)|$ goes to zero. If $|\varphi(z_n)| \leq c$, then $|\psi(z_n)|v(z_n)/v(\varphi(z_n)) \leq \|\psi\|_\infty v(z_n)/v(c)$, but $v(z_n)$ tends to zero. □

In [6, Proposition 2.3], the first author and Díaz-Madrigal proved the following result. A weighted composition operator $C_{\varphi,\psi} : H^\infty \to H^\infty$ is compact if and only if ψ belongs to H^∞ and for every sequence (z_n) in $\mathbb{D}$ such that $\varphi(z_n) \to b$ with $b \in \mathbb{T}$, we have that $\psi(z_n) \to 0$. Since the statement (a) of Corollary 4.6 does not imply statement (b), we have that the fact that $C_{\varphi,\psi} : H^\infty \to H^\infty$ is compact is not equivalent to the fact that, for every weight v, the operator $C_{\varphi,\psi}$ is compact on H_v^∞. On the other hand, it is worth mentioning that Bonet, Domański, Lindström, and Taskinen proved that $C_\varphi : H^\infty \to H^\infty$ is compact if and only if, for every weight v, the operator C_φ is compact on H_v^∞ [3, Corollary 3.8].

5. Non-compactness of weighted composition operators

In this section we prove that if a weighted composition operator is not compact, then it acts as an isomorphism on a big subspace. We give the proof with the *little spaces* because in the construction of the subspace where $C_{\varphi,\psi}$ is an isomorphism we have to check that a series converges and it is easier to see this convergence in H_v^∞ than in H_v^0.

THEOREM 5.1. *Let v and w be typical weights and suppose that the operator $C_{\varphi,\psi} : H_v^0 \to H_w^0$ is continuous. Then $C_{\varphi,\psi}$ is either compact or an isomorphism on a subspace isomorphic to c_0.*

PROOF. Suppose that $C_{\varphi,\psi}$ is not compact. Then, by Corollary 4.5, there exist $c > 0$ and a sequence (z_n) in $\mathbb{D}$, with $|z_n| \to 1$, such that $|\psi(z_n)|w(z_n) \geq c\widetilde{v}(\varphi(z_n))$. Since the function ψ belongs to H_w^0, $\widetilde{v}(\varphi(z_n)) \to 0$ and $|\varphi(z_n)| \to 1$. Thus, by [10, Corollary, page 204], there exists a subsequence of $(\varphi(z_n))$ (that we denote in the same way) which is an interpolating sequence. Moreover, we take a function $f_n \in B_{H_v^0}$ such that $|f_n(\varphi(z_n))| > 1/2\widetilde{v}(\varphi(z_n))$.

By [22, proof of Theorem III.E.4], there exist a sequence (h_k) in H^∞ and a constant $M > 0$ such that

$$h_k(\varphi(z_n)) = \delta_{k,n}, \quad \text{and} \quad \sum_{k=1}^{\infty} |h_k(z)| \leq M \quad \text{for all } z.$$

So $f_k h_k \in H_v^0$. Note that for every sequence (ξ_k) in c_0, $\sum_{k=1}^\infty \xi_k f_k h_k$ belongs to H_v^∞. The fact that this function belongs to H_v^0 is not so evident. Given $g \in L^1(\mathbb{D})$, we have

$$\sum_{k=1}^{\infty} |\langle g, f_k h_k \rangle| \leq \sum_{k=1}^{\infty} \int_{\mathbb{D}} |g f_v h_k| \, v dA \leq M \|g\|_1.$$

In particular, given $[g] \in L^1(\mathbb{D})/N$ we infer that

$$\sum_{k=1}^{\infty} |\langle [g], f_k h_k \rangle| \leq M \|[g]\|.$$

Thus, the series $\sum f_k h_k$ is weakly unconditionally Cauchy in H_v^0. Therefore, by [8, page 44], the map $T : c_0 \to H_v^0$ given by

$$T((\xi_k))(z) = \sum_{k=1}^{\infty} \xi_k f_k(z) h_k(z)$$

is well defined, linear and continuous. Further, we define a map $S : H_w^0 \to c_0$, by $S(f) = f(z_n)/(\psi(z_n) f_n(\varphi(z_n)))$. Now, $\|S(f)\| \leq 2\|f\|_w/c$ for all $f \in H_w^0$, hence S is well defined, linear and continuous. Finally, since $S \circ C_{\varphi,\psi} \circ T = Id_{c_0}$, $C_{\varphi,\psi}$ is an isomorphism on the subspace isomorphic to c_0 generated by $\{f_k h_k : k \in \mathbb{N}\}$. □

It is worth mentioning that the space H_v^0 has the property (V) [9, Examples III.1.4(i) and Theorem III.3.4]. That is, for every Banach space Y, each operator $T : H_v^0 \to Y$ is either weakly compact or an isomorphism on a subspace isomorphic to c_0. Bourgain showed that H^∞ also enjoys the property (V) (see [4]), but we do not know if H_v^∞ has it for every weight.

THEOREM 5.2. *Let v and w be weights. Suppose that the operator $C_{\varphi,\psi} : H_v^\infty \to H_w^\infty$ is continuous. Then $C_{\varphi,\psi}$ is either compact or an isomorphism on a subspace isomorphic to ℓ_∞.*

Theorem 5.2 was obtained by Bonet, Domański, and Lindström [2] for composition operators and by the first named author and Díaz-Madrigal for weighted composition operators defined on H^∞ [6].

6. Applications to composition operators on Bloch type spaces

In this section, weighted composition operators on H_v^∞ are used to solve questions on composition operators on Bloch type spaces. We begin with some definitions. For each $0 < p < \infty$, the *Bloch type space* is given by

$$\mathcal{B}_p = \left\{ f \in H(\mathbb{D}) : \sup_{z \in \mathbb{D}} |f'(z)| \left(1 - |z|^2\right)^p < \infty \right\}$$

and the *little Bloch type space* is given by

$$\mathcal{B}_p^0 = \left\{ f \in H(\mathbb{D}) : \lim_{|z| \to 1} |f'(z)| \left(1 - |z|^2\right)^p = 0 \quad \text{(uniform limit)} \right\}.$$

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It is a well-known fact that when endowed with the norm

$$\|f\|_p = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)| \left(1 - |z|^2\right)^p,$$

$\mathcal{B}_p$ is a Banach space and $\mathcal{B}_p^0$ is a closed subspace of $\mathcal{B}_p$. The classical Bloch space is the space $\mathcal{B}_1$ and $\mathcal{B}_1^0$ is the little Bloch space. Moreover, for $0 < p < 1$, $\mathcal{B}_p$ is the analytic Lipschitz space of order $1 - p$. That is, given $f \in H(\mathbb{D})$, $f \in \mathcal{B}_p$ if and only if there exists a constant $K > 0$ such that $|f(z) - f(\zeta)| \le K|z - \zeta|^{1-p}$ for all $z, \zeta \in \mathbb{D}$ [7, Theorem 4.1].

Bloch type spaces are connected to the study of the growth conditions of analytic functions and [24] is a good survey for results about these spaces. Composition operators defined on Bloch type spaces have been used by Jarchow and Riedl to characterize the nuclearity of composition operators between Hardy spaces [12].

We also work with the spaces defined by

$$\widetilde{\mathcal{B}}_p = \left\{ f \in H(\mathbb{D}) : f(0) = 0, \quad \sup_{z\in\mathbb{D}} |f'(z)| \left(1 - |z|^2\right)^p < \infty \right\}$$

and

$$\widetilde{\mathcal{B}}_p^0 = \left\{ f \in H(\mathbb{D}) : f(0) = 0, \quad \lim_{|z|\to 1} |f'(z)| \left(1 - |z|^2\right)^p = 0 \quad \text{(uniform limit)} \right\}.$$

These spaces are also called Bloch type spaces in the literature.

Consider the weight $v_p(z) = (1 - |z|^2)^p$. It is well known that the weight v_p is essential, in fact, it can be proved that $\widetilde{v}_p = v_p$ and, by [15, page 311], $H_{v_p}^0$ is isomorphic to c_0. The map $\Phi_p : \widetilde{\mathcal{B}}_p \to H_{v_p}^\infty$ given by $\Phi_p(f) = f'$ is an isometry (onto) and $\Phi_{p|\widetilde{\mathcal{B}}_p^0}$ is an isometry onto $H_{v_p}^0$. The aim of this section is to apply our results from the previous sections to Bloch type spaces.

The proof of the following lemma is routine (and probably known), so we omit it. We have to introduce some terminology. Given Banach spaces X and Y with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively, we denote by $X \oplus_1 Y$ the Banach space $X \oplus Y$ endowed with the norm $\|(x, y)\| = \|x\|_1 + \|y\|_2$.

LEMMA 6.1. *Let X and Y be Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. If $S : X \to X$ is bounded and $K : Y \to Y$ is compact, then*

$$\|S \oplus K\|_e = \|S\|_e,$$

where $S \oplus K : X \oplus_1 Y \to X \oplus_1 Y$ is defined by $(S \oplus K)(x, y) = (Sx, Ky)$.

Given $\alpha \in \mathbb{D}$, define

$$\phi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

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It is straightforward to obtain the following lemma, which together with Lemma 6.1, allows us to pass the results from $\widetilde{\mathscr{B}}_p$ to $\mathscr{B}_p$.

LEMMA 6.2. *Given $\alpha \in \mathbb{D}$, the composition operator $C_{\phi_\alpha} : \mathscr{B}_p \to \mathscr{B}_p$ is an isomorphism, its inverse is $C_{\phi_{-\alpha}}$ and $C_{\phi_\alpha|\mathscr{B}_p^0}$ is also an isomorphism onto $\mathscr{B}_p^0$.*

THEOREM 6.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function.*

(i) *The operator $C_\varphi : \mathscr{B}_p \to \mathscr{B}_q$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} |\varphi'(z)| \frac{\left(1-|z|^2\right)^q}{\left(1-|\varphi(z)|^2\right)^p} < \infty.$$

(ii) *If $q \geq p \geq 1$, then $C_\varphi : \mathscr{B}_p \to \mathscr{B}_q$ is always bounded.*

(iii) *The operator $C_\varphi : \mathscr{B}_p^0 \to \mathscr{B}_q^0$ is bounded if and only if $\varphi \in \mathscr{B}_q^0$ and*

$$\sup_{z\in\mathbb{D}} |\varphi'(z)| \frac{\left(1-|z|^2\right)^q}{\left(1-|\varphi(z)|^2\right)^p} < \infty.$$

(iv) *If $q \geq p \geq 1$, then $C_\varphi : \mathscr{B}_p^0 \to \mathscr{B}_q^0$ is bounded if and only if $\varphi \in \mathscr{B}_q^0$.*

PROOF. (i). Take $\alpha = \varphi(0)$. By Lemma 6.2, $C_\varphi : \mathscr{B}_p \to \mathscr{B}_q$ is bounded if and only if $C_{\phi_\alpha\circ\varphi} = C_\varphi \circ C_{\phi_\alpha} : \mathscr{B}_p \to \mathscr{B}_q$ is bounded. Since $\phi_\alpha \circ \varphi(0) = 0$, $C_{\phi_\alpha\circ\varphi}$ is bounded if and only if $C_{\phi_\alpha\circ\varphi|\widetilde{\mathscr{B}}_p}$ is bounded. Moreover, $C_{\phi_\alpha\circ\varphi|\widetilde{\mathscr{B}}_p}$ is bounded if and only if $\Phi_q \circ C_{\phi_\alpha\circ\varphi|\widetilde{\mathscr{B}}_p} \circ (\Phi_p)^{-1} : H_{v_p}^\infty \to H_{v_q}^\infty$ is bounded. But, $\Phi_q \circ C_{\phi_\alpha\circ\varphi|\widetilde{\mathscr{B}}_p} \circ (\Phi_p)^{-1} = C_{\phi_\alpha\circ\varphi,(\phi_\alpha\circ\varphi)'}$ and by Proposition 3.1, this weighted composition operator is bounded if and only if

$$\sup_{z\in\mathbb{D}} \left|(\phi_\alpha\circ\varphi)'(z)\right| \frac{\left(1-|z|^2\right)^q}{\left(1-|(\phi_\alpha\circ\varphi)(z)|^2\right)^p} < \infty. \tag{1}$$

A straightforward computation shows that

$$\left|\phi_\alpha\circ\varphi'(z)\right| \frac{\left(1-|z|^2\right)^q}{\left(1-|(\phi_\alpha\circ\varphi)(z)|^2\right)^p} = \frac{\left(1-|\alpha|^2\right)^{1-p}}{|1-\overline{\alpha}\varphi(z)|^{2(1-p)}} \left|\varphi'(z)\right| \frac{\left(1-|z|^2\right)^q}{\left(1-|\varphi(z)|^2\right)^p}. \tag{2}$$

Now, by (2), and the fact that $0 < 1-|\alpha| \leq |1-\overline{\alpha}\varphi(z)| \leq 1+|\alpha|$, we infer that (1) is satisfied if and only if

$$\sup_{z\in\mathbb{D}} |\varphi'(z)| \frac{\left(1-|z|^2\right)^q}{\left(1-|\varphi(z)|^2\right)^p} < \infty.$$

(iii). Arguing as in the proof of (i), (iii) follows from Proposition 3.2 and noting that, by (2), $\phi_\alpha \circ \varphi \in \widetilde{\mathscr{B}}_p^0$ if and only if $\varphi \in \mathscr{B}_p^0$.

(ii) and (iv). Suppose that $\varphi(0) = 0$. We have that $(1-|z|^2)/(1-|\varphi(z)|^2) \le 1$. Moreover, the Schwarz-Pick lemma implies that $|\varphi'(z)| \le (1-|\varphi(z)|^2)/(1-|z|^2)$. So,

$$\sup_{z\in\mathbb{D}} |\varphi'(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^p} \le \sup_{z\in\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{p-1} (1-|z|^2)^{q-p} \le 1,$$

and we apply (i) and (iii) to finish the proof in this case.

If $\varphi(0) \ne 0$, take $\alpha = \varphi(0)$. Then $C_{\phi_\alpha \circ \varphi}$ is bounded. Lemma 6.2 concludes the proof. □

The boundedness of the operator $C_\varphi : \mathcal{B}_p \to \mathcal{B}_p$ was studied by Madigan [16], when $p < 1$ and by Madigan and Matheson, when $p = 1$ [17].

THEOREM 6.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function such that the operator $C_\varphi : \mathcal{B}_p \to \mathcal{B}_q$ is bounded.*

(i) *If $\varphi(0) = 0$, then the essential norm of C_φ is given by*

$$\|C_\varphi\|_e = \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\varphi'(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^p}.$$

(ii) *The operator C_φ is compact if and only if*

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} |\varphi'(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^p} = 0.$$

(iii) *If $q > p \ge 1$, then $C_\varphi : \mathcal{B}_p \to \mathcal{B}_q$ is always compact.*

(iv) *The operator C_φ is either compact or an isomorphism on a subspace isomorphic to ℓ_∞.*

PROOF. (i). The map $\Psi_p : \tilde{\mathcal{B}}_p \oplus_1 \mathbb{C} \to \mathcal{B}_p$ given by $\Psi_p(f, \lambda) = \lambda + f$ is an isometry and satisfies $\Psi_q^{-1} \circ C_{\phi_\alpha \circ \varphi} \circ \Psi_p = (C_{\phi_\alpha \circ \varphi | \tilde{\mathcal{B}}_p}, Id_{\mathbb{C}})$. So, by Lemma 6.1, we have that

$$\|C_\varphi\|_e = \|\Psi_q^{-1} \circ C_\varphi \circ \Psi_p\|_e = \left\|C_{\varphi|\tilde{\mathcal{B}}_p}\right\|_e.$$

Note that

$$\left\|C_{\varphi|\tilde{\mathcal{B}}_p}\right\|_e = \left\|\Phi_q \circ C_{\varphi|\tilde{\mathcal{B}}_p} \circ (\Phi_p)^{-1}\right\|_e = \|C_{\varphi,\varphi'}\|_e,$$

where $C_{\varphi,\varphi'} : H^\infty_{v_p} \to H^\infty_{v_q}$. Finally, using Theorem 4.2, we have that

$$\|C_{\varphi,\varphi'}\|_e = \lim_{r\to 1} \sup_{|\varphi(z)|>r} |\varphi'(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^p}.$$

(ii). Take $\alpha = \varphi(0)$ and consider the composition operator $C_{\phi_\alpha \circ \varphi} : \mathscr{B}_p \to \mathscr{B}_q$. By Lemma 6.2, C_φ is a compact operator if and only if $C_{\phi_\alpha \circ \varphi}$ is compact and $C_{\phi_\alpha \circ \varphi}$ is compact if and only if it satisfies

$$\lim_{r \to 1} \sup_{|(\phi_\alpha \circ \varphi)(z)| > r} \left|(\phi_\alpha \circ \varphi)'(z)\right| \frac{\left(1 - |z|^2\right)^q}{\left(1 - |(\phi_\alpha \circ \varphi)(z)|^2\right)^p} = 0. \tag{3}$$

Using (2), an easy computation shows that (3) is equivalent to

$$\lim_{r \to 1} \sup_{|(\phi_\alpha \circ \varphi)(z)| > r} \left|\varphi'(z)\right| \frac{\left(1 - |z|^2\right)^q}{\left(1 - |\varphi(z)|^2\right)^p} = 0.$$

Now, since ϕ_α maps circles from $\mathbb{D}$ in circles in $\mathbb{D}$, we have that

$$\lim_{r \to 1} \sup_{|(\phi_\alpha \circ \varphi)(z)| > r} \left|\varphi'(z)\right| \frac{\left(1 - |z|^2\right)^q}{\left(1 - |\varphi(z)|^2\right)^p} = \lim_{r \to 1} \sup_{|\varphi(z)| > r} \left|\varphi'(z)\right| \frac{\left(1 - |z|^2\right)^q}{\left(1 - |\varphi(z)|^2\right)^p}.$$

(iii). By Theorem 6.3, the operator $C_\varphi : \mathscr{B}_p \to \mathscr{B}_p$ is continuous and it is enough to show that the injection from $\mathscr{B}_p$ into $\mathscr{B}_q$ is compact. But this injection is the composition operator with the function $\psi(z) = z$ and we have that

$$\lim_{r \to 1} \sup_{|\psi(z)| > r} \left|\psi'(z)\right| \frac{\left(1 - |z|^2\right)^q}{\left(1 - |\psi(z)|^2\right)^p} = \lim_{r \to 1} \sup_{|z| > r} \left(1 - |z|^2\right)^{q-p} = 0.$$

So, by (ii), C_ψ is compact.

(iv). If C_φ is not compact, $C_{\phi_\alpha \circ \varphi | \tilde{\mathscr{B}}_p}$ is also not compact. So the weighted composition operator $\Phi_q \circ C_{\phi_\alpha \circ \varphi | \tilde{\mathscr{B}}_p} \circ (\Phi_p)^{-1} : H^\infty_{v_p} \to H^\infty_{v_q}$ is not compact and by Theorem 5.2, $C_{\phi_\alpha \circ \varphi | \tilde{\mathscr{B}}_p}$ must be an isomorphism on a subspace isomorphic to ℓ_∞. But then, C_φ has the same property. □

The compactness of the operator $C_\varphi : \mathscr{B}_1 \to \mathscr{B}_1$ was studied by Madigan and Matheson [17]. Montes-Rodríguez [18] obtained the essential norm of a composition operator defined on the Bloch space. Let us point out that Shapiro obtained a different characterization of the compactness of the composition operators $C_\varphi : \mathscr{B}_p \to \mathscr{B}_p$, when $0 < p < 1$ (see [21]).

A sufficient condition was obtained in [16, Theorem B] to assure that a composition operator $C_\varphi : \mathscr{B}_p \to \mathscr{B}_p$ is completely continuous ($0 < p < 1$). Using Theorem 6.4(iv), we have that $C_\varphi : \mathscr{B}_p \to \mathscr{B}_q$ is completely continuous if and only if

$$\lim_{r \to 1} \sup_{|\varphi(z)| > r} \left|\varphi'(z)\right| \frac{\left(1 - |z|^2\right)^q}{\left(1 - |\varphi(z)|^2\right)^p} = 0$$

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(compare with [16, Theorem B]). Moreover, Theorem 6.4(iv) implies that $C_\varphi : \mathscr{B}_p \to \mathscr{B}_q$ is compact if and only C_φ it is weakly compact. This result was obtained, for $p = q = 1$, by Liu, Saksman, and Tylli in [13, Corollary 5].

The proof of Theorem 6.4 can be adapted to obtain the following result.

THEOREM 6.5. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function such that the operator $C_\varphi : \mathscr{B}_p^0 \to \mathscr{B}_q^0$ is bounded.*

(i) *If $\varphi(0) = 0$, the essential norm of C_φ is given by*

$$\|C_\varphi\|_e = \lim_{r\to 1} \sup_{|z|>r} |\varphi'(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^p}.$$

(ii) *The operator C_φ is compact if and only if*

$$\lim_{|z|\to 1} |\varphi'(z)| \frac{(1-|z|^2)^q}{(1-|\varphi(z)|^2)^p} = 0 \quad \textit{(uniform limit)}.$$

(iii) *If $q > p \geq 1$, then $C_\varphi : \mathscr{B}_p^0 \to \mathscr{B}_q^0$ is always compact.*

(iv) *The operator C_φ is either compact or an isomorphism on a subspace isomorphic to c_0.*

The compactness of the operator $C_\varphi : \mathscr{B}_1^0 \to \mathscr{B}_1^0$ was studied by Madigan and Matheson [17]. Montes-Rodríguez [18] obtained the essential norm of a composition operator defined on the little Bloch space. Again, by Theorem 6.5(iv), $C_\varphi : \mathscr{B}_p^0 \to \mathscr{B}_q^0$ is weakly compact if and only if C_φ is compact. This result was obtained in [17] for the case $p = q = 1$.

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