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m-embedded Subgroups and p-nilpotency of Finite Groups

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Abstract. Let A be a subgroup of a finite group G and $\Sigma = \{G_0 \le G_1 \le \cdots \le G_n\}$ some subgroup series of G. Suppose that for each pair (K, H) such that K is a maximal subgroup of H and $G_{i-1} \le K < H \le G_i$, for some i, either $A \cap H = A \cap K$ or $AH = AK$. Then A is said to be Σ-embedded in G. And A is said to be m-embedded in G if G has a subnormal subgroup T and a $\{1 \le G\}$-embedded subgroup C in G such that $G = AT$ and $T \cap A \le C \le A$. In this article, some sufficient conditions for a finite group G to be p-nilpotent are given whenever all subgroups with order p^k of a Sylow p-subgroup of G are m-embedded for a given positive integer k.

1 Introduction

All groups considered in this paper are finite, and G always denotes a group. The symbol $[A]B$ denotes the semidirect product of the groups A and B, where B is an operator group of A. The notions and notation are standard as in [3].

Let A be a subgroup of G, $K \le H \le G$, and let p be a prime. Then we say that A *covers* the pair (K, H) if $AH = AK$, and A *avoids* (K, H) if $A \cap H = A \cap K$. (K, H) is said to be a *maximal pair* of G if K is a maximal subgroup of H. In [2], the authors introduced the following concepts.

Definition 1.1 Let A be a subgroup of G and let $\Sigma = \{G_0 \le G_1 \le \cdots \le G_n\}$ be some subgroup series of G. Then we say that A is Σ-embedded in G if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \le K < H \le G_i$, for some i.

Definition 1.2 Let A be a subgroup of G.

(i) A is *m-embedded* in G if G has a subnormal subgroup T and a $\{1 \le G\}$-embedded subgroup C in G such that $G = AT$ and $T \cap A \le C \le A$.
(ii) A is *nearly m-embedded* in G if G has a subgroup T and a $\{1 \le G\}$-embedded subgroup C in G such that $G = AT$ and $T \cap A \le C \le A$.

Guo and Skiba in [2] get the following result.

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Theorem 1.3 ([2, Theorem 4.1]) *Let p be a prime dividing $|G|$ such that $(|G|, p-1) = 1$, and let P be a Sylow p-subgroup of G with $|P| = p^n$. Then G is p-nilpotent if and only if either the Sylow p-subgroups of G have order p or there is an integer k such that $1 \leq k < n$ and every subgroup of G of order p^k and every subgroup of G of order 4 (if $p^k = 2$ and P is non-abelian) are m-embedded in G.*

A celebrated theorem of Frobenius [3, Satz. IV.5.8] asserts that G is p-nilpotent if $N_G(H)$ is p-nilpotent for every p-subgroup H of G. In this paper we replace some of the conditions of Frobenius' theorem and Theorem 1.3. We shall investigate the p-nilpotency of a group G if every subgroup H with order p^k of a Sylow p-subgroup of G is m-embedded in G for a fixed positive integer k, and $N_G(H)$ is p-nilpotent. Some interesting results related to the p-nilpotency of a finite group are obtained.

2 Preliminaries

Lemma 2.1 ([7, p. 59, Proposition 2.6]) *Let P be a p-subgroup of G, $N \trianglelefteq G$, and $(|N|, p) = 1$. Then $N_{G/N}(PN/N) = N_G(P)N/N$.*

Lemma 2.2 ([2, Lemma 2.3]) *Let $M \leq G$, N and R be normal subgroups of G.*

(i) *If $E \leq V$ and M is $\{E \leq G\}$-embedded in G, then $M \cap V$ is $\{E \leq V\}$-embedded in V.*
(ii) *If $R \leq N$ and M is $\{R \leq G\}$-embedded in G, then NM is $\{R \leq G\}$-embedded in G and NM/N is $\{1 \leq G/N\}$-embedded in G/N.*

Lemma 2.3 ([2, Lemma 2.13]) *Let U be a m-embedded subgroup of G and let N be a normal subgroup of G.*

(i) *If $U \leq H \leq G$, then U is m-embedded in H.*
(ii) *If $N \leq U$, then U/N is m-embedded in G/N.*
(iii) *Let π be a set of primes, let U be a π-subgroup, and let N be a π'-subgroup. Then UN/N is m-embedded in G/N.*

Lemma 2.4 ([2, Lemma 2.14]) *Let P be a normal non-identity p-subgroup of G with $|P| = p^n$ and $P \cap \Phi(G) = 1$. Suppose that either every maximal subgroup of P is nearly m-embedded in G or there is an integer k such that $1 \leq k < n$ and the subgroups of P of order p^k are m-embedded in G. Then some maximal subgroup of P is normal in G.*

Lemma 2.5 ([3, III, 5.2 and IV, 5.4]) *Suppose p is a prime and G is not a p-nilpotent group, but its proper subgroups are all p-nilpotent.*

(i) *G has a normal Sylow p-subgroup P and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*
(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
(iii) *The exponent of P is p or 4.*

3 Main Results

Theorem 3.1 *Let G be a group and let P be a Sylow p-subgroup of G, where p is an odd prime. If each maximal subgroup P_1 of P is m-embedded in G and $N_G(P_1)$ is p-nilpotent, then G is p-nilpotent.*

Proof Assume that the result is false and let G be a counterexample of minimal order. Then we have the following series of conclusions.

(**a**) $O_{p'}(G) = 1$: Suppose that $O_{p'}(G) \neq 1$. Consider $G/O_{p'}(G)$. Let $K/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $K = K \cap PO_{p'}(G) = (K \cap P)O_{p'}(G)$. Let $P_1 = K \cap P$. It is easy to see that P_1 is a maximal subgroup of P. By the hypothesis, P_1 is m-embedded in G and $N_G(P_1)$ is p-nilpotent. By Lemma 2.1, we have $N_{G/O_{p'}(G)}(P_1O_{p'}(G))/O_{p'}(G) = N_G(P_1)O_{p'}(G)/O_{p'}(G)$. So $N_G(P_1)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent. By Lemma 2.3(iii), $K/O_{p'}(G)$ is m-embedded in $G/O_{p'}(G)$. Then $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, which implies that G is p-nilpotent, a contradiction.

(**b**) Let W be a subgroup of G such that $P \leq W < G$; then W is p-nilpotent : Let P_1 be a maximal subgroup of P. Obviously $N_W(P_1) \leq N_G(P_1)$. By the hypothesis, we have $N_W(P_1)$ is p-nilpotent and by Lemma 2.3(i), P_1 is m-embedded in W. Hence W satisfies the hypothesis of the theorem. The minimality of G implies that W is p-nilpotent.

(**c**) $L = O_p(G)$ is the unique minimal normal subgroup of G, $G/O_p(G)$ is p-nilpotent, and $\Phi(G) = 1$: Since G is not p-nilpotent, by the Glauberman–Thompson Theorem, $N_G(Z(J(P)))$ is not p-nilpotent, where $J(P)$ is the Thompson subgroup of P. Noticing that $Z(J(P))$ is a characteristic subgroup of P, and $P \leq N_G(P) \leq N_G(Z(J(P)))$. By (b), we have $N_G(Z(J(P))) = G$, and so $O_p(G) \neq 1$. Let L be a minimal normal subgroup of G contained in $O_p(G)$. If $L = P$, then obviously G/L is p-nilpotent. If L is a maximal subgroup of P, then by the hypothesis, $G = N_G(L)$ is p-nilpotent, a contradiction. Hence we may assume that $|P{:}L| \geq p^2$. Let P_1/L be a maximal subgroup of P/L. Then P_1 is a maximal subgroup of P. By the hypothesis and Lemma 2.3(ii), P_1/L is m-embedded in G/L. Suppose that $N_{G/L}(P_1/L) = K/L$. Then $P_1/L \trianglelefteq K/L$, so $P_1 \trianglelefteq K$, hence $K \leq N_G(P_1)$. Clearly, $N_G(P_1)/L \leq K/L$. Thus $N_{G/L}(P_1/L) = N_G(P_1)/L$. By the hypothesis, we get that $N_{G/L}(P_1/L)$ is p-nilpotent. Then G/L satisfies the hypothesis of the theorem, so the choice of G yields that G/L is p-nilpotent. The uniqueness of L and $\Phi(G) = 1$ are obvious. Now by [4, Lemma 2.6], we have $L = O_p(G)$.

(**d**) $C_G(O_p(G)) \leq O_p(G)$ and $G = PQ$, where Q is a Sylow q-subgroup of G with $q \neq p$: By (c), G is p-solvable. So $C_G(O_p(G)) \leq O_p(G)$ follows from (a) and [5, Theorem 9.3.1]. For each prime $q \in \pi(G)$ and $q \neq p$, there exists a Sylow q-subgroup Q of G such that $G_1 = PQ$ is a subgroup of G by [1, Theorem 6.3.5]. If $G_1 < G$, then (b) forces G_1 to be p-nilpotent, and so $Q \trianglelefteq G_1$. Thus we have $LQ = L \times Q$. It follows that $Q \leq C_G(L) = C_G(O_p(G)) \leq O_p(G)$, a contradiction. Hence $G_1 = G$; that is, $G = PQ$.

(e) $|L| = p$ and $P \cap M$ is a maximal subgroup of P: By (c), $\Phi(G) = 1$. Then G has a maximal subgroup M such that $G = ML$ and $M \cap L = 1$. Clearly, $P = L(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. By the hypothesis, there are a subnormal subgroup T of G and a $\{1 \leq G\}$-embedded subgroup C of G such that $G = P_1 T$ and $P_1 \cap T \leq C \leq P_1$. Thus C either covers or avoids (M, G). But $CM \leq P_1 M \neq G$, hence $C \cap M = C$; that is, $C \leq M$. By [2, Lemma 2.3], C is subnormal in G. Then $C \leq O_p(G) = L$. Hence, $C \leq M \cap L = 1$, and then $|T|_p = p$. Since $|G:T|$ is a power of p, $O^p(G) \leq T$. By the minimality of L, we have $L \leq O^p(G) \leq T$, thus $|L| = p$, and so $P \cap M$ is a maximal subgroup of P.

(f) The final contradiction : Let Q_1 be a Sylow q-subgroup of M such that $M = (P \cap M)Q_1$. If $p < q$, then by [6, Lemma 2.8], $O_p(G)Q_1$ is p-nilpotent, and so $Q_1 \leq C_G(O_p(G))$, which contradicts (d). So $q < p$. Then by (c) and (d), we have $F(G) = L = C_G(L)$. It follows that $M \cong G/L = N_G(L)/C_G(L)$, which is isomorphic to a subgroup of $\mathrm{Aut}(L)$. Because $|L| = p$ by (e), $\mathrm{Aut}(L)$ is a cyclic group of order $p-1$. It follows that M is cyclic, and so $Q_1 \leq N_G(P \cap M)$. Since $P \cap M$ is a maximal subgroup of P, we have $P \cap M \trianglelefteq P$ and $G = PM = PQ_1 \leq N_G(P \cap M)$. Now by the hypothesis, $G = N_G(P \cap M)$ is p-nilpotent, the final contradiction.

This completes the proof. ■

Theorem 3.2 *Let G be a group and let P be a Sylow p-subgroup of G, where p is an odd prime. If P has a subgroup D with $1 < |D| < |P|$ such that all subgroups H of P with order $|H| = |D|$ are m-embedded in G and $N_G(H)$ is p-nilpotent, then G is p-nilpotent.*

Proof Assume that the result is false and let G be a counterexample of minimal order. Now, arguing as in the proof of Theorem 3.1, the following statements (a) and (b) about G are true.

(a) $O_{p'}(G) = 1$.

(b) Let W be a subgroup of G such that $P \leq W < G$; then W is p-nilpotent.

Again, we have a series of conclusions.

(c) $|P:D| > p$ and $|D| > p$: That $|P:D| > p$ follows from Theorem 3.1. Now assume that $|D| = p$. By Lemma 2.3, it is easy to see that each proper subgroup of G satisfies the hypothesis. By the choice of G, we have that each proper subgroup of G is p-nilpotent. So by Lemma 2.5(i), $G = [P]Q$, where Q is a Sylow q-subgroup of G and $q \neq p$. Denote $\Phi = \Phi(P)$. Let X/Φ be a subgroup of P/Φ of order p, $x \in X \setminus \Phi$, and $S = \langle x \rangle$. Then S is of order p by Lemma 2.5(iii). By the hypothesis, S is m-embedded in G, then there are a subnormal subgroup T of G and a $\{1 \leq G\}$-embedded subgroup C of G such that $G = ST$ and $S \cap T \leq C \leq S$. Since S has order p, if $S \cap T = 1$, then $|G:T| = p$. Since T is p-nilpotent, G is p-nilpotent, a contradiction. Thus $T = G$ and then $S = C$ is $\{1 \leq G\}$-embedded in G. It follows that $X/\Phi = S\Phi/\Phi$ is $\{1 \leq G/\Phi\}$-embedded in G/Φ by Lemma 2.2. Now Lemmas 2.4 and 2.5 imply that $|P/\Phi| = p$. It follows immediately that P is cyclic. So P has a unique minimal subgroup, say P_1. Then P_1 is a characteristic subgroup of P. Since

$P \trianglelefteq G$, we get $P_1 \trianglelefteq G$. It follows from the hypothesis that $G = N_G(P_1)$ is p-nilpotent, a contradiction. Hence $|D| > p$.

(d) $O_p(G) \neq 1$. Let L be a minimal normal subgroup of G contained in $O_p(G)$; then $|L| < |D|$: Since G is not p-nilpotent, by the Glauberman–Thompson Theorem, $N_G(Z(J(P)))$ is not p-nilpotent, where $J(P)$ is a Thompson subgroup of P. Noticing that $Z(J(P))$ is a characteristic subgroup of P, $N_G(P) \leq N_G(Z(J(P)))$. By (b), we have $N_G(Z(J(P))) = G$ and then $O_p(G) \neq 1$. If $|L| = |D|$, then by the hypothesis, $G = N_G(L)$ is p-nilpotent, a contradiction. Suppose that $|L| > |D|$. Since $L \leq O_p(G)$, L is elementary abelian. By Lemma 2.4, L has a maximal subgroup that is normal in G, contrary to the minimality of L. Hence $|L| < |D|$.

(e) G/L is p-nilpotent, and L is the unique minimal normal subgroup of G and $\Phi(G) = 1$: By (d) and Lemma 2.3, it is easy to see that G/L satisfies the hypothesis of the theorem, so the choice of G yields that G/L is p-nilpotent. The uniqueness of L and $\Phi(G) = 1$ are obvious.

(f) The final contradiction : By (e), G has a maximal subgroup M such that $G = ML$, $M \cap L = 1$, and $M \cong G/L$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by L and M, the uniqueness of L yields $O_p(G) \cap M = 1$, and so $L = O_p(G)$. Clearly, $P = L(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Then $P = LP_1$.

If $M \cap P = 1$, then $L = P$, a contradiction. Now we suppose that $1 < |M \cap P| \leq |D|$. Pick $H \leq P$ such that $M \cap P \leq H$ and $|H| = |D|$. By the hypothesis, there are a subnormal subgroup T of G and a $\{1 \leq G\}$-embedded subgroup C of G such that $G = HT$ and $H \cap T \leq C \leq H$. If $T < G$, since $|G{:}T|$ is a power of p and T is subnormal in G, there exists a normal subgroup V of G such that $T \leq V$ and $|G{:}V| = p$. It follows that $P \cap V$ is a Sylow p-subgroup of V and it is a maximal subgroup of P. By (c), $|P{:}D| > p$, then $|D| < |P \cap V|$. Now by the hypothesis, all subgroups H of $P \cap V$ with order $|H| = |D|$ are m-embedded in G then in V by Lemma 2.3(i), and $N_V(H) \leq N_G(H)$ is p-nilpotent. Thus V satisfies the hypothesis. The choice of G yields that V is p-nilpotent, we have G is p-nilpotent, a contradiction. Hence $T = G$, so $H = C$ is a $\{1 \leq G\}$-embedded subgroup. By [2, Lemma 2.3], H is subnormal in G. Then $H \leq O_p(G) = L$. So $M \cap P \leq M \cap H \leq M \cap L = 1$, a contradiction.

Suppose that $|M \cap P| > |D|$. Then we can choose a subgroup H of $M \cap P$ such that $|H| = |D|$. Using a similar argument as above, we can get $H \leq O_p(G) = L$, so $H \leq M \cap P \cap L = 1$, the final contradiction.

This completes the proof. ■

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