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RESEARCH ARTICLE

Some cases of Kudla's modularity conjecture for unitary Shimura varieties

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Abstract

We use the method of Bruinier–Raum to show that symmetric formal Fourier–Jacobi series, in the cases of norm-Euclidean imaginary quadratic fields, are Hermitian modular forms. Consequently, combining a theorem of Yifeng Liu, we deduce Kudla's conjecture on the modularity of generating series of special cycles of arbitrary codimension for unitary Shimura varieties defined in these cases.

Contents

1. Introduction

Fourier–Jacobi expansions of automorphic forms, first defined in [1], are among the major tools to study the subject. They played prominent roles for Siegel modular forms in the proof of the Saito–Kurokawa conjecture [2, 3, 4, 5, 6] and recently in the work of Bruinier–Raum [7] on Kudla's modularity conjecture for orthogonal Shimura varieties. Fourier–Jacobi expansions are also available for unitary groups. For example, for holomorphic automorphic forms on unitary groups $U(2, 1)$ over totally real number fields, Shintani developed a theory (1979) on Fourier–Jacobi expansions that was reformulated by Murase–Sugano in [8]; as an application of Shintani's theory, a nonvanishing criterion for the unitary Kudla lift ([9]) of a holomorphic cusp form on $U(1, 1)$ was found in [10]. In this paper, we define a formal analogue of Fourier–Jacobi expansions for the unitary group $U(g, g)$, which is called a symmetric formal Fourier–Jacobi series after Bruinier–Raum, and show that they define Hermitian modular forms [11, 12, 13] in the cases of norm-Euclidean imaginary quadratic fields.

Our major motivation to study such symmetric formal Fourier–Jacobi series arises from the role they play for the Kudla conjecture [14] on the modularity of generating series of special cycles, for both orthogonal and unitary Shimura varieties. In the unitary case, Kudla's conjecture predicts that the special cycles on a unitary Shimura variety should be the Fourier coefficients of some Hermitian modular form. Beyond the case of codimension 1, assuming the absolute convergence of the generating series, Liu [15] proved the unitary Kudla conjecture. Recently, Maeda [16] gave another proof of Liu's result and generalised it assuming the Beilinson-Bloch conjecture. In this paper, we show that the unitary Kudla conjecture is true unconditionally in the cases of norm-Euclidean imaginary quadratic fields. We do this by combining a fact from Liu's theorem (that the generating series is a symmetric formal Fourier–Jacobi series) and our modularity result that such a series defines a Hermitian modular form. This is a method that has already been successfully applied to prove the orthogonal Kudla conjecture over $\mathbb{Q}$ in the work of Bruinier–Raum [7]. In the unitary case of imaginary quadratic fields, the scope of this method is limited to the cases of norm-Euclidean imaginary quadratic fields that we treat in this paper. In these cases, however, there are several places where additional new ideas are needed; for instance, the boundary geometry of toroidal compactifications of Hermitian modular varieties is more subtle than in the Siegel case.

1.1. The modularity result

Let $E/\mathbb{Q}$ be an imaginary quadratic field. For integers g, k, l such that $1 \le l \le g-1$, every Hermitian modular form f of degree g, weight k has a Fourier–Jacobi expansion of cogenus l. More precisely, if we write the variable $\tau \in \mathbb{H}_g$ in the Hermitian upper half-space $\mathbb{H}_g$ as

$$\tau = \begin{pmatrix} \tau_1 & w \\ z & \tau_2 \end{pmatrix}$$

for $\tau_1 \in \mathbb{H}_{g-l}$, $\tau_2 \in \mathbb{H}_l$, $w \in \mathrm{Mat}_{g-l,l}(\mathbb{C})$ and $z \in \mathrm{Mat}_{l,g-l}(\mathbb{C})$, then f has a Fourier–Jacobi expansion of the form

$$f(\tau) = \sum_{m \in \mathrm{Herm}_l(E)_{\ge 0}} \phi_m(\tau_1, w, z) e(m\tau_2),$$

where the sum runs over all the $l \times l$ positive semidefinite Hermitian matrices m with entries in E and $e(x) = \exp\left(2\pi i \cdot \mathrm{Tr}(x)\right)$ for a square matrix x. Moreover, the coefficients ϕ_m are Hermitian Jacobi forms of degree $g - l$, weight k and index m that satisfy certain symmetry condition for their ordinary Fourier coefficients from the modularity of f.

The fact that any Hermitian modular form is a formal Fourier–Jacobi series satisfying this symmetry condition and absolute convergence motivates the notion of symmetric formal Fourier–Jacobi series, namely the series satisfying the symmetry condition without assuming absolute convergence. Conversely, if a symmetric formal Fourier–Jacobi series converges absolutely, then it defines a Hermitian modular form. Then one natural question is, does every symmetric formal Fourier–Jacobi series automatically converge absolutely?

Such a rigidity result was already considered by Ibukiyama–Poor–Yuen [17] for Siegel paramodular forms. For Siegel modular forms, Bruinier [18] and Raum [19] resolved independently the case of degree 2 and arbitrary type over $\mathbb{Q}$. In their joint work [7], Bruinier–Raum proved the general case of higher degree and arbitrary type over $\mathbb{Q}$. In this paper, we confirm the Hermitian counterpart in the cases of norm-Euclidean imaginary quadratic fields $E = \mathbb{Q}(\sqrt{d})$, namely for $d \in \{-1, -2, -3, -7, -11\}$.

Theorem 1.1. *Every symmetric formal Fourier–Jacobi series of arbitrary arithmetic type for the unitary group* $\mathrm{U}(g, g)(\mathbb{Z})$*, defined in the cases of norm-Euclidean imaginary quadratic fields E, converges absolutely to a Hermitian modular form.*

Remark 1.2. A more detailed version is presented in Theorem 5.7.

Remark 1.3. After uploading a first version of this paper on arXiv, the author learned that Yuxiang Wang partially proved similar results in his thesis work [20]. Note that Wang claims to prove his results for all imaginary quadratic fields. However, in the proof of Lemma 4.6 on page 39 of his thesis, the choice of r is not justified and in fact cannot be made in the general case. One of the first counterexamples arises from the case of $E = \mathbb{Q}(\sqrt{-5})$, where in Wang's notations for an arbitrarily fixed $a \in \frac{1}{m}(\mathcal{O}^{\#})^g/\mathcal{O}^g$, one cannot even choose $r = (r_1, \ldots, r_g)$ such that $\frac{1}{m}r \equiv a \pmod{\mathcal{O}^g}$ and that $|r_g|^2 < m^2$ (but Wang claims $|r_g|^2 \leq \frac{m^2}{D}$ for $D = 20$).

1.2. Application: The unitary Kudla conjecture

The modularity of generating series of geometric cycles has been studied since the first construction of such modular generating series in the work of Hirzebruch–Zagier [21]. In a long collaboration, Kudla–Millson [22, 23, 24] defined a special family of locally symmetric cycles of Riemannian locally symmetric spaces X, which are called special cycles and proved the modularity of their generating series valued in cohomology classes. It turns out to be particularly interesting when X is a Shimura variety, as the analogous generating series valued in Chow groups can be defined, and it is natural to ask if they are already modular at this level.

In the case of Shimura varieties of orthogonal type over a totally real number field, Kudla raised this question in his seminal work [14]. Inspired by the work of Gross–Kohnen–Zagier [25] on the images of Heegner points in the Jacobian of a modular curve, Borcherds [26] proved the modularity of generating series of Heegner (special) divisors valued in the first Chow group, by employing his work [27] on the construction of a family of meromorphic modular functions via regularised theta lift. Building on Borcherds' work, Zhang [28] proved that the generating series of special cycles valued in Chow groups are modular, assuming absolute convergence. Subsequently, Bruinier–Raum [7] completed the proof of Kudla's modularity conjecture over $\mathbb{Q}$. Most recently, over an arbitrary totally real field of degree d, assuming the Beilinson–Bloch conjecture on the injectivity of the Abel–Jacobi maps, Kudla [29] proved the modularity conjecture for orthogonal Shimura varieties of signature $((m, 2)^{d_+}, (m + 2, 0)^{d-d_+})$; and Maeda [30] independently proved the modularity conjecture in more general cases, assuming the absolute convergence of the generating functions and the Beilinson–Bloch conjecture.

In the case of Shimura varieties of unitary type and codimension 1 (special divisors), the conjecture was verified by Liu in [15]. In recent preprints of Bruinier–Howard–Kudla–Rapoport–Yang (to appear in Astérisque) [31, 32], generating series of special divisors valued in the Chow group and the

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arithmetic Chow group are defined on the compactified integral model; consequently, their modularity result is proved and more arithmetic applications are found, including relations between derivatives of L-functions and arithmetic intersection pairings à la Gross–Zagier and a special case of Colmez's conjecture on the Faltings heights of abelian varieties with complex multiplication. However, the cases of higher codimension remain open problems. In the cases of norm-Euclidean imaginary quadratic fields, we show in this paper that the convergence assumption in Theorem 3.5 of Liu's work [15] can be dropped for arbitrary codimension. The following theorem is proved in Section 6.

Theorem 1.4. *The unitary Kudla conjecture is true for open Shimura varieties over norm-Euclidean imaginary quadratic fields.*

Since modular forms of a fixed weight are finite-dimensional, an immediate consequence of Kudla's modularity conjecture is that the $\mathbb{C}$-ranks of the special cycles in the complexification of the Chow groups are explicitly bounded from above, even though we don't know in general whether the Chow groups are finite-dimensional in the cases of higher codimension. Furthermore, as an appealing consequence, relations between Fourier coefficients of certain Hermitian modular forms give rise to the corresponding relations between special cycles, which are otherwise not accessible in the literature.

1.3. Structure of the proof

The proof of our main result can be separated into three parts.

In the first part (Section 3), we show that every symmetric formal Fourier–Jacobi series f of genus g, cogenus 1 and trivial type is algebraic over the graded algebra of Hermitian modular forms of genus g. This part is where the norm-Euclidean condition is required.

In the second part (Section 4), we prove in two steps that every such f converges on the whole Hermitian upper half-space. First, the algebraicity of f allows us to show the local convergence of f in a neighbourhood of any toroidal boundary of the Hermitian modular variety. Then we show that f can be analytically continued to the whole space. We develop a technique of embedding the Siegel upper half-space to the Hermitian one in a generic yet rational way to pass to the Siegel case, which was resolved by Bruinier–Raum.

Finally, in the third part (Section 5), to cover all arithmetic types and arbitrary cogenus, we argue in two steps by induction. First, fixing the values of genus and cogenus, we consider a natural pairing of (meromorphic) Hermitian modular forms and (meromorphic) symmetric formal Fourier–Jacobi series. Then we increase the cogenus from 1 by induction, using the machinery introduced in Section 2.

2. Hermitian modular forms and symmetric formal Fourier–Jacobi series

Our aim in this section is to provide preliminaries for this paper and introduce the notion of symmetric formal Fourier–Jacobi series in the Hermitian case. First we briefly recall Hermitian modular forms in Subsections 2.1 and 2.2 and introduce notations that are used throughout the paper. Then we define Hermitian Jacobi forms of higher degrees and recall a key lemma connecting the Hermitian modular group to a certain Jacobi group in Subsections 2.3 and 2.4. Based on the notations introduced in Subsections 2.1 and 2.2, we recall Fourier–Jacobi expansions in Subsection 2.5 and introduce their formal analogue, symmetric formal Fourier–Jacobi series, in Subsection 2.6. Finally, we study a formal analogue of the theta decomposition of Jacobi forms and prove a few propositions in Subsections 2.7 and 2.8, which are used in Section 3 and the proof of Proposition 5.5.

2.1. Hermitian modular groups

Historically, in the context of Hermitian modular forms, the unitary group $\mathrm{U}(g, g)$ is defined as a classical group: for instance, in [11]. In most parts of Sections 2–5, following all prior literature in this field, we employ this classical notion as opposed to the perspective of algebraic groups but adopt the notation from the latter for coherence of this paper. Let $E/\mathbb{Q}$ be an imaginary quadratic field, specified at the beginning of each section, and let $\mathrm{U}(g,g)(\mathbb{R})$ denote the classical unitary group of signature (g,g) with entries in $\mathbb{C}$, which in fact corresponds to the real points of a reductive group. Similarly, let $\mathrm{U}(g,g)(\mathbb{Q})$ denote the classical unitary group with entries in E and $\mathrm{U}(g,g)(\mathbb{Z})$ denote the one with entries in the ring of integers $\mathcal{O}_E \subseteq \mathbb{C}$ for a fixed embedding $\iota : E \hookrightarrow \mathbb{C}$. In other words, the classical group $\mathrm{U}(g,g)(\mathbb{R})$ (respectively, $\mathrm{U}(g,g)(\mathbb{Q})$ and $\mathrm{U}(g,g)(\mathbb{Z})$) is the subgroup of $\mathrm{GL}_{2g}(\mathbb{C})$ (respectively, $\mathrm{GL}_{2g}(E)$ and $\mathrm{GL}_{2g}(\mathcal{O}_E)$) whose elements γ satisfy the equation

$$\gamma^* J_{g,g} \gamma = J_{g,g}\,, \tag{2.1}$$

for the $2g \times 2g$ matrix

$$J_{g,g} = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}. \tag{2.2}$$

Writing γ furthermore in blocks of $g \times g$ matrices

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we can characterise these elements γ by the condition

$$a^*c = c^*a,\ b^*d = d^*b, \text{ and } a^*d - c^*b = I_g. \tag{2.3}$$

Similar to the Siegel modular group $\mathrm{Sp}_g(\mathbb{Z}) = \mathrm{U}(g,g)(\mathbb{Z}) \cap \mathrm{Mat}_{2g}(\mathbb{Z})$, we call $\mathrm{U}(g,g)(\mathbb{Z})$ the Hermitian modular group of degree g. The notation $\mathrm{Herm}_n(A)$ stands for the set of $n \times n$ Hermitian matrices with entries in the ring $A \subseteq \mathbb{C}$. For a matrix $x \in \mathrm{Herm}_n(A)$, the notation $x \geq 0$ (respectively, $x > 0$) means x is positive semidefinite (respectively, positive definite), and the set of all such matrices is denoted by $\mathrm{Herm}_n(A)_{\geq 0}$ (respectively, $\mathrm{Herm}_n(A)_{>0}$).

2.2. *Hermitian modular forms*

Basic notions on Hermitian modular forms can be found in [11, 12, 13]. For completeness, we recall them briefly in the case of integral weight and arbitrary arithmetic type.

We define the Hermitian upper half-space of degree g, denoted by $\mathbb{H}_g$, to be the set of matrices $\tau \in \mathrm{Mat}_g(\mathbb{C})$ such that the Hermitian matrix

$$y := \frac{1}{2i}(\tau - \tau^*) \tag{2.4}$$

is positive definite. By this definition, the Siegel upper half-space $\mathcal{H}_g$ is the set of all the symmetric matrices in $\mathbb{H}_g$. Recall that the symplectic group $\mathrm{Sp}_g(\mathbb{R}) = \mathrm{U}(g,g)(\mathbb{R}) \cap \mathrm{Mat}_{2g}(\mathbb{R})$ acts on $\mathcal{H}_g$ biholomorphically. Similarly, $\mathrm{U}(g,g)(\mathbb{R})$ acts on $\mathbb{H}_g$ biholomorphically via Möbius transformations

$$\tau \longmapsto \gamma\tau := (a\tau + b)(c\tau + d)^{-1} \tag{2.5}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{U}(g,g)(\mathbb{R})$. We define the factor of automorphy j via the formula

$$j(\gamma, \tau) := \det\,(c\tau + d)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{U}(g,g)(\mathbb{R})$ and $\tau \in \mathbb{H}_g$. It is well known that $j(\gamma,\tau)$ is always nonzero and satisfies the 1-cocycle condition

$$j(\gamma_1\gamma_2, \tau) = j(\gamma_1, \gamma_2\tau)\, j(\gamma_2, \tau)$$

for all $\gamma_1, \gamma_2 \in \mathrm{U}(g,g)(\mathbb{R})$ and $\tau \in \mathbb{H}_g$.

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Let g, k be integers such that $g \geq 1$ and $\big(\rho, V(\rho)\big)$ a finite-dimensional complex representation of $\mathrm{U}(g,g)(\mathbb{Z})$ that factors through a finite quotient, also called an (arithmetic) type. We define the space of (vector-valued) Hermitian modular forms of degree g, weight k and (arithmetic) type ρ, to be the space of holomorphic (vector-valued) functions $f : \mathbb{H}_g \longrightarrow V(\rho)$ satisfying the modularity condition

$$f(\gamma\tau) = j(\gamma,\tau)^k \rho(\gamma) f(\tau)$$

for all $\gamma \in \mathrm{U}(g,g)(\mathbb{Z})$, and if $g = 1$, being also holomorphic at the cusp of $\mathrm{U}(1,1)(\mathbb{Z})$.

If ρ is trivial, $\mathrm{M}_k^{(g)}(\rho)$ is just a direct sum of copies of classical (scalar valued) Hermitian modular forms, and the latter is denoted by $\mathrm{M}_k^{(g)}$ in this paper. For a finite-index subgroup $\Gamma \subseteq \mathrm{U}(g,g)(\mathbb{Z})$, the space of classical modular forms for Γ is denoted by $\mathrm{M}_k(\Gamma)$. The graded module of Hermitian modular forms of genus g and type ρ is denoted by

$$\mathrm{M}_\bullet^{(g)}(\rho) := \bigoplus_{k\in\mathbb{Z}} \mathrm{M}_k^{(g)}(\rho).$$

Note that if ρ is trivial, $\mathrm{M}_\bullet^{(g)}(\rho)$ is a graded algebra and we write $\mathrm{M}_\bullet^{(g)}$ for it.

Remark 2.1. In the case of half-integral weight, there exist multiplier systems on some congruence subgroups $\Gamma \subseteq \mathrm{U}(g,g)(\mathbb{Z})$ by the general construction of Deligne [33, 34] and the work of Prasad–Rapinchuk [35] and Prasad [36]. For the aim of the unitary Kudla conjecture in this paper, we work with Hermitian modular forms of integral weights.

2.3. Hermitian Jacobi groups

Let g, l be positive integers. Consider a discrete Heisenberg group

$$H_{\mathcal{O}_E}^{(g,l)} := \big\{[(\lambda,\mu),\kappa] : \lambda,\mu \in \mathrm{Mat}_{l,g}(\mathcal{O}_E), \kappa \in \mathrm{Mat}_l(\mathcal{O}_E), \kappa + \mu\lambda^* \in \mathrm{Herm}_l(\mathcal{O}_E)\big\},$$

with the following group law:

$$[(\lambda,\mu),\kappa][(\lambda',\mu'),\kappa'] = [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda\mu'^* - \mu\lambda'^*].$$

Note that the condition $\kappa + \mu\lambda^* \in \mathrm{Herm}_l(\mathcal{O}_E)$ is equivalent to $\kappa - \lambda\mu^* \in \mathrm{Herm}_l(\mathcal{O}_E)$. There is a natural action of the Hermitian modular group $\mathrm{U}(g,g)(\mathbb{Z})$ on the Heisenberg group $H_{\mathcal{O}_E}^{(g,l)}$ via matrix multiplication from the right, and we define the Hermitian Jacobi group $\Gamma^{(g,l)}$ of genus g and cogenus l to be the semidirect product $\Gamma^{(g,l)} = \mathrm{U}(g,g)(\mathbb{Z}) \ltimes H_{\mathcal{O}_E}^{(g,l)}$ with the associated group law. More explicitly, the group law of the Jacobi group $\Gamma^{(g,l)}$ reads

$$\big(\gamma,[(\lambda,\mu),\kappa]\big)\big(\gamma',[(\lambda',\mu'),\kappa']\big) = \big(\gamma\gamma',[(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda\gamma'\mu'^* - \mu\gamma'\lambda'^*]\big).$$

Based on the standard embeddings of Heisenberg groups, we define an embedding

$$\Gamma^{(g,l)} \hookrightarrow \mathrm{U}(g+l,g+l)(\mathbb{Z}) \tag{2.6}$$

of the Jacobi group into the Hermitian modular group via the formula

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [(\lambda,\mu),\kappa]\right) \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & I_l & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & I_l \end{pmatrix}\begin{pmatrix} I_g & 0 & 0 & \mu^* \\ \lambda & I_l & \mu & \kappa \\ 0 & 0 & I_g & -\lambda^* \\ 0 & 0 & 0 & I_l \end{pmatrix}. \tag{2.7}$$

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To relate the embedded image of the Jacobi group to the whole Hermitian modular group, we record the following fact. First we define an embedding $\mathrm{rot} : \mathrm{GL}_g(\mathcal{O}_E) \longrightarrow \mathrm{U}(g,g)(\mathbb{Z})$ of groups via $u \longmapsto \mathrm{rot}(u) := \left(\begin{smallmatrix} u & 0 \\ 0 & u^{*-1} \end{smallmatrix}\right)$.

Lemma 2.2. *Let* $g \geq 2$ *be an integer, and let* E *be an imaginary quadratic field. Then the Hermitian modular group* $\mathrm{U}(g,g)(\mathbb{Z})$ *is generated by the embedded subgroup* $\mathrm{rot}\big(\mathrm{GL}_g(\mathcal{O}_E)\big)$ *and the embedded Jacobi group* $\Gamma^{(g-1,1)}$ *of cogenus* 1 *under equation (2.6).*

Proof. By Theorem 2.1 and Corollary 2.3 in [37] (for an English version, see (1.13) and (1.14) in the thesis of Wernz [38]), the group $\mathrm{U}(g,g)(\mathbb{Z})$ is generated by the embedded subgroup $\mathrm{rot}\big(\mathrm{GL}_g(\mathcal{O}_E)\big)$, the matrix $J_{g,g}$ defined in equation (2.2) and the parabolic subgroup

$$P := \left\{ \left(\begin{smallmatrix} I_g & H \\ 0 & I_g \end{smallmatrix}\right) : H \in \mathrm{Herm}_g(\mathcal{O}_E) \right\}.$$

First note that the parabolic subgroup P is already in the embedded Jacobi group $\Gamma^{(g-1,1)}$ by putting $l = 1$, $a = d = I_{g-1}$, $c = 0_{g-1}$, $\lambda = 0_{1\times(g-1)}$ in the embedding in equation (2.7) and noticing that the condition $\kappa + \mu\lambda^* \in \mathrm{Herm}_l(\mathcal{O}_E)$ in the definition of the Jacobi group becomes $\kappa \in \mathbb{Z}$ in this case.

Then we consider the matrix $J_{g,g}$. For each integer j such that $1 \leq j \leq g$, let

$$\iota^{(j)} : \mathrm{U}(1,1)(\mathbb{Z}) \longrightarrow \mathrm{U}(g,g)(\mathbb{Z})$$

be the jth diagonal embedding sending an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\iota^{(j)}(\gamma)$ whose 2×2 submatrix indexed by $(j,j), (j,j+g), (j+g,j), (j+g,j+g)$ is equal to γ, and the remaining entries of $\iota^{(j)}(\gamma)$ are those of the identity matrix I_{2g}. Note that the matrix $J_{g,g}$ can be decomposed into

$$J_{g,g} = \iota^{(g)}(\gamma)\iota^{(g-1)}(\gamma)\cdots\iota^{(1)}(\gamma)$$

for $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{U}(1,1)(\mathbb{Z})$. Furthermore, for any $\gamma \in \mathrm{U}(1,1)(\mathbb{Z})$ and any $j \in \{2,3,\ldots,g\}$, we have the relations

$$\iota^{(j)}(\gamma) = \mathrm{rot}(u_j)\iota^{(1)}(\gamma)\mathrm{rot}(u_j)$$

for $u_j \in \mathrm{GL}_g(\mathcal{O}_E)$ defined by putting its 2×2 submatrix indexed by $(1,1), (1,j), (j,1), (j,j)$ to be $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and the remaining entries to be those of the identity matrix I_g. Finally, for any $\gamma \in \mathrm{U}(1,1)(\mathbb{Z})$, it is clear that $\iota^{(1)}(\gamma)$ is an element in the embedded Jacobi group $\Gamma^{(g-1,1)}$ (in fact, this element lies in the embedded Jacobi group of arbitrary cogenus, not only the one of cogenus 1), as desired. □

Remark 2.3. From this proof, we see that the statement of Lemma 2.2 actually holds for all imaginary quadratic fields E and Jacobi groups of arbitrary cogenus. This might be an attractive feature of Hermitian modular groups for future investigation, even though we only need Jacobi groups of cogenus 1 and a few cases of imaginary quadratic fields in this paper.

2.4. Hermitian Jacobi forms

Jacobi forms are closely related to Fourier–Jacobi expansions of automorphic forms. In the case of degree $(1,1)$ associated to Siegel modular forms, Eichler–Zagier developed a theory of Jacobi forms in their monograph [39]. The case of degree $(1,1)$ associated to Hermitian modular forms over imaginary quadratic fields was treated by Haverkamp [40, 41]. Cases of higher degrees were investigated in [42, 43, 44], and in the spirit of Eichler–Zagier a theory was built up by Ziegler [45]. To fit the presentation of this paper and to simplify the expressions, we define Hermitian Jacobi forms in terms of Hermitian modular forms via the embedding of Jacobi groups into Hermitian modular groups.

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Let g, l be positive integers. We define the Hermitian Jacobi upper half-space $\mathbb{H}_{g,l}$ of genus g and cogenus l to be $\mathbb{H}_{g,l} := \mathbb{H}_g \times \mathrm{Mat}_{g,l}(\mathbb{C}) \times \mathrm{Mat}_{l,g}(\mathbb{C})$, and there is a (biholomorphic) action of the Jacobi group $\Gamma^{(g,l)}$ on $\mathbb{H}_{g,l}$ via the formula

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [(\lambda, \mu), \kappa]\right)(\tau, w, z)$$
$$=\Big((a\tau+b)(c\tau+d)^{-1}, \big(a-(a\tau+b)(c\tau+d)^{-1}c\big)(w+\tau\lambda^*+\mu^*), (z+\lambda\tau+\mu)(c\tau+d)^{-1}\Big).$$

Note that the Hermitian upper half-space $\mathbb{H}_{g+l}$ projects to the Hermitian Jacobi upper half-space $\mathbb{H}_{g,l}$ by restricting to the corresponding matrix blocks.

For an imaginary quadratic field $E/\mathbb{Q}$, we recall the dual lattice $\mathcal{O}_E^\#$ of the lattice of integers $\mathcal{O}_E$, also known as the inverse different ideal of E, which in our case can be explicitly written as $\mathcal{O}_E^\# = \frac{1}{\sqrt{D_E}}\mathcal{O}_E$, where $D_E(<0)$ is the discriminant of E. We say a Hermitian matrix m is semi-integral (over E) if the diagonal entries of m are in the ring $\mathbb{Z}$ and the off-diagonal entries are in the dual lattice $\mathcal{O}_E^\#$.

Let k be an integer, and let $m \in \mathrm{Herm}_l(E)$ be an $l \times l$ Hermitian matrix with entries in E. Let ρ be a complex representation of the Jacobi group $\Gamma^{(g,l)}$. Recall that we use the notation $e(x) = \exp\big(2\pi i \cdot \mathrm{Tr}(x)\big)$ for a square matrix x throughout this paper. We say a holomorphic vector-valued function $\phi : \mathbb{H}_{g,l} \longrightarrow V(\rho)$ is a Hermitian Jacobi form of genus g, weight k, index m and type ρ, if the function $f : \mathbb{H}_{g+l} \longrightarrow V(\rho)$ defined via

$$f\left(\begin{pmatrix} \tau & w \\ z & \tau' \end{pmatrix}\right) = \phi(\tau, w, z)e(m\tau')$$

transforms as a Hermitian modular form for the image of $\Gamma^{(g,l)}$ under the embedding in equation (2.6), of degree $g+l$, weight k and type ρ, and if $g = 1$, the function $\phi(\tau, \tau r + r', s\tau + s')$ also being bounded (under any norm as $\dim V(\rho) < \infty$) at the cusp for all row vectors r, r' and column vectors s, s' of dimension l over E. The space of Hermitian Jacobi forms of genus g, weight k, index m and type ρ is denoted by $\mathrm{J}^{(g)}_{k,m}(\rho)$. To see the similarity between Hermitian Jacobi forms and Siegel Jacobi forms, we spell out the transformation formulae for Hermitian Jacobi forms $\phi(\tau, w, z)$ explicitly using the embedding in equation (2.7). For any element $\big(\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big), [(\lambda, \mu), \kappa]\big) \in \Gamma^{(g,l)}$ in the Jacobi group and any point $(\tau, w, z) \in \mathbb{H}_{g,l}$ in the Hermitian Jacobi upper half-space, the equation for invariance under the embedded Hermitian modular subgroup

$$\phi\Big((a\tau+b)(c\tau+d)^{-1}), \big(a-(a\tau+b)(c\tau+d)^{-1}c\big)w, z(c\tau+d)^{-1}\Big)$$
$$= \det\,(c\tau+d)^k e\big(mz(c\tau+d)^{-1}cw\big)\rho\Big(\big(\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big), [(0,0),0]\big)\Big)\phi(\tau, w, z)$$

and the equation for invariance under the embedded discrete Heisenberg group

$$\phi\big(\tau, w+\tau\lambda^*+\mu^*, z+\lambda\tau+\mu\big) = e\big(-m(\lambda\tau\lambda^*+\lambda w+z\lambda^*+\mu\lambda^*+\kappa)\big)\rho\big(I_{2g}, [(\lambda,\mu),\kappa]\big)\phi(\tau, w, z)$$

hold for a Jacobi form $\phi \in \mathrm{J}^{(g)}_{k,m}(\rho)$.

When the context is clear, for a complex representation ρ of the Hermitian modular group $\mathrm{U}(g+l, g+l)(\mathbb{Z})$, we also write ρ for its restriction to the image of the Jacobi group $\Gamma^{(g,l)}$. Conversely, for a complex representation ρ' of the Jacobi group $\Gamma^{(g,l)}$, we also write ρ' for its restriction to the subgroup $\mathrm{U}(g,g)(\mathbb{Z})$.

2.5. Fourier–Jacobi expansions

Let g, l be integers such that $1 \leq l \leq g-1$. Fourier–Jacobi expansions of Hermitian modular forms are partial Fourier expansions with respect to the matrix block τ_2 of the variable $\tau = \left(\begin{smallmatrix} \tau_1 & z \\ w & \tau_2 \end{smallmatrix}\right)$. More precisely, for a Hermitian modular form f of degree g, weight k and a type ρ factoring through a finite quotient of $\mathrm{U}(g,g)(\mathbb{Z})$, there is a Fourier expansion ([11]) in the form of

$$f(\tau) = \sum_{t \in \mathrm{Herm}_g(E)_{\geq 0}} c(f;t) e(t\tau) \tag{2.8}$$

for $c(f;t) \in V(\rho)$, where the matrices t appearing in the Fourier expansion depend on the type ρ. In particular, when ρ is the trivial representation, the sum is supported on semi-integral positive semidefinite Hermitian matrices t. We write

$$\tau = \left(\begin{smallmatrix} \tau_1 & w \\ z & \tau_2 \end{smallmatrix}\right) \in \mathbb{H}_g \quad \text{and} \quad t = \left(\begin{smallmatrix} n & r \\ r^* & m \end{smallmatrix}\right) \in \mathrm{Herm}_g(E)_{\geq 0}$$

for $\tau_1 \in \mathbb{H}_{g-l}, \tau_2 \in \mathbb{H}_l, w \in \mathrm{Mat}_{g-l,l}(\mathbb{C}), z \in \mathrm{Mat}_{l,g-l}(\mathbb{C}), n \in \mathrm{Herm}_{g-l}(E)_{\geq 0}, r \in \mathrm{Mat}_{g-l,l}(E)$ and $m \in \mathrm{Herm}_l(E)_{\geq 0}$. Consequently, we arrange the sum in equation (2.8) into the form

$$f(\tau) = \sum_{m \in \mathrm{Herm}_l(E)_{\geq 0}} \phi_m(\tau_1, w, z) e(m\tau_2) \tag{2.9}$$

for functions $\phi_m : \mathbb{H}_{g-l,l} \longrightarrow V(\rho)$ with the ordinary Fourier expansion

$$\phi_m(\tau_1, w, z) = \sum_{\substack{n \in \mathrm{Herm}_{g-l}(E)_{\geq 0} \\ r \in \mathrm{Mat}_{g-l,l}(E)}} c(\phi_m; n, r) e(n\tau_1 + rz) e(r^* w), \tag{2.10}$$

whose Fourier coefficients $c(\phi_m; n, r)$ are determined by the Fourier coefficients of f via the equation

$$c(\phi_m; n, r) = c\left(f; \begin{pmatrix} n & r \\ r^* & m \end{pmatrix}\right). \tag{2.11}$$

We say equation (2.9) is the Fourier–Jacobi expansion of f of cogenus l and ϕ_m is the mth Fourier–Jacobi coefficient of f, or the Fourier–Jacobi coefficient of index m. Furthermore, it follows from the definition that any Jacobi form $\phi \in \mathrm{J}^{(g)}_{k,m}(\rho)$ has the Fourier expansion in the form of equation (2.10).

2.6. Symmetric formal Fourier–Jacobi series

We motivate in this subsection the notion of symmetric formal Fourier–Jacobi series for the Hermitian modular group, by combining two features of the Fourier–Jacobi expansion of a Hermitian modular form: each Fourier–Jacobi coefficient is a Hermitian Jacobi form, and they satisfy a $\mathrm{GL}_g(\mathcal{O}_E)$-symmetry condition. A priori, for formal series, these two features 'almost' characterise 'formal modularity': that is, invariance under the action of the Hermitian modular group without assuming absolute convergence. The group-theoretic Lemma 2.2 amounts to saying that formal modularity can be reformulated in terms of two conditions: invariance under the embedded Jacobi group of cogenus 1, and invariance under the embedded subgroup $\mathrm{rot}\big(\mathrm{GL}_g(\mathcal{O}_E)\big)$.

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It is clear that the invariance of a formal Fourier series f in the form of equation (2.8) under the weight k, type ρ-slash action of $\mathrm{rot}\big(\mathrm{GL}_g(\mathcal{O}_E)\big)$ is equivalent to a symmetry condition for the formal Fourier coefficients of f, which reads

$$\rho\left(\begin{pmatrix} u & 0 \\ 0 & u^{*-1} \end{pmatrix}\right) c(f; u^* t u) = \big(\det u^*\big)^k c(f; t) \tag{2.12}$$

for all $u \in \mathrm{GL}_g(\mathcal{O}_E)$.

Moreover, the invariance of f under the weight k, type ρ-slash action of the embedded Jacobi group $\Gamma^{(g-l,l)}$ under equation (2.6) amounts to saying that f can be rearranged into a formal series in the form of equation (2.9), such that each coefficient ϕ_m is a formal Hermitian Jacobi form of genus $g-l$, weight k, index k and type ρ.

Combining these two aspects, for a complex representation ρ of $\mathrm{U}(g,g)(\mathbb{Z})$, we define a symmetric formal Fourier–Jacobi series f of degree g, cogenus l, weight k and type ρ to be a formal series of Hermitian Jacobi forms $\phi_m \in \mathrm{J}^{(g-l)}_{k,m}(\rho)$ in the form of equation (2.9), such that its Fourier coefficients $c(f;t)$ defined by the equation

$$c\left(f; \begin{pmatrix} n & r \\ r^* & m \end{pmatrix}\right) = c(\phi_m; n, r)$$

satisfy the symmetry condition in equation (2.12).

We write $\mathrm{FM}^{(g,l)}_k(\rho)$ for the vector space of such symmetric formal Fourier–Jacobi series, and when the cogenus $l = 1$ or ρ is the 1-dimensional trivial representation, we suppress them in the notation. Furthermore, we let

$$\mathrm{FM}^{(g)}_\bullet(\rho) := \bigoplus_{k \in \mathbb{Z}} \mathrm{FM}^{(g)}_k(\rho)$$

denote the graded module over the graded ring $\mathrm{M}^{(g)}_\bullet$ of classical Hermitian modular forms, and note that $\mathrm{FM}^{(g)}_\bullet$ is actually a graded algebra over $\mathrm{M}^{(g)}_\bullet$.

Remark 2.4. Of course, readers will not confuse the degree g in $\mathrm{FM}^{(g,l)}_k(\rho)$ for the genus g in $\mathrm{J}^{(g)}_{k,m}(\rho)$. In fact, for a symmetric formal Fourier–Jacobi series $f \in \mathrm{FM}^{(g,l)}_k(\rho)$, the mth coefficient ϕ_m of f is a Jacobi form of genus $g-l$ and index m, more precisely $\phi_m \in \mathrm{J}^{(g-l)}_{k,m}(\rho)$.

2.7. Formal Fourier–Jacobi coefficients

Just as Fourier–Jacobi coefficients of arbitrary cogenus are attached to a Hermitian modular form, we associate formal Fourier–Jacobi coefficients of various cogenus to a formal Fourier–Jacobi series.

Let $1 \le l' < l < g$ be positive integers, and let $f \in \mathrm{M}^{(g)}_k(\rho)$ be a Hermitian modular form. For $\tau = \left(\begin{smallmatrix} \tau_1 & w \\ z & \tau_2 \end{smallmatrix}\right) \in \mathbb{H}_g$, we refine the decomposition of the matrix τ further into blocks

$$\tau = \begin{pmatrix} \tau_{11} & w_{11} & w_{12} \\ z_{11} & \tau_{12} & w_{22} \\ z_{12} & z_{22} & \tau_2 \end{pmatrix},$$

so that the sizes of τ_{11}, τ_{12} and τ_2 are $(g-l) \times (g-l)$, $(l-l') \times (l-l')$ and $l' \times l'$, respectively, and the sizes of the off-diagonal blocks are determined in the ordinary way. We write w_1 for the $(g-l) \times l$ matrix $(w_{11}\, w_{12})$ and z_1 for the $l \times (g-l)$ matrix $\left(\begin{smallmatrix} z_{11} \\ z_{12} \end{smallmatrix}\right)$.

For a formal Fourier series f in the form of equation (2.8), the formal Fourier–Jacobi expansion of cogenus l is given by equation (2.9), with the mth formal Fourier–Jacobi coefficient ψ_m defined in

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equations (2.10) and (2.11), where we replace the notation ϕ_m by ψ_m to emphasise that it is a formal series. In particular, for a symmetric formal Fourier–Jacobi series $f \in \mathrm{FM}_k^{(g,l)}(\rho)$ with Fourier–Jacobi coefficients ϕ_m, and for each Hermitian matrix $m' \in \mathrm{Herm}_{l'}(E)_{\geq 0}$, the formal Fourier–Jacobi coefficient $\psi_{m'}$ of index m' is related to ϕ_m via an equation of formal Fourier-series, namely

$$\psi_{m'}(\tau_1, w, z) = \sum_{\substack{n' \in \mathrm{Herm}_{l-l'}(E)_{\geq 0} \\ r' \in \mathrm{Mat}_{l-l', l'}(E)}} \phi_{\left(\begin{smallmatrix} n' & r' \\ r'^* & m' \end{smallmatrix}\right)}(\tau_{11}, w_1, z_1) e(n'\tau_{12} + r' z_{22}) e(r'^* w_{22}). \tag{2.13}$$

When $l' = 1$, we need to use the following fact about the formal Fourier–Jacobi coefficient ψ_0 of index $0 \in \mathrm{Herm}_1(E)_{\geq 0}$. Note that the following fact also holds for arbitrary l', but we only need the case $l' = 1$ in the proof of Proposition 5.5.

Proposition 2.5. *Let $f \in \mathrm{FM}_k^{(g,l)}$ be a symmetric formal Fourier–Jacobi series of trivial type. Then the formal Fourier–Jacobi coefficient ψ_0 defined by equation (2.13) is a symmetric formal Fourier–Jacobi series of degree $g - 1$ and cogenus $l - 1$ (under an identification), namely $\psi_0 \in \mathrm{FM}_k^{(g-1,l-1)}$.*

Proof. For any Hermitian matrix $\left(\begin{smallmatrix} n' & r' \\ r'^* & 0 \end{smallmatrix}\right)$ that occurs in equation (2.13), it is positive semidefinite, hence $r' = 0$. Moreover, for any index $n' \in \mathrm{Herm}_{l-1}(E)_{\geq 0}$, as $\phi_{\left(\begin{smallmatrix} n' & 0 \\ 0 & 0 \end{smallmatrix}\right)}$ is a Hermitian Jacobi form, by the transformation law it must be a constant function in w_{12} and z_{12}. In particular, $\phi_{\left(\begin{smallmatrix} n' & 0 \\ 0 & 0 \end{smallmatrix}\right)}$ can be identified as a Jacobi form of genus $g - l$ and index n', which we write as $\phi_{n'}$. Therefore, ψ_0 can be identified as a formal Fourier–Jacobi series

$$\psi_0(\tau_1) = \sum_{n' \in \mathrm{Herm}_{l-1}(E)_{\geq 0}} \phi_{n'}(\tau_{11}, w_{11}, z_{11}) e(n'\tau_{12}).$$

The symmetry condition of the formal Fourier–Jacobi series ψ_0 under the action of the embedded subgroup $\mathrm{GL}_{g-1}(\mathcal{O}_E)$ follows from the one of f under the action of $\mathrm{GL}_g(\mathcal{O}_E)$. □

2.8. Formal theta decomposition

We show in this subsection that the classical theta decomposition of Jacobi forms has a formal analogue, namely the formal theta decomposition of formal Fourier–Jacobi coefficients.

Let g, l be arbitrary positive integers, and let $m \in \mathrm{Herm}_l(E)_{>0}$ be a semi-integral positive definite Hermitian matrix, which we also view as an integral positive definite quadratic form. Let $\Delta_g(m)$ denote the finite abelian group

$$\Delta_g(m) = \mathrm{Mat}_{g,l}(\mathcal{O}_E{}^{\#}) / \mathrm{Mat}_{g,l}(\mathcal{O}_E) m,$$

and let $\rho_m^{(g)}$ be the Weil representation of $\mathrm{U}(g,g)(\mathbb{Z})$ associated with $\Delta_g(m)$ on $\mathbb{C}[\Delta_g(m)]$. Recall that the theta series $\theta_{m,s}^{(g)} : \mathbb{H}_{g,l} \longrightarrow \mathbb{C}$ of genus g, integral quadratic form m and shift $s \in \Delta_g(m)$ is defined by

$$\theta_{m,s}^{(g)}(\tau, w, z) = \sum_{r \in s + \mathrm{Mat}_{g,l}(\mathcal{O}_E) m} e(r m^{-1} r^* \tau + rz) e(r^* w). \tag{2.14}$$

It is a classical result (Proposition 3.5 of [43]; see also Theorem 5.1 of [39]) that the vector-valued theta series $\left(\theta_{m,s}^{(g)}\right)_{s \in \Delta_g(m)}$ transforms as a vector-valued Hermitian Jacobi form of weight l and type $\left(\rho_m^{(g)}\right)^{\vee}$. Furthermore, any Hermitian Jacobi form $\phi \in \mathrm{J}_{k,m}^{(g)}$ can be uniquely written as a sum

$$\phi(\tau, w, z) = \sum_{s \in \Delta_g(m)} h_s(\tau) \theta_{m,s}^{(g)}(\tau, w, z) \tag{2.15}$$

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for some holomorphic functions h_s, and these functions are the components of a vector-valued Hermitian modular form of type $\rho_m^{(g)}$.

Let $1 \leq l' < l < g$ be positive integers. For a symmetric formal Fourier–Jacobi series $f \in \mathrm{FM}_k^{(g,l)}$ with Fourier–Jacobi coefficients ϕ_m for $m \in \mathrm{Herm}_l(E)_{\geq 0}$, we construct the formal theta decomposition of its formal Fourier–Jacobi coefficients. Let $m' \in \mathrm{Herm}_{l'}(E)_{>0}$ be a positive definite Hermitian matrix, and let $\psi_{m'}$ be the m'th formal Fourier–Jacobi coefficient of f. Since the formal Fourier coefficients of f satisfy by definition the symmetry condition in equation (2.12), setting $u = \begin{pmatrix} I_{g-l'} & 0 \\ \lambda^* & I_{l'} \end{pmatrix}$ for $\lambda \in \mathrm{Mat}_{g-l',l'}(\mathcal{O}_E)$ in equation (2.12), we find a symmetry condition for the formal Fourier coefficients of $\psi_{m'}$, namely

$$c(\psi_{m'}; n' + r'm'^{-1}r'^*, r') = c(\psi_{m'}; n' + r''m'^{-1}r''^*, r'') \tag{2.16}$$

for all $n' \in \mathrm{Herm}_{g-l'}(E)_{\geq 0}$ and all $r', r'' \in \mathrm{Mat}_{g-l',l'}(E)$ such that $r'' = r' + \lambda m'$ for some $\lambda \in \mathrm{Mat}_{g-l',l'}(\mathcal{O}_E)$. In particular, for each $s' \in \Delta_{g-l'}(m')$, equation (2.16) defines a formal Fourier series $h_{m',s'}$ in τ_1 via its formal Fourier coefficients

$$c(h_{m',s'}; n') := c(\psi_{m'}; n' + r'm'^{-1}r'^*, r'), \tag{2.17}$$

for any choice of $r' \in s' + \mathrm{Mat}_{g-l',l'}(\mathcal{O}_E)m'$ and all $n' \in \mathrm{Herm}_{g-l'}(E)_{\geq 0}$. It then follows that

$$\psi_{m'}(\tau_1, w, z) = \sum_{s' \in \Delta_{g-l'}(m')} h_{m',s'}(\tau_1)\theta_{m',s'}^{(g-l')}(\tau_1, w, z) \tag{2.18}$$

holds as an equation of formal Fourier series.

Our aim is to show that the vector-valued formal Fourier series $(h_{m',s'})_{s'}$ is in fact a symmetric formal Fourier–Jacobi series. This fact can be proved for arbitrary cogenus l'; for simplicity, we only do the case $l' = l - 1$, which is the only case that we need in Section 5. We separate our proof into the following two lemmas and state the conclusion after that.

Lemma 2.6. *Let $m' \in \mathrm{Herm}_{l'}(E)_{>0}$ be a positive definite Hermitian matrix. Then the formal vector-valued Fourier series $h_{m'} = (h_{m',s'})_{s' \in \Delta_{g-l'}(m')}$ defined by equation (2.17) satisfies the symmetry condition in equation (2.12) for the Weil representation $\rho = \rho_{m'}^{(g-l')}$.*

Proof. Setting $u = \begin{pmatrix} u' & 0 \\ 0 & I_{l'} \end{pmatrix}$ for $u' \in \mathrm{GL}_{g-l'}(\mathcal{O}_E)$ in the symmetry condition in equation (2.12) of $f \in \mathrm{FM}_k^{(g,l)}$, we find a symmetry condition

$$c(\psi_{m'}; u'^*n'u', u'^*r') = (\det u'^*)^k c(\psi_{m'}; n', r') \tag{2.19}$$

for all $n' \in \mathrm{Herm}_{g-l'}(E)_{\geq 0}$, $r' \in \mathrm{Mat}_{g-l',l'}(\mathcal{O}_E)$ and $u' \in \mathrm{GL}_{g-l'}(\mathcal{O}_E)$. Combining the definition in equation (2.17) of $h_{m',s'}$ and equation (2.19), we see that

$$c(h_{m',u'^*s'}; u'^*n'u') = (\det u'^*)^k c(h_{m',s'}; n')$$

holds for all $n' \in \mathrm{Herm}_{g-l'}(E)_{\geq 0}$, $s' \in \Delta_{g-l'}(m')$ and $u' \in \mathrm{GL}_{g-l'}(\mathcal{O}_E)$, as desired. □

Next, for the formal Fourier series $h_{m',s'}$ defined by equation (2.17), in line with our goal to prove Proposition 2.8, we only need to consider their cogenus 1 formal Fourier–Jacobi expansions

$$h_{m',s'}(\tau_1) = \sum_{n \in \mathbb{Q}_{\geq 0}} \psi_{m',s',n}(\tau_{11}, w_{11}, z_{11})e(n\tau_{12}), \tag{2.20}$$

and relate their formal Fourier–Jacobi coefficients $\psi_{m',s',n}$ to the Fourier–Jacobi coefficients ϕ_m of $f \in \mathrm{FM}_k^{(g,l)}$. In fact, equations (2.13) and (2.18) provide such a relation. Inserting equation (2.20) in

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equation (2.18), and writing $s' \in \Delta_{g-l'}(m')$ in the form $\left(\begin{smallmatrix} s'_1 \\ s'_2 \end{smallmatrix}\right)$ for $s'_1 \in \Delta_{g-l}(m')$ and $s'_2 \in \Delta_1(m')$, we obtain the following result after comparing the formal Fourier coefficient at $e(n'\tau_{12} + r'z_{22})e(r'^{*}w_{22})$ with equation (2.13) and simplifying the expression.

Lemma 2.7. *Let equation (2.20) be the formal Fourier–Jacobi expansion of cogenus 1 of $h_{m',s'}$. Then for any shift $s'_2 \in \Delta_1(m')$, every representative $r' \in s'_2 + \mathrm{Mat}_{1,l'}(\mathcal{O}_E)m'$ and every integer $n' \geq 0$, the partial theta decomposition of Fourier–Jacobi coefficients*

$$\phi_{\left(\begin{smallmatrix} n' & r' \\ r'^{*} & m' \end{smallmatrix}\right)}(\tau_{11}, w_1, z_1) = \sum_{s'_1 \in \Delta_{g-l}(m')} \psi_{m', \left(\begin{smallmatrix} s'_1 \\ s'_2 \end{smallmatrix}\right), n' - r'm'^{-1}r'^{*}}(\tau_{11}, w_{11}, z_{11}) \theta^{(g-l)}_{m', s'_1}(\tau_{11}, w_{12} + w_{11}r'm'^{-1}, z_{12} + m'^{-1}r'^{*}z_{11}) \tag{2.21}$$

holds as an equation of formal Fourier series. In particular, every formal Fourier–Jacobi coefficient $\psi_{m',s',n}$ in equation (2.20) is a holomorphic function in τ_{11}, w_{11} and z_{11}.

Proposition 2.8. *Let $g > l$ be positive integers, and fix $l' = l - 1$. Let $f \in \mathrm{FM}^{(g,l)}_k$ be a symmetric formal Fourier–Jacobi series, and let $m' \in \mathrm{Herm}_{l'}(E)_{>0}$ be a semi-integral positive definite Hermitian matrix. Then the formal Fourier–Jacobi coefficient $\psi_{m'}$ of f has a formal theta expansion*

$$\psi_{m'}(\tau_1, w, z) = \sum_{s' \in \Delta_{g-l'}(m')} h_{m',s'}(\tau_1) \theta^{(g-l')}_{m',s'}(\tau_1, w, z),$$

where the coefficients are vector-valued symmetric formal Fourier–Jacobi series, more precisely

$$(h_{m',s'})_{s'} \in \mathrm{FM}^{(g-l')}_{k-l'}(\rho^{(g-l')}_{m'}).$$

Proof. By Lemmas 2.6 and 2.7, it suffices to prove the modularity of the vector-valued function

$$(\psi_{m',s',n})_{s' \in \Delta_{g-l'}(m')}.$$

We analyse both sides of equation (2.21) in Lemma 2.7. On the left-hand side, by the assumption $f \in \mathrm{FM}^{(g,l)}_k$, the function transforms with weight k and index $\left(\begin{smallmatrix} n' & r' \\ r'^{*} & m' \end{smallmatrix}\right)$ under the action of the Hermitian Jacobi group $\Gamma^{(g-l,l)}$. On the right-hand side, the vector-valued theta function transforms by the restriction of $(\rho^{(g)}_{m'})^{\vee}$ to $\Gamma^{(g-l,l')}$, of weight l'. Arguing as in Section 3 of [45], we conclude that the vector-valued holomorphic function $(\psi_{m',s',n})_{s' \in \Delta_{g-l'}(m')}$ transforms by the restriction of $\rho^{(g)}_{m'}$ to $\Gamma^{(g-l,1)}$, of weight $k - l'$, which completes the proof. □

Remark 2.9. Note that the cases for l' being other than $l-1$ might need more technical effort, as in these cases, the relations analogous to equation (2.21) between formal Fourier–Jacobi coefficients $\psi_{m',s',n}$ and the Fourier–Jacobi coefficients ϕ_m involve multiple copies of $\Delta_g(m)$ in general when we define the Weil representation and theta functions. This is why we only prove the case $l' = l - 1$, which already suffices for this paper.

Finally, we state the classical result mentioned from the beginning. Let $\phi \in \mathrm{J}^{(g)}_{k,m}(\rho)$ be a Hermitian Jacobi form; then its Fourier coefficients $c(\phi; n, r)$ satisfy a symmetry condition similar to equation (2.16), which then defines a formal Fourier series h_s via

$$c(h_s; n) := c(\phi; n + rm^{-1}r^{*}, r)$$

for each $s \in \Delta_g(m)$ and any choice of $r \in s + \frac{1}{N}\mathrm{Mat}_{g,l}(\mathcal{O}_E)m$ for a suitable positive integer $N = N(\rho)$. Then we have an equation of formal Fourier series similar to equation (2.15), and the vector-valued holomorphic function $(h_s)_{s \in \Delta_g(m)}$ satisfying the equation is unique. Similar to the formal theta

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decomposition, we follow the lines of [45], Section 3 and obtain the classical theta decomposition in the Hermitian case.

Proposition 2.10. *Let g and l be positive integers. Let $m \in \mathrm{Herm}_l(E)_{>0}$ be a positive definite Hermitian matrix. Let $\rho_m^{(g)}$ be the Weil representation of $\mathrm{U}(g,g)(\mathbb{Z})$ on $\mathbb{C}[\Delta_g(m)]$, and let ρ be a complex finite-dimensional representation of the Jacobi group $\Gamma^{(g,l)}$ that factors through a finite quotient. We also write ρ for its restriction to the subgroup $\mathrm{U}(g,g)(\mathbb{Z})$. Then there is an isomorphism*

$$\begin{aligned} \mathrm{J}_{k,m}^{(g)}(\rho) &\cong \mathrm{M}_{k-l}^{(g)}(\rho_m^{(g)} \otimes \rho) \\ \phi &\longmapsto (h_s)_{s \in \Delta_g(m)}, \end{aligned}$$

sending a Hermitian Jacobi form ϕ to the components of its theta decomposition.

3. Symmetric formal Fourier–Jacobi series are algebraic over Hermitian modular forms

The aim of this section is to prove Theorem 3.15. Throughout this section, let $E/\mathbb{Q}$ be a norm-Euclidean imaginary quadratic field over which the Hermitian modular forms are defined. In fact, except for Lemma 3.5 and its corollaries, the whole section also works for $E/\mathbb{Q}$ being an arbitrary imaginary quadratic field. The proof of Theorem 3.15 is based on the following asymptotic analysis: on the one hand, the dimension formula in Proposition 3.13 provides an asymptotic lower bound as in Corollary 3.14 for the total dimension of Hermitian modular forms of bounded weight; on the other hand, the embedding constructed in Lemma 3.3 gives rise to an asymptotic upper bound as in Corollary 3.12 for the total dimension of symmetric formal Fourier–Jacobi series of bounded weight; but the orders of these two bounds in terms of the weight coincide, due to the nonvanishing of the lower slope bound for Hermitian modular forms, defined in Subsection 3.1 and proven in Subsection 3.2.

3.1. Vanishing orders and slope bounds

Let $t \in \mathrm{Herm}_g(E)$. We say that t represents a rational number h if there is a nonzero integral element $\omega \in \mathcal{O}_E^g \setminus \{0\}$ such that $\omega^* t \omega = h$. For a nonzero vector-valued formal series

$$f(\tau) = \sum_{t \in \mathrm{Herm}_g(E)} c(f;t) e(t\tau) \tag{3.1}$$

for $\tau \in \mathbb{H}_g$, we define its *vanishing order* to be

$$\mathrm{ord} f := \inf \left\{ h \in \mathbb{Q} : h \text{ can be represented by some } t \in \mathrm{Herm}_g(E) \text{ such that } c(f;t) \neq 0 \right\}.$$

If $f = 0$, we set $\mathrm{ord} f := \infty$ by convention.

Lemma 3.1. *Let f_1, f_2 be any two formal series in the form of equation (3.1). Then their vanishing orders satisfy*

$$\mathrm{ord}(f_1 \otimes f_2) \geq \mathrm{ord} f_1 + \mathrm{ord} f_2.$$

Proof. The assertion is clear if $f_1 = 0$ or $f_2 = 0$. Assume now f_1 and f_2 are both nonzero, and let $t \in \mathrm{Herm}_g(E)$ be a Hermitian matrix such that $c(f_1 \otimes f_2; t) \neq 0$. Since

$$c(f_1 \otimes f_2; t) = \sum_{\substack{0 \leq t_1, t_2 \in \mathrm{Herm}_g(E), \\ t_1 + t_2 = t}} c(f_1; t_1) \otimes c(f_2; t_2),$$

there must be some pair (t_1, t_2) such that $t_1 + t_2 = t$, $c(f_1; t_1) \neq 0$ and $c(f_2; t_2) \neq 0$, as desired. □

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Similarly, we define the *vanishing order* of a nonzero vector-valued formal series

$$\phi(\tau, w, z) = \sum_{\substack{t \in \mathrm{Herm}_g(E), \\ r \in \mathrm{Mat}_{g,l}(E)}} c(\phi; t, r) e(t\tau + rz) e(r^* w) \tag{3.2}$$

for $(\tau, w, z) \in \mathbb{H}_{g,l}$ to be

$$\begin{aligned} \mathrm{ord}\, \phi := \inf \big\{ h \in \mathbb{Q} : h \text{ can be represented by some } t \in \mathrm{Herm}_g(E) \text{ such that} \\ c(\phi; t, r) \neq 0 \text{ for some } r \in \mathrm{Mat}_{g,l}(E) \big\}. \end{aligned}$$

If $\phi = 0$, we set $\mathrm{ord}\, \phi := \infty$ by convention. Spaces of (vector-valued) Hermitian modular (respectively, Hermitian Jacobi) forms of genus g, weight k, (index m,) and type ρ with vanishing order at least $o \in \mathbb{Q}$ in their Fourier expansions are denoted by $\mathrm{M}_k^{(g)}(\rho)[o]$ (respectively, $\mathrm{J}_{k,m}^{(g)}(\rho)[o]$).

Let f be a classical (scalar-valued) Hermitian modular form of weight k. If $\mathrm{ord} f \neq 0$, the *slope* of f is defined as

$$\omega(f) := \frac{k}{\mathrm{ord} f},$$

and if $\mathrm{ord} f = 0$, we define $\omega(f) = +\infty$. The *lower slope bound* for classical Hermitian modular forms of degree g is defined as

$$\omega_g := \inf_{\substack{k \in \mathbb{Z}, \\ f \in \mathrm{M}_k^{(g)} \setminus \{0\}}} \omega(f).$$

It is well known that the lower slope bound ω_g for classical Hermitian modular forms is strictly positive, which is also shown in Corollary 3.7.

3.2. Embedding and vanishing

Lemma 3.2. *Let $g \geq 2$ be an integer, and let l be an integer such that $1 \leq l \leq g-1$ and d be a rational number. For every complex finite-dimensional representation ρ of $\mathrm{U}(g,g)(\mathbb{Z})$ factoring through a finite quotient, the linear map*

$$\begin{aligned} \mathrm{M}_k^{(g)}(\rho)[d] &\longrightarrow \mathrm{FM}_k^{(g,l)}(\rho)[d] \\ f &\longmapsto \sum_{0 \leq m \in \mathrm{Herm}_l(E)} \phi_m(\tau_1, w, z) e(m\tau_2) \end{aligned}$$

given by the Fourier–Jacobi expansion of cogenus l is an embedding.

Proof. This is clear from the definition of Fourier–Jacobi expansion. □

Lemma 3.3. *Let $g \geq 2$ be an integer, and let ρ be a complex finite-dimensional representation of $\mathrm{U}(g,g)(\mathbb{Z})$ that factors through a finite quotient. Let $N = N(\rho)$ be a positive integer such that the set of vanishing orders $\big\{\mathrm{ord} f : f \in \mathrm{FM}_k^{(g)}(\rho)\big\}$ is contained in $\frac{1}{N}\mathbb{Z}_{\geq 0}$. Then for every rational number $d \in \frac{1}{N}\mathbb{Z}_{\geq 0}$, the dimension of symmetric formal Fourier–Jacobi series of vanishing order at least d can be bounded above by*

$$\dim \mathrm{FM}_k^{(g)}(\rho)[d] \leq \sum_{m \in \left(\frac{1}{N}\mathbb{Z}\right)_{\geq d}} \dim \mathrm{J}_{k,m}^{(g-1)}(\rho)[m].$$

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Proof. For any $d \in \frac{1}{N}\mathbb{Z}_{\geq 0}$ and every $f \in \mathrm{FM}_k^{(g)}(\rho)[d]$, by picking a (noncanonical) $\mathbb{C}$-linear section

$$l_m : \mathrm{FM}_k^{(g)}(\rho)[m]/\mathrm{FM}_k^{(g)}(\rho)\left[m + \frac{1}{N}\right] \longrightarrow \mathrm{FM}_k^{(g)}(\rho)[m]$$

for each $m \in \left(\frac{1}{N}\mathbb{Z}\right)_{\geq d}$, we define recursively $f_m \in \mathrm{FM}_k^{(g)}(\rho)[m]$ for all $m \in \left(\frac{1}{N}\mathbb{Z}\right)_{\geq d}$ by putting $f_d := f$ and

$$f_{m+\frac{1}{N}} := f_m - l_m([f_m]) \in \mathrm{FM}_k^{(g)}(\rho)\left[m + \frac{1}{N}\right].$$

Then we define a $\mathbb{C}$-linear map

$$\begin{aligned} i : \mathrm{FM}_k^{(g)}(\rho)[d] &\longrightarrow \prod_{m \in \left(\frac{1}{N}\mathbb{Z}\right)_{\geq d}} \mathrm{FM}_k^{(g)}(\rho)[m]/\mathrm{FM}_k^{(g)}(\rho)\left[m + \frac{1}{N}\right] \\ f &\longmapsto ([f_m])_m, \end{aligned}$$

which we claim to be injective. Indeed, for any $f \in \ker i$, since $[f_m] = [0]$ implies $l_m([f_m]) = 0$ for each m, we have $f = f_n \in \mathrm{FM}_k^{(g)}(\rho)[n]$ for all $n \geq d$. In particular, the vanishing order of f is not finite, hence $f = 0$ and we find the map i is an embedding. Moreover, by sending $f \in \mathrm{FM}_k^{(g)}(\rho)[m]$ to its mth Fourier–Jacobi coefficient, we obtain a linear map

$$\mathrm{FM}_k^{(g)}(\rho)[m] \longrightarrow \mathrm{J}_{k,m}^{(g-1)}(\rho)[m] \tag{3.3}$$

whose kernel is $\mathrm{FM}_k^{(g)}(\rho)\left[m + \frac{1}{N}\right]$. To see this fact, recall that a symmetric formal Fourier–Jacobi series $f \in \mathrm{FM}_k^{(g)}(\rho)[m]$ can be written as

$$f(\tau) = \sum_{m' \in \left(\frac{1}{N}\mathbb{Z}\right)_{\geq m}} \phi_{m'}(\tau_1, w, z) e(m'\tau_2)$$

for $\tau = \begin{pmatrix} \tau_1 & w \\ z & \tau_2 \end{pmatrix} \in \mathbb{H}_g$. If f is in the kernel of the map in equation (3.3), then its mth Fourier–Jacobi coefficient ϕ_m vanishes, and by the symmetry relation in equation (2.12) for symmetric formal Fourier–Jacobi series we find that

$$c\left(f; u^* \begin{pmatrix} n & r \\ r^* & m \end{pmatrix} u\right) = 0$$

for all $u \in \mathrm{GL}_g(\mathcal{O}_E)$, $n \in \mathrm{Herm}_{g-1}(E)_{\geq 0}$ and $r \in \mathrm{Mat}_{g-1,1}(E)$. This implies that for any index $t \in \mathrm{Herm}_g(E)_{\geq 0}$ such that t can represent the rational number m, the Fourier coefficient $c(f;t)$ vanishes, hence the vanishing order of f satisfies $\operatorname{ord} f \geq m + \frac{1}{N}$, and the kernel of the map in equation (3.3) is contained in $\mathrm{FM}_k^{(g)}(\rho)\left[m + \frac{1}{N}\right]$. The other direction is clear.

Finally, the map in equation (3.3) induces an embedding

$$j : \prod_{m \in \left(\frac{1}{N}\mathbb{Z}\right)_{\geq d}} \mathrm{FM}_k^{(g)}(\rho)[m]/\mathrm{FM}_k^{(g)}(\rho)\left[m + \frac{1}{N}\right] \hookrightarrow \prod_{m \in \left(\frac{1}{N}\mathbb{Z}\right)_{\geq d}} \mathrm{J}_{k,m}^{(g-1)}(\rho)[m].$$

The desired inequality follows by composing the embeddings i, j and the $\mathbb{C}$-dimension function. □

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Proposition 3.4. *Let g be a positive integer, and let ρ be a complex finite-dimensional representation of $\mathrm{U}(g,g)(\mathbb{Z})$ that factors through a finite quotient. For every rational number $o > \frac{k}{\omega_g}$, we have*

$$\mathrm{M}_k^{(g)}(\rho)[o] = \{0\}.$$

Proof. If $\omega_g = 0$, the statement holds automatically. We assume $\omega_g > 0$. If ρ is the trivial representation of dimension 1, the assertion is clear from the definition of ω_g. In the general case, it suffices to show that for any $f \in \mathrm{M}_k^{(g)}(\rho)[o]$ and $v \in V(\rho)^\vee$, we have $v \circ f = 0$. For this we consider a finite product

$$f_v := \prod_{\gamma \in \ker(\rho) \backslash \mathrm{U}(g,g)(\mathbb{Z})} v \circ f|_k \gamma. \tag{3.4}$$

It is clear that $f_v \in M_{|\rho|k}^{(g)}$, where $|\rho|$ is the index of $\ker(\rho)$ in $\mathrm{U}(g,g)(\mathbb{Z})$. Note that each factor $v \circ f|_k\gamma = v \circ \rho(\gamma) f$ in equation (3.4) is a linear combination of components of the vector-valued function f and $\mathrm{ord} f \geq o$; therefore $f_v \in M_{|\rho|k}^{(g)}\big[|\rho|o\big]$ by Lemma 3.1. But $|\rho|o > \frac{|\rho|k}{\omega_g}$ by the assumption $o > \frac{k}{\omega_g}$, as shown in the trivial case $M_{|\rho|k}^{(g)}\big[|\rho|o\big]$ vanishes, so $v \circ f|_k\gamma = 0$ for some γ. Taking the slash action of γ^{-1} on each side, we find $v \circ f = 0$, as desired. □

Recall that there are exactly 5 norm-Euclidean imaginary quadratic fields $E = \mathbb{Q}(\sqrt{d})$, which are given by $d \in \big\{-1, -2, -3, -7, -11\big\}$. For each of these fields E, there is a maximal uniform constant $c_E > 0$, such that for any $\beta \in E$, there is an integer $\alpha \in \mathcal{O}_E$ satisfying $\big(N_{E/\mathbb{Q}}(\beta - \alpha)\big)^2 \leq 1 - c_E$. In particular, we have the following corollary.

Lemma 3.5. *For each $s \in \Delta_g(m)$, there is an element $r \in s + m\mathcal{O}_E^g$ such that its components r_i satisfy $|r_i|^2 \leq (1 - c_E)m^2$.*

Let g be a positive integer and $r \in E^g$ be a column vector of dimension g. In analogy to the ordinary vanishing order for Jacobi forms, we define the rth vanishing order $\mathrm{ord}_r \phi$ for a formal series ϕ in the form of equation (3.2) to be the infimum of rational numbers $m \in \mathbb{Q}$ such that there is some Hermitian positive semidefinite matrix $t \in \mathrm{Herm}_g(E)_{\geq 0}$ satisfying $c(\phi; t, r) \neq 0$ and the (g,g)th entry $t_{g,g} = m$. It is clear that $\mathrm{ord}\, \phi \leq \mathrm{ord}_r \phi$ for every $r \in E^g$, and the equation $\mathrm{ord}_r\, (f\psi) = \mathrm{ord} f + \mathrm{ord}_r \psi$ holds for every component f of a vector-valued Hermitian modular form of degree g and every component ψ of a vector-valued Hermitian Jacobi form of degree g, arbitrary cogenus and any type that factors through a finite quotient of the Jacobi group.

Recall that our goal in this section is to estimate the dimension of the space of symmetric formal Fourier–Jacobi series $\mathrm{FM}_k^{(g)}(\rho)$ of degree g and cogenus 1. Using the rth vanishing order, we deduce the following analogous result on vanishing orders for Hermitian Jacobi forms of cogenus 1 from Hermitian modular forms.

Proposition 3.6. *Let g be a positive integer, and let ρ be a complex finite-dimensional representation of $\Gamma^{(g,1)}$, which factors through a finite quotient. For every rational number $m > c_E^{-1}\frac{k}{\omega_g}$, we have*

$$\mathrm{J}_{k,m}^{(g)}(\rho)[m] = 0.$$

Proof. If $\omega_g = 0$, the statement is automatically true. We assume $\omega_g > 0$. Let $\phi \in \mathrm{J}_{k,m}^{(g)}(\rho)[m]$ be a vector-valued Jacobi form of cogenus 1. By Proposition 2.10, this Jacobi form ϕ has a unique theta decomposition

$$\phi(\tau, w, z) = \sum_{s \in \Delta_g(m)} h_s(\tau) \theta_{m,s}^{(g)}(\tau, w, z). \tag{3.5}$$

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For every $s \in \Delta_g(m)$ and every $r \in s + m\mathcal{O}_E^g$, by equation (3.5) and the definition and properties of the rth vanishing order of Jacobi forms, we have

$$m \leq \operatorname{ord} \phi \leq \operatorname{ord}_r \phi = \operatorname{ord}_r \left(h_s \theta_{m,s}^{(g)}\right) = \operatorname{ord}(h_s) + \operatorname{ord}_r \left(\theta_{m,s}^{(g)}\right), \tag{3.6}$$

where the first equality is due to the fact that the only summand in the sum in equation (3.5) that yields the vector r in the Fourier expansion of ϕ is $h_s\theta_{m,s}^{(g)}$.

On the other hand, by Lemma 3.5, for each $s \in \Delta_g(m)$, there is some element $r(s) \in s+m\mathcal{O}_E^g$ whose entries r_i satisfy $|r_i|^2 \leq (1-c_E)m^2$ for each i. It then follows from the definition that $\operatorname{ord}_r(\theta_{m,s}^{(g)}) \leq (1-c_E)m$ for $r = r(s)$, and hence $\operatorname{ord}(h_s) \geq c_E m$ by the inequality in equation (3.6). Since this holds for every s, by the assumption $m > c_E^{-1}\frac{k}{\omega_g}$, we have

$$h = (h_s)_{s\in\Delta_g(m)} \in \mathrm{M}_{k-1}^{(g)}(\rho_m^{(g)} \otimes \rho)[o]$$

for some $o > \frac{k}{\omega_g} > \frac{k-1}{\omega_g}$, which implies $h = 0$ by Proposition 3.4 and hence $\phi = 0$, as desired. □

Corollary 3.7. *For every positive integer g, the slope bound satisfies* $\omega_g \geq 12c_E^{g-1} > 0$.

Proof. We argue by induction. For $g = 1$, the upper half-space $\mathbb{H}_1$ is just the Poincaré upper half plane, and the modular group $\mathrm{U}(1,1)(\mathbb{Z})$ can be identified with $(\mathrm{U}(1) \cap \mathcal{O}_E) \times \mathrm{SL}_2(\mathbb{Z})$. Hence $\mathrm{M}_k^{(1)}$ is just a subspace of the elliptic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k, and it follows from [46], Page 9, Proposition 2 that $\omega_1 \geq 12$.

Now assume the assertion holds for $g = n-1$ with $n \in \mathbb{Z}_{\geq 2}$, and we have to show that $\omega_n \geq c_E\omega_{n-1}$. Combining Lemma 3.2, Lemma 3.3, the induction hypothesis and Proposition 3.6 for $g = n-1$, we see that the inequalities

$$\dim \mathrm{M}_k^{(n)}[d] \leq \dim \mathrm{FM}_k^{(n)}[d] \leq \sum_{m=d}^{\infty} \dim \mathrm{J}_{k,m}^{(n-1)}[m] = \sum_{m=d}^{\lfloor c_E^{-1}\frac{k}{\omega_{n-1}}\rfloor} \dim \mathrm{J}_{k,m}^{(n-1)}[m]$$

hold for every positive integer d and every weight k. In particular, if $d > c_E^{-1}\frac{k}{\omega_{n-1}}$, then the space $\mathrm{M}_k^{(g)}[d]$ vanishes, which implies $\omega_n \geq c_E\omega_{n-1}$. □

3.3. Asymptotic dimensions and algebraicity of $\mathrm{FM}_\bullet^{(g)}$ over $\mathrm{M}_\bullet^{(g)}$

Lemma 3.8. *Let ρ be a complex finite-dimensional representation of* $\mathrm{U}(1,1)(\mathbb{Z})$ *that factors through a finite quotient. Then*

$$\dim \mathrm{M}_k^{(1)}(\rho) \ll \dim V(\rho)k.$$

Proof. As $\mathrm{M}_k^{(1)}(\rho)$ is a subspace of elliptic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k and some arithmetic type, the result follows from [47], Theorem 2.5. □

Lemma 3.9. *Let ρ be a complex finite-dimensional representation of the Jacobi group* $\Gamma^{(1,1)}$ *that factors through a finite quotient. Then*

$$\dim \mathrm{J}_{k,m}^{(1)}(\rho) \ll \dim V(\rho)km^2.$$

Proof. Combining Proposition 2.10 and Lemma 3.8, we obtain the desired asymptotic bound. □

Remark 3.10. The case of trivial type ρ was also proved by Haverkamp in [41], Theorems 1 and 3.

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Proposition 3.11. *Let $g \geq 2$ be an integer and ρ be a complex finite-dimensional representation of* $\mathrm{U}(g,g)(\mathbb{Z})$ *that factors through a finite quotient. Then*

$$\dim \mathrm{FM}_k^{(g)}(\rho) \ll_{g,\rho} \dim V(\rho) k^{g^2}.$$

Proof. Let $N = N(\rho)$ be a positive integer such that the set of vanishing orders $\{\mathrm{ord} f : f \in \mathrm{FM}_k^{(g)}(\rho)\}$ is contained in $\frac{1}{N}\mathbb{Z}_{\geq 0}$. We prove the assertion by induction. For $g = 2$, combining Lemma 3.3, Proposition 3.6 and Lemma 3.9, we have asymptotic inequalities

$$\dim \mathrm{FM}_k^{(2)}(\rho) \leq \sum_{\substack{0 \leq m \leq c_E^{-1} \frac{k}{\omega_1} \\ m \in \frac{1}{N}\mathbb{Z}}} \dim \mathrm{J}_{k,m}^{(1)}(\rho)[m] \ll \dim V(\rho) k \sum_{\substack{0 \leq m \leq c_E^{-1} \frac{k}{\omega_1} \\ m \in \frac{1}{N}\mathbb{Z}}} m^2 \ll_\rho \dim V(\rho) k^4.$$

Assume the assertion is true for $g = n - 1$ with $n \geq 3$, and we show it for $g = n$. Applying Lemma 3.3, Proposition 3.6, Proposition 2.10 and Lemma 3.2 successively, we have

$$\dim \mathrm{FM}_k^{(n)}(\rho) \leq \sum_{\substack{0 \leq m \leq c_E^{-1} \frac{k}{\omega_{n-1}} \\ m \in \frac{1}{N}\mathbb{Z}}} \dim \mathrm{J}_{k,m}^{(n-1)}(\rho)[m] \leq \sum_{\substack{0 \leq m \leq c_E^{-1} \frac{k}{\omega_{n-1}} \\ m \in \frac{1}{N}\mathbb{Z}}} \dim \mathrm{FM}_{k-1}^{(n-1)}(\rho_m^{(n-1)} \otimes \rho). \tag{3.7}$$

Furthermore, the induction hypothesis and the rank of $\Delta_{n-1}(m)$ for the Weil representation imply that

$$\dim \mathrm{FM}_{k-1}^{(n-1)}(\rho_m^{(n-1)} \otimes \rho) \ll_{n-1} \dim V(\rho) m^{2(n-1)} k^{(n-1)^2}. \tag{3.8}$$

Finally, we combine the inequalities in equation (3.7) with the one in equation (3.8) and apply Corollary 3.7 for $g = n - 1$ to see that the sums in equation (3.7) has $\mathcal{O}(k)$ terms, whence the desired assertion for $g = n$. □

Corollary 3.12. *Let g be a positive integer. Then*

$$\dim \mathrm{FM}_{\leq k}^{(g)} \ll_g k^{g^2+1}.$$

To give an asymptotic lower bound for the dimension of Hermitian modular forms $\mathrm{M}_k^{(g)}$ of degree g and weight k, we use a dimension formula for Hermitian cusp forms obtained from the Selberg trace formula. For $k \geq 2g$, we define the following positive constant:

$$C(k,g) := 2^{-g^2-g} \pi^{-g^2} \prod_{0 \leq i,j \leq g-1} (k - 2g + 1 + i + j).$$

Let $Y_g := \mathrm{U}(g,g)(\mathbb{Z}) \backslash \mathbb{H}_g$ denote the Hermitian modular variety. For $\tau \in Y_g$ we introduce a Bergman kernel function

$$S(k,g,\tau) := \sum_{\gamma \in \mathrm{U}(g,g)(\mathbb{Z}) / Z(\mathrm{U}(g,g)(\mathbb{Z}))} \overline{j(\gamma,\tau)}^{-k} \left(\det\left(\frac{1}{2i}(\tau - (\gamma\tau)^*)\right)\right)^{-k},$$

where $Z(\mathrm{U}(g,g)(\mathbb{Z}))$ is the centre of $\mathrm{U}(g,g)(\mathbb{Z})$. The sum is well-defined and absolutely convergent by general theory of Bergman kernel functions, see [48] for detailed discussions. We define the (weight-k normalised) hyperbolic volume form to be

$$\mathrm{d}\tau := \left(\det\left(\frac{1}{2i}(\tau - \tau^*)\right)\right)^{k-2g} \mathrm{d}_E \tau,$$

where $\mathrm{d}_E \tau$ is the (complex) Euclidean measure.

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Proposition 3.13 ([49], Page 62). *Let $k > 4g - 2$ be an even integer. Then we have the following dimension formula for the space of Hermitian cusp forms:*

$$\dim S_k^{(g)}(\mathrm{U}(g,g)(\mathbb{Z})) = C(k,g) \int_{Y_g} S(k,g,\tau)\mathrm{d}\tau.$$

We observe the asymptotic behaviour of $C(k,g)$ and the fact that the values of $\int_{Y_g} S(k,g,\tau)\mathrm{d}\tau$ when k varies are bounded from below by some positive constant independent of k. Since the space of Hermitian cusp forms $S_k^{(g)}(\mathrm{U}(g,g)(\mathbb{Z}))$ is contained in the space of Hermitian modular forms $\mathrm{M}_k^{(g)}$, we deduce an asymptotic lower bound for the dimension of Hermitian modular forms $\mathrm{M}_{\leq k}^{(g)}$ of degree g and weight at most k.

Corollary 3.14. *Let g be a positive integer. Then*

$$\dim \mathrm{M}_{\leq k}^{(g)} \gg_g k^{g^2+1}.$$

Theorem 3.15. *For every integer $g \geq 2$, the graded algebra $\mathrm{FM}_\bullet^{(g)}$ is an algebraic extension over the graded algebra $\mathrm{M}_\bullet^{(g)}$.*

Proof. It suffices to show that for an arbitrary $f \in \mathrm{FM}_{k_0}^{(g)}$ of weight k_0, the set $\{f^i : i \in \mathbb{N}\}$ is of finite rank over the graded algebra $\mathrm{M}_\bullet^{(g)}$. Indeed, for any positive integers k, n such that $k \geq nk_0$, by the fact that $\sum_{i=0}^n \mathrm{M}_{\leq k}^{(g)} f^i \subseteq \mathrm{FM}_{\leq k+nk_0}^{(g)}$ combined with Corollary 3.12 and Corollary 3.14, we have the following asymptotic inequalities

$$\dim_{\mathbb{C}} \Big(\sum_{i=0}^n \mathrm{M}_{\leq k}^{(g)} f^i \Big) \leq \dim_{\mathbb{C}} \mathrm{FM}_{\leq k+nk_0}^{(g)} \ll_g (k+nk_0)^{g^2+1} \ll_g k^{g^2+1} \ll_g \dim_{\mathbb{C}} \mathrm{M}_{\leq k}^{(g)},$$

which imply that there is a constant $C = C(g)$ such that for any large n and k satisfying $k \geq nk_0$, we have $\dim_{\mathbb{C}} \big(\sum_{i=0}^n \mathrm{M}_{\leq k}^{(g)} f^i \big) \leq C \dim_{\mathbb{C}} \mathrm{M}_{\leq k}^{(g)}$. In particular, the rank of the set $\{f^i : i \in \mathbb{N}\}$ over the graded algebra $\mathrm{M}_\bullet^{(g)}$ is bounded above by C, as desired. □

4. Hermitian modular forms are algebraically closed in symmetric formal Fourier–Jacobi series

In this section, we work in the case of an arbitrary imaginary quadratic field $E/\mathbb{Q}$. We prove Theorem 4.14 based on the following strategy: in Subsection 4.1, we study the geometry of a toroidal boundary of the Hermitian modular variety and show that every prime divisor on the toroidal compactification intersects any open neighbourhood of the toroidal boundary; we then recall the general construction of toroidal compactifications and the Weierstrass preparation theorem to show the local convergence of a symmetric formal Fourier–Jacobi series on an open neighbourhood of the toroidal boundary in Subsection 4.2; combining these two aspects, we conclude our argument in Subsection 4.3 to show the global convergence of symmetric formal Fourier–Jacobi series.

4.1. Toroidal boundaries of Hermitian modular varieties

The aim of this subsection is to prove Theorem 4.6 based on the following strategy: the analogous result in the Siegel case for the Satake boundary, Lemma 4.5, was proven in the work of Bruinier–Raum; to reduce to the Siegel case, we embed the Siegel upper half-space into the Hermitian upper half-space both in a generic way and over $\mathbb{Q}$ via a density argument; finally, we compare the behaviour of various boundaries under the constructed maps. We also point out a cohomological approach to the theorem at the end of this subsection.

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The theory of toroidal compactifications was developed by Ash–Mumford–Rapoport–Tai [50]. A brief discussion of the construction of partial compacitifications of Siegel modular varieties can be found, for instance, in [51] and [46], and the book [52] contains more details in an accessible way. Let $E/\mathbb{Q}$ be a fixed imaginary quadratic field with an embedding $E \hookrightarrow \mathbb{C}$. The unitary group $\mathrm{U}(g,g) := \mathrm{U}_{E/\mathbb{Q}}(g,g)$ is a semi-simple algebraic group defined over $\mathbb{Q}$, and the group $\mathrm{U}(g,g)(\mathbb{R})^o$ is the connected component of $\mathrm{Aut}(\mathbb{D}_g)$, for the Hermitian symmetric domain $\mathbb{D}_g = \{\sigma \in \mathrm{Mat}_g(\mathbb{C}) : \sigma^*\sigma < I_g\} \cong \mathbb{H}_g$. Here we define a certain toroidal compactification of the Hermitian modular variety $Y_g = \mathrm{U}(g,g)(\mathbb{Z})\backslash\mathbb{H}_g$ associated to an admissible collection of polyhedra for a 0-dimensional cusp, since this determines the cases for the other cusps as well. For the arithmetic group $\mathrm{GL}_g(\mathcal{O}_E) \subseteq \mathrm{U}(g,g)(\mathbb{R})^o$, let $\{\sigma_\alpha\}_\alpha$ be a $\mathrm{GL}_g(\mathcal{O}_E)$-admissible collection of polyhedra (see, for instance, (7.3) in [52]). Then we obtain a toroidal compactification X_g of the Hermitian modular variety $Y_g = \mathrm{U}(g,g)(\mathbb{Z})\backslash\mathbb{H}_g$ associated to this set of data. We choose a collection of polyhedra $\{\sigma_\alpha\}_\alpha$ such that the resulting toroidal compactification X_g is nonsingular in the orbifold sense. We write $\partial Y_g := X_g \setminus Y_g$ for the toroidal boundary.

Let $\overline{Y_g}$ be the Satake compactification of Y_g, and let

$$\pi : X_g \longrightarrow \overline{Y_g}$$

be the natural map, which exists by the definition of the minimal compactification $\overline{Y_g}$. Let $\phi : \mathbb{H}_g \longrightarrow Y_g$ be the quotient map. Recall that $\mathrm{U}(g,g)(\mathbb{R})$ acts transitively on the Hermitian upper half-space $\mathbb{H}_g$ via Möbius transformations. In particular, for each $\gamma \in \mathrm{U}(g,g)(\mathbb{R})$, we obtain an embedding of the Siegel upper half-space $\mathcal{H}_g$ into $\mathbb{H}_g$ by sending $\tau \in \mathcal{H}_g$ to $\gamma\tau$, and the images $\gamma\mathcal{H}_g \subseteq \mathbb{H}_g$ for $\gamma \in \mathrm{U}(g,g)(\mathbb{R})$ cover $\mathbb{H}_g$. Let $\mathcal{Y}_g := \mathrm{Sp}_g(\mathbb{Z})\backslash\mathcal{H}_g$ and $\overline{\mathcal{Y}_g}$ denote the Siegel modular variety of degree g and its Satake compactification, respectively.

Lemma 4.1. *Let D be a prime divisor on Y_g. Then there is some rational element $\gamma \in \mathrm{U}(g,g)(\mathbb{Q})$ such that $\phi(\gamma\mathcal{H}_g)$ intersects D transversally at some point.*

Proof. For any $\gamma \in \mathrm{U}(g,g)(\mathbb{R})$, $\phi(\gamma\mathcal{H}_g)$ intersects D transversally if and only if $\mathcal{H}_g$ intersects $\gamma^{-1}\phi^{-1}(D)$ transversally. It is a general fact that $\mathrm{U}(g,g)(\mathbb{R})$ acts transitively on the holomorphic unit tangent bundle of $\mathbb{H}_g$ via $\gamma(\tau,v) := (\gamma\tau, \widehat{\mathrm{d}\gamma_\tau v})$, where $\widehat{\mathrm{d}\gamma_\tau v}$ denotes the unit vector associated to the pushforward of v by the smooth map γ at τ. Since $\phi^{-1}(D)$ is an embedded Hermitian submanifold of codimension 1 in $\mathbb{H}_g$, and Möbius transformations are conformal with respect to the Hermitian metric, there is $\gamma_0 \in \mathrm{U}(g,g)(\mathbb{R})$ such that $\mathcal{H}_g$ intersects $\gamma_0^{-1}\phi^{-1}(D)$ transversally at some point. In particular, there is an open neighbourhood $W \subseteq \mathrm{U}(g,g)(\mathbb{R})$ of γ_0 such that for all $\gamma \in W$, $\mathcal{H}_g$ intersects $\gamma^{-1}\phi^{-1}(D)$ transversally at some point. The density of $\mathrm{U}(g,g)(\mathbb{Q})$ in $\mathrm{U}(g,g)(\mathbb{R})$ then implies the assertion. □

We say that a point τ in the Siegel upper half-space $\mathcal{H}_g$ is an *elliptic fixed point* if its stabiliser under the action of $\mathrm{Sp}_g(\mathbb{Z})$ on $\mathcal{H}_g$ is larger than the centre of $\mathrm{Sp}_g(\mathbb{Z})$.

Lemma 4.2. *Let $g \geq 2$ be an integer. Then the set of elliptic fixed points under the action of $\mathrm{Sp}_g(\mathbb{Z})$ has codimension at least $g-1$ in $\mathcal{H}_g$. Moreover, for the case $g = 2$, the set of elliptic fixed points of codimension 1 is given by the $\mathrm{Sp}_2(\mathbb{Z})$-translates of the divisors $\{z = 0\}$ and $\{z = 1/2\}$ in $\mathcal{H}_2$ for $\left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathcal{H}_2$.*

Proof. See the proof of Theorem 1.2 in [18]. □

Lemma 4.3. *For every $\gamma \in \mathrm{U}(g,g)(\mathbb{Q})$, there is a positive integer $N = N(\gamma)$ such that*

$$\begin{aligned}[\gamma] : \mathcal{Y}_g(N) = \Gamma(N)\backslash\mathcal{H}_g &\longrightarrow Y_g = \mathrm{U}(g,g)(\mathbb{Z})\backslash\mathbb{H}_g \\ [\tau] &\longmapsto [\gamma\tau]\end{aligned}$$

is a well-defined continuous map and induces a covering map onto the image with possible ramification at the orbits of elliptic fixed points. Furthermore, the map can be continuously extended to $\overline{[\gamma]} : \overline{\mathcal{Y}_g(N)} \longrightarrow$

$\overline{Y_g}$ on the Satake compactification, and for every open neighbourhood V of the Satake boundary $\overline{Y_g} \setminus Y_g$, there is an open neighbourhood U of the Satake boundary $\overline{\mathcal{Y}_g(N)} \setminus \mathcal{Y}_g(N)$, such that $\overline{[\gamma]}(U) \subseteq V$.

Proof. As $\gamma \in \mathrm{U}(g,g)(\mathbb{Q})$, there are positive integers N_1, N_2 such that $N_1\gamma$ and $N_2\gamma^{-1}$ are integral. Let $N := N_1N_2$. Then for any $\gamma_N \in \Gamma(N)$, we have

$$\gamma\gamma_N\gamma^{-1} - I_{2g} = \gamma(\gamma_N - I_{2g})\gamma^{-1} \in \gamma N \mathrm{Sp}_g(\mathbb{Z})\gamma^{-1} \subseteq \mathrm{U}(g,g)(\mathbb{Z}),$$

hence $\gamma\Gamma(N)\gamma^{-1} \subseteq \mathrm{U}(g,g)(\mathbb{Z})$. In other words, the map $[\gamma]$ is well defined. It is clear that the induced surjective map from the complement of the orbits $[\tau]$ of elliptic fixed points τ onto the image is a covering map (whose degree is equal to the index of the projective group of $\Gamma(N)$ in the projective group of $\gamma^{-1}\mathrm{U}(g,g)(\mathbb{Z})\gamma \cap \mathrm{Sp}_g(\mathbb{Z})$). Furthermore, the continuous extension can be defined with respect to the topology of the Satake compactification, and the boundary is mapped to the boundary under $\overline{[\gamma]}$. We put $U := \overline{[\gamma]}^{-1}(V)$, which satisfies the desired property. □

Lemma 4.4. *For each positive integer N, the natural map*

$$\phi_N : \mathcal{Y}_g(N) = \Gamma(N)\backslash\mathcal{H}_g \longrightarrow \mathcal{Y}_g = \mathrm{Sp}_g(\mathbb{Z})\backslash\mathcal{H}_g$$

can be continuously extended to $\overline{\phi_N} : \overline{\mathcal{Y}_g(N)} \longrightarrow \overline{\mathcal{Y}_g}$ on the Satake compactification. For every open neighbourhood U of the Satake boundary $\overline{\mathcal{Y}_g(N)} \setminus \mathcal{Y}_g(N)$, there is an open neighbourhood V of the Satake boundary $\overline{\mathcal{Y}_g} \setminus \mathcal{Y}_g$ such that $\overline{\phi_N}^{-1}(V) \subseteq U$.

Proof. Similar to Lemma 4.3, the first assertion is clear. For the second one, since $\overline{\phi_N}$ maps the boundary to the boundary, it suffices to find an open neighbourhood V' of the boundary in $\mathcal{Y}_g$ such that $\phi_N^{-1}(V') \subseteq U'$ for $U' = U \bigcap \mathcal{Y}_g(N)$. Since ϕ_N is a covering map of finite degree n (the index of the projective group of $\Gamma(N)$ in the projective group of $\mathrm{Sp}_g(\mathbb{Z})$) with possible ramification at the orbits of elliptic fixed points, the restriction on the boundary neighbourhood $\phi_N^{-1}\left(\phi_N(U')\right) \longrightarrow \phi_N(U')$ is also so. Let U'_i be ith sheet over $\phi_N(U')$ for $i = 1, 2, \ldots, n$. Then $V := V' \bigcup (\overline{\mathcal{Y}_g} \backslash \mathcal{Y}_g)$ for $V' := \bigcap_{i=1}^n \phi_N(U' \cap U'_i)$ satisfies the desired property. □

Lemma 4.5. *Assume $g \geq 2$, and let D be a prime divisor on $\overline{Y_g}$. Then D intersects the Satake boundary $\overline{\mathcal{Y}_g} \setminus \mathcal{Y}_g$.*

Proof. The assertion is proved in the second paragraph of the proof of Proposition 4.1 of [7], with D replaced by D'. □

Theorem 4.6. *Assume $g \geq 2$, and let D be a prime divisor on X_g. Let $U \subseteq X_g$ be an open neighbourhood of the boundary ∂Y_g. Then $D \cap U$ is a nontrivial divisor on U.*

Proof. Recall the maps defined in this subsection:

$$X_g \xrightarrow{\pi} \overline{Y_g} \xleftarrow[\overline{[\gamma]}]{} \overline{\mathcal{Y}_g(N)} \xrightarrow{\overline{\phi_N}} \overline{\mathcal{Y}_g}$$

for $\gamma \in \mathrm{U}(g,g)(\mathbb{Q})$. If D is contained in the boundary $\partial Y_g = X_g \setminus Y_g$, the assertion is clear. Otherwise D intersects Y_g, so the pushforward $D_1 := \pi_*(D) = \overline{\pi(D)}$ is a prime divisor on $\overline{Y_g}$ and $D'_1 := D_1 \cap Y_g$ is a prime divisor on Y_g.

By Lemma 4.1, there is $\gamma \in \mathrm{U}(g,g)(\mathbb{Q})$, which we fix from now on, such that $\phi(\gamma\mathcal{H}_g)$ intersects D'_1 transversally at some point. In particular, the intersection has codimension 1 in $\phi(\gamma\mathcal{H}_g)$ at some point. Let $N := N(\gamma)$. By Lemma 4.3, the continuous map $[\gamma] : [\tau] \longmapsto \phi(\gamma\tau)$ induces a covering map onto the image with possible ramification at the orbits of elliptic fixed points. By Lemma 4.2, since irreducible components are equidimensional, $\overline{[\gamma]}^{-1}(D_1)$ contains a prime divisor D_2 on $\overline{\mathcal{Y}_g(N)}$.

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Since the Satake boundary has codimension at least 2 for $g \geq 2$, D_2 must intersect $\mathcal{Y}_g(N)$. As ϕ_N induces a covering map onto the image with possible ramification at the orbits of elliptic fixed points, it follows that the pushforward $D_3 := \overline{\phi_N}_*(D_2)$ is a prime divisor on $\overline{\mathcal{Y}_g}$.

Finally we analyse the boundary intersection. Given an arbitrary open neighbourhood $U \subseteq X_g$ of the toroidal boundary ∂Y_g, the open neighbourhood $U_1 = \overline{Y_g} \setminus (X_g \setminus U) \subseteq \overline{Y_g}$ of the Satake boundary $\overline{Y_g} \setminus Y_g$ satisfies $\pi^{-1}(U_1) = U$. By Lemma 4.3, there is an open neighbourhood $U_2 \subseteq \overline{\mathcal{Y}_g(N)}$ of the boundary $\overline{\mathcal{Y}_g(N)} \setminus \mathcal{Y}_g(N)$ such that $\overline{[\gamma]}(U_2) \subseteq U_1$. Finally, by Lemma 4.4, there is an open neighbourhood $U_3 \subseteq \overline{\mathcal{Y}_g}$ of the boundary $\overline{\mathcal{Y}_g} \setminus \mathcal{Y}_g$ such that $\pi^{-1}(U_3) \subseteq U_2$. By Lemma 4.5, D_3 intersects U_3, and it follows from the aforementioned choice of open neighbourhoods that D_2 intersects U_2, D_1 intersects U_1 and D intersects U, as desired. □

Remark 4.7. There is another approach to prove Theorem 4.6 using results from stable cohomology, at least for large degree g. We sketch a proof here for reference. Following the strategy of the proof of Proposition 4.1 in [7], it suffices to show that the Picard group of $\overline{Y_g}$ has rank at most 1. Since $\overline{Y_g} \setminus Y_g$ has codimension at least 2 in $\overline{Y_g}$ for $g \geq 2$, we have $\mathrm{Pic}(\overline{Y_g}) = \mathrm{Pic}(Y_g)$. By the exponential sheaf sequence, we have ([53] Appendix B) an exact sequence

$$0 \longrightarrow H^1(Y_g, \mathbb{Z}) \longrightarrow H^1(Y_g, \mathcal{O}_{Y_g}) \longrightarrow \mathrm{Pic}(Y_g) \longrightarrow H^2(Y_g, \mathbb{Z}) \longrightarrow \cdots. \tag{4.1}$$

Since $\mathbb{H}_g$ is simply connected, by a standard argument we find that $\pi_1(Y_g)$ is the projective group of $\mathrm{U}(g,g)(\mathbb{Z})$. Hence by the Hurewicz theorem and the universal coefficient theorem, we find that $H^1(Y_g, \mathbb{Z})$ and $H^1(Y_g, \mathcal{O}_{Y_g})$ vanish. By equation (4.1), we only need to show that $H^2(Y_g, \mathbb{Z})$ has rank at most 1. Applying results in [54] (Chapter VII), this amounts to computing the rank for the degree-2 part in the graded ring $H^*(\mathrm{U}(g,g)(\mathbb{Z}), \mathbb{Z})$, which is canonically isomorphic to $H^*(A(\mathbb{H}_g; \mathbb{Z})^{\mathrm{U}(g,g)(\mathbb{Z})})$, where $A^q(M; E)$ denotes the space of smooth E-valued differential q-forms on M for $q \geq 0$. Alternatively, we can compute the lower degree parts of $H^*(\mathrm{U}(g,g)(\mathbb{Z}), \mathbb{Z})$ using Theorem 7.5, the table under 10.6 and Theorem 11.1 in [55]. In particular, we can apply these results to deduce Theorem 4.6 for sufficiently large g. For small g, one needs more explicit results of computing the second degree information in $H^*(A(\mathbb{H}_g; \mathbb{Z})^{\mathrm{U}(g,g)(\mathbb{Z})})$, which we skip here.

4.2. Local convergence at the boundary

Proposition 4.8 ([56], Theorem 102). *Let k be a field of characteristic* 0 *and R be a regular ring containing k. Suppose that (1) for any maximal ideal $\mathfrak{m}$ of R, the residue field $R/\mathfrak{m}$ is algebraic over k and* $\mathrm{ht}\,\mathfrak{m} = n$, *and (2) there exist $D_1, \ldots, D_n \in \mathrm{Der}_k(R)$ and $x_1, \ldots, x_n \in R$ such that $D_i x_j = \delta_{ij}$. Then R is excellent.*

Remark 4.9. As pointed out in the remark after Theorem 102 in the book, convergent power series rings over $\mathbb{C}$ are examples of regular rings to which the theorem applies.

Proposition 4.10 ([7], Proposition 4.2). *Let A be a local integral domain with maximal ideal $\mathfrak{m}$, and let $\hat{A}$ be the completion of A with respect to $\mathfrak{m}$. If A is henselian and excellent, then A is algebraically closed in $\hat{A}$.*

Proposition 4.11. *Let $f \in \mathrm{FM}_k^{(g)}$ be a symmetric formal Fourier–Jacobi series of cogenus* 1. *If f is algebraic over the graded algebra $\mathrm{M}_\bullet^{(g)}$, then there is an open neighbourhood U of the boundary $\partial Y_g = X_g \setminus Y_g$ such that f converges absolutely and defines a holomorphic function on U.*

Proof. This is the Hermitian counterpart of Lemma 4.3 in [7], and we follow the proof there. Let $x \in \partial Y_g$ be a toroidal boundary point, and let $Q \in \mathrm{M}_\bullet^{(g)}[X]$ be a polynomial such that $Q(f) = 0$. Consider germs of holomorphic functions at the point x, and we write $\mathcal{O}$ for the structure sheaf $\mathcal{O}_{X_g}$. Note that the polynomial Q defines a polynomial $Q_x \in \mathcal{O}_x[X]$ and the symmetric formal Fourier–Jacobi series f defines an element $f_x \in \hat{\mathcal{O}}_x$ in the completion of the local ring $\mathcal{O}_x$, such that $Q_x(f_x) = 0$.

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By general theory of toroidal compactifications and the Weierstrass preparation theorem, the local ring $\mathcal{O}_x$ is henselian. On the other hand, Proposition 4.8 implies that the ring $\mathcal{O}_x$ is excellent. Therefore, by Proposition 4.10 and the assumption that f_x is algebraic over $\mathcal{O}_x$, we conclude that f converges absolutely in a neighbourhood of x. Varying the boundary point x, we find that f converges in an open neighbourhood of the whole toroidal boundary $\partial Y_g = X_g \setminus Y_g$. □

4.3. *Analytic continuation and algebraic closedness of* $\mathrm{M}_\bullet^{(g)}$ *in* $\mathrm{FM}_\bullet^{(g)}$

Proposition 4.12 (Lemma 4.4, [7]). *Let N be a positive integer and $W \subseteq \mathbb{C}^N$ be a domain. Let $P(\tau, X) \in \mathcal{O}(W)[X]$ be a monic irreducible polynomial with discriminant $\Delta_P (\in \mathcal{O}(W))$. Let $V \subseteq W$ be an open subset that intersects every irreducible component of the divisor $D = \mathrm{div}(\Delta_P)$. If f is an analytic function on V such that $P(\tau, f(\tau)) = 0$ on V, then f has an analytic continuation to the entire domain W.*

Proposition 4.13. *Let $g \geq 2$ be an integer. If $f \in \mathrm{FM}_k^{(g)}$ is integral over the graded algebra $\mathrm{M}_\bullet^{(g)}$, then $f \in \mathrm{M}_k^{(g)}$.*

Proof. Let $P(\tau, X) \in \mathrm{M}_\bullet^{(g)}[X]$ be a monic irreducible polynomial over the graded algebra $\mathrm{M}_\bullet^{(g)}$ with coefficients of pure weights, such that $P(\tau, f(\tau)) = 0$ formally. By Proposition 4.11, there is an open neighbourhood U of the boundary $\partial Y_g = X_g \setminus Y_g$ such that f converges absolutely and defines a holomorphic function on the preimage $V \subseteq \mathbb{H}_g$ of U, say $f|_V$. By assumption, $f|_V$ satisfies $P(\tau, f|_V(\tau)) = 0$ on V. Furthermore, the weights of the coefficients of P form an arithmetic progression, hence Δ_P is a nonzero Hermitian modular form by the quasi-homogeneity of the discriminant. In particular, $\mathrm{div}(\Delta_P)$ defines a divisor on X_g, hence every irreducible component of $\mathrm{div}(\Delta_P)$ intersects U by Theorem 4.6. Consequently, $f|_V$ has an analytic continuation f to $\mathbb{H}_g$ by Proposition 4.12. The modularity of f follows from the assumption that $f \in \mathrm{FM}_k^{(g)}$ combined with Lemma 2.2. □

Theorem 4.14. *For every integer $g \geq 2$, the graded algebra $\mathrm{M}_\bullet^{(g)}$ is algebraically closed in the graded algebra $\mathrm{FM}_\bullet^{(g)}$.*

Proof. First we show that if a finite sum $f = \sum_k f_k \in \mathrm{FM}_\bullet^{(g)}$ for $f_k \in \mathrm{FM}_k^{(g)}$ is algebraic over $\mathrm{M}_\bullet^{(g)}$, then so is each f_k. Indeed, as f satisfies a polynomial equation over $\mathrm{M}_\bullet^{(g)}$, the highest weight piece f_{k_0} of f must satisfy the equation defined by the highest weight part, so f_{k_0} is also algebraic over $\mathrm{M}_\bullet^{(g)}$, and we proceed by induction. We assume now that $f \in \mathrm{FM}_k^{(g)}$ and hf is integral over $\mathrm{M}_\bullet^{(g)}$ for some Hermitian modular form h. By Proposition 4.13, hf defines a Hermitian modular form. In particular, f defines a meromorphic function on $\mathbb{H}_g$.

Furthermore, Proposition 4.11 implies that f is holomorphic on $\pi^{-1}(U) \subseteq \mathbb{H}_g$ for some open neighbourhood U of ∂Y_g under the natural map $\pi : \mathbb{H}_g \longrightarrow X_g$. If the polar set of f is nonempty, then the pushforward of it under π does not intersect U, which is a contradiction to Theorem 4.6. Hence f is holomorphic on $\mathbb{H}_g$, and defines a Hermitian modular form, as desired. □

5. Modularity of symmetric formal Fourier–Jacobi series

In this section, we establish the main result, Theorem 5.7, via a few steps of reduction based on the machinery introduced in Section 2. Let $E/\mathbb{Q}$ be a norm-Euclidean imaginary quadratic fields, over which the Hermitian modular forms are defined.

Theorem 5.1. *Let $g \geq 2$ be an integer. Then*

$$\mathrm{FM}_\bullet^{(g)} = \mathrm{M}_\bullet^{(g)}.$$

Proof. Combining Theorem 3.15 and Theorem 4.14, we complete the proof. □

We say that a point τ in the Hermitian upper half-space $\mathbb{H}_g$ is an *elliptic fixed point* under the action of $\mathrm{U}(g,g)(\mathbb{Z})$ if its stabiliser under the action of $\mathrm{U}(g,g)(\mathbb{Z})$ on $\mathbb{H}_g$ is larger than the centre $Z := Z\big(\mathrm{U}(g,g)(\mathbb{Z})\big)$. We call $\gamma \in \mathrm{U}(g,g)(\mathbb{Z})$ an elliptic element if $\gamma \notin Z$ and has a fixed point in $\mathbb{H}_g$.

Lemma 5.2. *Let $g \geq 2$ be an integer. Then the set of elliptic fixed points under the action of $\mathrm{U}(g,g)(\mathbb{Z})$ has codimension at least g in $\mathbb{H}_g$. In particular, every holomorphic function on the complement of the elliptic fixed points has an analytic continuation to the entire space $\mathbb{H}_g$.*

Proof. We work with the open unit disk model, where the unitary group

$$\mathrm{U} := \left\{\gamma \in \mathrm{GL}_{2g}(\mathcal{O}_E) : \gamma^* \begin{pmatrix} I_g & 0_g \\ 0_g & -I_g \end{pmatrix} \gamma = \begin{pmatrix} I_g & 0_g \\ 0_g & -I_g \end{pmatrix}\right\}$$

acts on $\mathbb{D}_g = \big\{\sigma \in \mathrm{Mat}_g(\mathbb{C}) : \sigma^*\sigma < I_g\big\} (\cong \mathbb{H}_g)$. First we observe that conjugate elements in U have the same fixed points up to U-translations, which have the same codimension. For every elliptic element γ, since γ is of finite order, it is U-conjugate to some elements in the standard maximal compact subgroup $\mathrm{U}_g(\mathbb{Z}) \times \mathrm{U}_g(\mathbb{Z})$. Therefore, it suffices to prove the assertion for diagonal matrices $\gamma = \mathrm{diag}(\lambda_1, \lambda_2, \dots \lambda_{2g}) \in \mathrm{U} \setminus Z(\mathrm{U})$. In fact, for any fixed diagonal matrix $\gamma = \mathrm{diag}(\lambda_i)_i \in \mathrm{U} \setminus Z(\mathrm{U})$, a point $w = (w_{i,j})_{i,j} \in \mathbb{D}_g$ is fixed by γ if and only if the equations $\lambda_i w_{i,j} = \lambda_{j+g} w_{i,j}$ hold for all $i, j \in \big\{1, 2, \dots, g\big\}$. In particular, any elliptic point fixed by γ lies in the hyperplane cut out by the equations $w_{i,j} = 0$ for the pairs (i,j) such that $\lambda_i \neq \lambda_{j+g}$. But the number of these pairs for such γ is always at least g, as desired. The rest is clear from Riemann's first extension theorem. □

Recall that for any $f \in \mathrm{M}_k^{(g)}(\rho)$, the values that f can take are in the $\mathrm{U}(g,g)(\mathbb{Z})$-invariant subspace

$$V(\rho,k) := \Big\{v \in V(\rho) : \forall \gamma \in Z \left(j(\gamma,\tau)^k \rho(\gamma) v = v\right)\Big\}.$$

In the next proposition, we relate the space of values of Hermitian modular forms of weight k and type ρ to $V(\rho,k)$.

Proposition 5.3. *Let $g \in \mathbb{Z}_{\geq 2}$, and let ρ be a finite-dimensional representation of $\mathrm{U}(g,g)(\mathbb{Z})$. Then for any integer k, there exists some integer k_0 such that for every $\tau \in \mathbb{H}_g$ that is not an elliptic fixed point, the values of $\mathrm{M}_{k_0}^{(g)}(\rho)$ at τ linearly span the space $V(\rho,k)$.*

Proof. Recall that $\overline{Y_g}$ denote the Satake compactification of $Y_g = \mathrm{U}(g,g)(\mathbb{Z})\backslash\mathbb{H}_g$. By the work of Satake and Baily-Borel, there is some positive integer k', such that the $\mathcal{O}_{\overline{Y_g}}$-module $\mathcal{F}_{k'}$ of weight-k' Hermitian modular forms of trivial type is a very ample invertible sheaf on X. Let $\mathcal{F}_{k,\rho}$ denote the $\mathcal{O}_{\overline{Y_g}}$-module of weight-$k$ Hermitian modular forms of type ρ, which is a coherent sheaf. By a result of Serre ([53], Theorem 5.17), for large enough integers n, the sheaf $\mathcal{F}_{k,\rho} \otimes_{\mathcal{O}_{\overline{Y_g}}} \mathcal{F}_{k'}^{\otimes n} = \mathcal{F}_{k+k'n,\rho}$ is generated by a finite number of global sections. We set $k_0 = k + k'n$ for some fixed n so that $V(\rho,k) = V(\rho,k_0)$ and that $\mathcal{F}_{k_0,\rho}$ is generated by finitely many $f_i \in \mathcal{F}_{k_0,\rho}(\overline{Y_g})$. Taking stalks at any point $\tau \in \mathbb{H}_g$ that is not an elliptic fixed point, we deduce that the stalk $\mathcal{F}_{k_0,\rho,\tau} = \mathcal{O}_{\overline{Y_g},\tau} \otimes_{\mathbb{C}} V(\rho,k)$ is generated by $\{f_{i,\tau}\}$. In particular, the values $f_i(\tau)$ linearly span $V(\rho,k)$. □

By restricting ρ to the invariant subspace $V(\rho,k)$ in the rest of this section, we may assume $V(\rho) = V(\rho,k)$. Let $g \geq 2$ be an integer and σ, σ' be two complex finite-dimensional representations of the Hermitian modular group $\mathrm{U}(g,g)(\mathbb{Z})$. Recall the canonical pairing $\langle\, , \rangle$ induced by evaluation

$$\langle\, , \rangle : \mathrm{M}_k^{(g)}(\sigma^\vee \otimes \sigma') \times \mathrm{FM}_{k'}^{(g)}(\sigma) \longrightarrow \mathrm{FM}_{k+k'}^{(g)}(\sigma').$$

Moreover, the pairing $\langle\, , \rangle$ can be defined for meromorphic Hermitian modular forms and meromorphic symmetric formal Fourier–Jacobi series; see [18], Definition 2.3 for further details. For this aim, we need to assume that ρ factors through a finite quotient.

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Proposition 5.4. *Let g, l be integers such that $g \geq 2$ and $1 \leq l \leq g-1$. If $\mathrm{FM}_\bullet^{(g,l)} = \mathrm{M}_\bullet^{(g)}$, then*

$$\mathrm{FM}_\bullet^{(g,l)}(\rho) = \mathrm{M}_\bullet^{(g)}(\rho)$$

for every complex finite-dimensional representation ρ of $\mathrm{U}(g,g)(\mathbb{Z})$ that factors through a finite quotient.

Proof. We follow the proof in [18], Theorem 1.2. Let $f \in \mathrm{FM}_k^{(g,l)}(\rho)$ be an arbitrary symmetric Fourier–Jacobi series of weight k, and let n be the dimension of $V(\rho)$. First we fix an integer k_0 by Lemma 5.3, such that at every τ that is not an elliptic fixed point, the values of $\mathrm{M}_{k_0}^{(g)}(\rho^\vee)$ generate the space $V(\rho^\vee)$, and we also fix such a point τ.

Let ρ_0^n be the trivial representation on $\mathbb{C}^n$. By assumption, there is some tuple of Hermitian modular forms $F = (F_i)_i \in \big(\mathrm{M}_{k_0}^{(g)}(\rho^\vee)\big)^n$ such that the values $F_i(\tau) \in V(\rho^\vee)$ for $i = 1, \ldots, n$ are linearly independent. This defines a Hermitian modular form of weight k_0 and type $\rho^\vee \otimes \rho_0^n$, also denoted by F, as well as a meromorphic Hermitian modular form F^{-1} of weight $-k_0$ valued in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, V(\rho))$ via the meromorphic continuation of the assignment $\tau' \longmapsto \big(F_i(\tau')\big)_i^{-1}$ on an open neighbourhood of τ such that the matrix $\big(F_i(\tau')\big)_i$ is nondegenerate for every τ' in that neighbourhood.

We consider the pairing $\widetilde{f} := \big\langle F^{-1}, \langle F, f\rangle\big\rangle$. By the assumption $\mathrm{FM}_\bullet^{(g,l)} = \mathrm{M}_\bullet^{(g)}$, it is clear that $\widetilde{f}$ is a meromorphic Hermitian modular form of genus g, weight k and type ρ, and its formal Fourier–Jacobi expansion matches f. Since $\widetilde{f}$ is holomorphic on an open neighbourhood of τ by our choice of F, varying τ we find f is holomorphic on the complement of the elliptic fixed points. By Lemma 5.2, f is holomorphic on $\mathbb{H}_g$, and its modularity follows from that of $\widetilde{f}$. □

Proposition 5.5. *Let $g \geq 2$ be an integer. Assume that $\mathrm{FM}_\bullet^{(g')}(\rho) = \mathrm{M}_\bullet^{(g')}(\rho)$ for every integer g' such that $2 \leq g' \leq g-1$ and for every complex finite-dimensional representation ρ of $\mathrm{U}(g',g')(\mathbb{Z})$ that factors through a finite quotient. If $\mathrm{FM}_\bullet^{(g)} = \mathrm{M}_\bullet^{(g)}$, then $\mathrm{FM}_\bullet^{(g,l)} = \mathrm{M}_\bullet^{(g)}$ for every integer l satisfying $1 \leq l \leq g-1$.*

Proof. We prove the statement by induction on the cogenus l. Recall the notation $\mathrm{FM}_\bullet^{(g)} = \mathrm{FM}_k^{(g,1)}$, so the case $l = 1$ is automatically true for all $g \geq 2$. For the case $l \geq 2$, we fix $l' := l - 1$. Assuming $\mathrm{FM}_\bullet^{(g',l')} = \mathrm{M}_\bullet^{(g')}$ for all $g' \geq l$, we have to show that $\mathrm{FM}_\bullet^{(g,l)} = \mathrm{M}_\bullet^{(g)}$ for all $g \geq l+1$.

Let $f \in \mathrm{FM}_k^{(g,l)}$ be an arbitrary symmetric Fourier–Jacobi series of cogenus l. To show that $f \in \mathrm{FM}_\bullet^{(g,l')} = \mathrm{M}_\bullet^{(g)}$, we consider its formal Fourier–Jacobi expansion of cogenus l' defined in Subsection 2.7. In fact, for $\tau = \left(\begin{smallmatrix} \tau_1 & w \\ z & \tau_2 \end{smallmatrix}\right) \in \mathbb{H}_g$, where $\tau_2 \in \mathbb{H}_{l'}$, the formal Fourier–Jacobi expansion

$$f(\tau) = \sum_{m' \in \mathrm{Herm}_{l'}(E)_{\geq 0}} \psi_{m'}(\tau_1, w, z) e(m'\tau_2) \tag{5.1}$$

holds as an identity of formal Fourier series and is invariant under the weight-k slash action of the Hermitian modular group $\mathrm{U}(g,g)(\mathbb{Z})$, as $f \in \mathrm{FM}_k^{(g,l)}$. Therefore, it suffices to show the convergence of each $\psi_{m'}$.

If m' is nondegenerate, then $\psi_{m'}$ admits a formal theta decomposition with coefficients $(h_{m',s'})_{s'} \in \mathrm{FM}_{k-l'}^{(g-l')}(\rho_{m'}^{(g-l')})$, by Proposition 2.8. The assumption implies that $(h_{m',s'})_{s'}$ converges, and hence so does $\psi_{m'}$.

If m' is degenerate, we observe that for $0 \in \mathrm{Herm}_1(E)_{\geq 0}$ and for every $m'' \in \mathrm{Herm}_{l'-1}(E)_{\geq 0}$, ψ_0 and $\psi_{\left(\begin{smallmatrix} m'' & 0 \\ 0 & 0 \end{smallmatrix}\right)}$ satisfy a relation in the form of equation (2.13). By Proposition 2.5 and the induction assumption, we have $\psi_0 \in \mathrm{M}_k^{(g-1)}$, hence $\psi_{\left(\begin{smallmatrix} m'' & 0 \\ 0 & 0 \end{smallmatrix}\right)}$ must be a Hermitian Jacobi form. Furthermore, for any degenerate matrix $m' \in \mathrm{Herm}_{l'}(E)_{\geq 0}$, there is a suitable matrix $u' \in \mathrm{GL}_{l'}(\mathcal{O}_E)$ such that $u'^* m' u'$ is of the form $\left(\begin{smallmatrix} m'' & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By the invariance of both sides in equation (5.1) under the embedded action of $\left(\begin{smallmatrix} I_{g-l'} & 0 \\ 0 & u' \end{smallmatrix}\right) \in \mathrm{GL}_g(\mathcal{O}_E)$, the convergence of $\psi_{m'}$ is then reduced to that of $\psi_{\left(\begin{smallmatrix} m'' & 0 \\ 0 & 0 \end{smallmatrix}\right)}$. □

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Remark 5.6. As is shown in the proof, we may only assume in the proposition that $\mathrm{FM}_\bullet^{(g')}(\rho) = \mathrm{M}_\bullet^{(g')}(\rho)$ for every integer g' such that $2 \le g' \le g-1$ and for every Weil representation ρ of $\mathrm{U}(g',g')(\mathbb{Z})$.

Theorem 5.7. *Let $E/\mathbb{Q}$ be a norm-Euclidean imaginary quadratic field over which the Hermitian modular groups are defined. Let g, l be integers such that $g \ge 2$ and $1 \le l \le g-1$. Let k be an integer and ρ be a finite-dimensional representation of $\mathrm{U}(g,g)(\mathbb{Z})$ that factors through a finite quotient. Then*

$$\mathrm{FM}_k^{(g,l)}(\rho) = \mathrm{M}_k^{(g)}(\rho).$$

Proof. Combining Theorem 5.1 and a successive application of Propositions 5.4, 5.5 and 5.4 again, we complete the proof. □

6. Application: Kudla's modularity conjecture for unitary Shimura varieties

In this section, we discuss more details for an application of Theorem 5.7 in the context of Kudla's conjecture for unitary Shimura varieties, following Section 1 in [57] and Section 3 in [15]. We adopt the adelic language for the benefit of not handling the class number problem explicitly.

Let $E/\mathbb{Q}$ be an imaginary quadratic field, and fix a complex embedding $E \hookrightarrow \mathbb{C}$. Let $n \ge 2$ be an integer. Let V be an E-Hermitian space of signature $(n-1,1)$ and $H = \mathrm{U}(V)$ be the unitary group of V, which is a reductive group over $\mathbb{Q}$. Let $\mathbb{D}$ be the Hermitian symmetric space attached to $H(\mathbb{R})$, which in a projective model consists of all negative $\mathbb{C}$-lines in $V(\mathbb{C})$. For any compact open subgroup $K \subseteq H(\mathbb{A}_f)$, these data determine a Shimura variety $X_K = \mathrm{Sh}_K(H,\mathbb{D})$, which is a smooth quasi-projective variety of dimension $n-1$ defined over the field E whose complex points $X_K(\mathbb{C})$ can be identified with the locally symmetric space

$$H(\mathbb{Q})\backslash\big(\mathbb{D} \times H(\mathbb{A}_f)/K\big).$$

We then proceed to define special cycles. Let g be an integer such that $1 \le g \le n-1$. For a g-tuple $\lambda = (\lambda_i)_i \in V(E)^g$, we define $V_\lambda := \{v \in V : \forall i (v \perp \lambda_i)\}$, $H_\lambda := \mathrm{U}(V_\lambda)$ and $\mathbb{D}_\lambda := \{w \in \mathbb{D} : \forall i (w \perp \lambda_i)\}$. For every compact open subgroup $K \subseteq H(\mathbb{A}_f)$ and every coset $\gamma \in K\backslash H(\mathbb{A}_f)$, these data determine a 'translated' algebraic cycle $Z_K(\lambda,\gamma)$ on X_K over E at level K via the map

$$\begin{aligned} H_\lambda(\mathbb{Q})\backslash\big(\mathbb{D}_\lambda \times H_\lambda(\mathbb{A}_f)/H_\lambda(\mathbb{A}_f) \cap \gamma^{-1}K\gamma\big) &\longrightarrow H(\mathbb{Q})\backslash\big(\mathbb{D} \times H(\mathbb{A}_f)/K\big) = X_K(\mathbb{C}),\\ [(w,\gamma')] &\longmapsto [(w,\gamma'\gamma^{-1})]. \end{aligned}$$

Note that $Z_K(\lambda,\gamma)$ vanishes unless the $g \times g$ Hermitian matrix $Q(\lambda) := \frac{1}{2}\big((\lambda_i,\lambda_j)\big)_{i,j}$ is positive semidefinite and $\mathrm{rk}(Q(\lambda)) = \mathrm{rk}\,\mathrm{span}_E\{\lambda_i : 1 \le i \le g\}$, where the ranks are defined over E. Given a nice weight function, special cycles are a family of weighted linear combinations of these translated cycles over essentially integral tuples λ, indexed by the values of $Q(\lambda)$. More precisely, this family is defined as follows. Let $T \in \mathrm{Herm}_g(E)_{\ge 0}$ be a Hermitian positive semidefinite matrix and $\varphi \in \mathcal{S}_{K,g}$ be a K-invariant Schwartz–Bruhat function on the finite adelic points $V(\mathbb{A}_{E,f})^g$ over E. Recall that V can be defined as an algebraic variety over E. We define Ω_T to be the set-valued functor $\{\lambda \in V^g(-) : Q(\lambda) = T\}$. For any T such that $\Omega_T(E)$ is nonempty, we define the special cycle indexed by T as follows: we fix a rational tuple $\lambda_0 \in \Omega_T(E)$. Then for every K-orbit $\lambda \in K\backslash\Omega_T(\mathbb{A}_{E,f})$, there is a unique shift element $\gamma \in K\backslash H(\mathbb{A}_f)$ such that $\gamma\lambda_0 = \lambda$, and we define the translated cycle $Z_K(\lambda)$ to be $Z_K(\lambda_0,\gamma)$. Finally, the special cycle $Z_K(T,\varphi)$ in X_K is defined to be the sum

$$Z_K(T,\varphi) = \sum_{\lambda \in K\backslash\Omega_T(\mathbb{A}_{E,f})} \varphi(\lambda) Z_K(\lambda),$$

which is a finite sum is due to the compactness of the set $\Omega_T(\mathbb{A}_{E,f}) \cap \mathrm{supp}(\varphi)$. Moreover, the special cycles $Z_K(T,\varphi)$ as K varies are compatible in the projective system under the étale covering maps of the corresponding Shimura varieties X_K, so we may write $Z(T,\varphi)$ for $Z_K(T,\varphi)$.

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We are in position to state the unitary Kudla modularity conjecture. Let $\mathcal{L}_{\mathbb{D}}$ be the tautological line bundle over $\mathbb{D}$: that is, the restriction to $\mathbb{D}$ of the line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(V(\mathbb{C}))$. Since the action of $H(\mathbb{R})$ on $\mathbb{D}$ lifts naturally to a certain action on $\mathcal{L}_{\mathbb{D}}$, the line bundle $\mathcal{L}_{\mathbb{D}}$ descends to a line bundle $\mathcal{L}_K$ over the Shimura variety X_K. Just like the special cycles $Z_K(T,\varphi)$, the line bundles $\mathcal{L}_K$ are also compatible in the projective system of Shimura varieties X_K as K varies, and we write $\mathcal{L}^\vee$ for the class of the dual bundle in the first Chow group $\mathrm{CH}^1(X_K)$. It is clear that $Z(T,\varphi)$ is a cycle of codimension $\mathrm{rk}(T)$, and we write $Z(T,\varphi)$ also for its class in the Chow group $\mathrm{CH}^{\mathrm{rk}(T)}(X_K)$. Finally, the generating series for special cycles on X_K is defined to be the formal sum

$$\psi^{\mathrm{CH}}_{g,\varphi}(\tau) := \sum_{T\in\mathrm{Herm}_g(E)_{\geq 0}} Z(T,\varphi)\cdot(\mathcal{L}^\vee)^{g-\mathrm{rk}(T)}q^T,$$

where the dot '$\cdot$' denotes the product in the Chow ring $\mathrm{CH}^\bullet(X_K)$ and $q^T = \exp\big(2\pi i\,\mathrm{tr}(T\tau)\big)$ for τ in the Hermitian upper half-space $\mathbb{H}_g$. The unitary Kudla modularity conjecture states that for any compact open subgroup K, the element $\psi^{\mathrm{CH}}_g(\tau) := (\varphi \longmapsto \psi^{\mathrm{CH}}_{g,\varphi}(\tau))$ in the set $\mathrm{Hom}(\mathcal{S}_{K,g}, \mathrm{CH}^g(X_K)_{\mathbb{C}})$ defines a vector-valued Hermitian modular form over $E/\mathbb{Q}$ of genus g, weight n and some congruence type ρ of $\mathrm{U}(g,g)(\mathbb{Z})$ arising from the Weil representation. In particular, for any linear functional $\ell\in\mathrm{CH}^g(X_K)^\vee_{\mathbb{C}}$, the termwise composition $\ell\circ\psi^{\mathrm{CH}}_{g,\varphi}(\tau)$ defines a scalar-valued Hermitian modular form of genus g and weight n for some congruence subgroup of $\mathrm{U}(g,g)(\mathbb{Z})$, as was originally stated by Kudla.

An immediate consequence of this conjecture is that the $\mathbb{C}$-span of special cycles $Z(T,\varphi)$ in the complexification $\mathrm{CH}^g(X_K)_{\mathbb{C}}$ is finite-dimensional, and a further appealing feature is that relations between Fourier coefficients of certain Hermitian modular forms give rise to the corresponding relations between special cycles $Z(T,\varphi)$ in $\mathrm{CH}^g(X_K)_{\mathbb{C}}$. See [7, 19] for this computational aspect in the orthogonal case. Under the assumption that the generating series $\psi^{\mathrm{CH}}_{g,\varphi}(\tau)$ is absolutely convergent, Liu proved the conjecture in Theorem 3.5 of [15]. Combining his result with Theorem 5.7, we obtain the following result.

Theorem 6.1. *The unitary Kudla conjecture is true for open Shimura varieties over norm-Euclidean imaginary quadratic fields.*

Proof. We claim that the generating series $\psi^{\mathrm{CH}}_g(\tau)$ defines a symmetric formal Fourier–Jacobi series of genus g, cogenus $g-1$, weight n and some congruence type ρ, and the statement then follows from Theorem 5.7 for $l = g-1$, $k = n$ and the type ρ. First, writing $\tau = \left(\begin{smallmatrix}\tau_1 & w\\ z & \tau_2\end{smallmatrix}\right)$, we consider the formal Fourier–Jacobi expansion

$$\psi^{\mathrm{CH}}_{g,\varphi}(\tau) = \sum_{m\in\mathrm{Herm}_{g-1}(E)_{\geq 0}} \phi_{m,\varphi}(\tau_1,w,z)e(m\tau_2),$$

where the mth formal Fourier–Jacobi coefficient $\phi_{m,\varphi}$ is given by

$$\phi_{m,\varphi}(\tau_1,w,z) = \sum_{\substack{n\in\mathbb{Q}_{\geq 0}\\ r\in\mathrm{Mat}_{1,g-1}(E)}} Z\left(\begin{pmatrix} n & r\\ r^* & m\end{pmatrix},\varphi\right)e(n\tau_1+rz)e(r^*w).$$

To verify our claim, we recall the first part of Theorem 3.5 in [15], which implies that $\phi_m := (\varphi \longmapsto \phi_{m,\varphi})$ defines a formal vector-valued Jacobi form of genus 1, weight n, index m and some congruence type ρ arising from Weil representation, hence it suffices to show the absolute convergence of each $\phi_{m,\varphi}$. We proceed as in Zhang's thesis [28]. Although connected components of a Shimura variety X_K are in general defined over cyclotomic extensions of $\mathbb{Q}$, they are linked via Galois action, and special cycles on them are Galois conjugate to each other. Therefore, without loss of generality, we may work with a connected Shimura variety in the classical setting as in [26], and unless otherwise stated, we use from now on the numbering and notation as in Chapter 2 of [28] for the counterparts that can be

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similarly defined in the unitary case. The Shimura variety X_K we work with can be written as X_Γ for a neat congruence subgroup Γ of $\mathrm{U}(L)$, where L is the unitary counterpart of the lattice defined in Theorem 2.5.

First, we note that our formal Fourier–Jacobi coefficients $\phi_{m,\varphi}$ correspond to $\theta_{(\lambda,\mu),t}$ defined in (2.7), where the functions F are defined as F_ι just before Theorem 2.9 for linear functionals ι on the Chow group CH^r_L and $1 \le r = g \le n-1$. By (2.10), which expresses $\theta_{(\lambda,\mu),t}$ as a finite sum of $\theta_{\lambda,\underline{x}}$ (defined just before (2.10)), and following the proof of Proposition 2.6, we see that each formal Fourier–Jacobi coefficient $\theta_{(\lambda,\mu),t}$ is absolutely convergent assuming the absolute convergence of the vector-valued generating q-series $\Theta_{F_{\underline{x}}} = \Theta_{F_{\iota,\underline{x}}}$ (defined just before Theorem 2.5) on $\mathbb{H}_1$ for every tuple $\underline{x} \in L^{\vee,r-1}$ with positive definite $\mathbb{Q}\underline{x}$ and $1 \le r \le n-1$. Following the proof of Theorem 2.9, for any linear functional ι on the Chow group CH^r_L, in the unitary case we also have the equation $\Theta_{F_{\iota,\underline{x}}} = \Theta_{F_{\iota\circ i\underline{x}}}$, where $i\underline{x}$ is the associated shifting by a power of dual line bundle defined in the middle of the proof. Since $\iota \circ i\underline{x}$ is a linear functional on the first Chow group $\mathrm{CH}^1_{L_{\underline{x}}}$, the absolute convergence of the counterpart of $\Theta_{F_{\iota\circ i\underline{x}}}$ in the unitary case follows from the second part of Theorem 3.5 in [15], which completes the proof of our claim. □

Remark 6.2. If the Hermitian space V is anisotropic, then X_K is compact, and our theorem is already complete in this case. If V is isotropic so that X_K is not compact, we only obtain the result for open Shimura varieties. On a toroidal compactification X_K, the modularity of the generating series of special cycles becomes a natural question after we define suitable boundary components of special cycles. The problem was posed in [57], and in the case of special divisors on toroidal compactifications of orthogonal Shimura varieties, Bruinier–Zemel recently proved the corresponding modularity conjecture in [58].

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Conflicts of Interest. None.

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