J. Appl. Probab. 1–17 (2025)
*doi:*10.1017/jpr.2025.17

LIMIT THEOREMS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

HUIJIE QIAO ,* *Southeast University*

Abstract

This work concerns stochastic differential equations with jumps. We prove convergence for solutions to a sequence of (possibly degenerate) stochastic differential equations with jumps when the coefficients converge in some appropriate sense. Then some special cases are analyzed and some concrete and verifiable conditions are given.

Keywords: Weak solutions; martingale solutions; Fokker–Planck equations; superposition principles

2010 Mathematics Subject Classification: Primary 60H10
Secondary 60J76

1. Introduction

Stochastic differential equations (SDEs) with jumps appear naturally in various applied fields, and more and more people pay attention to them. For example, [6] systematically discusses the martingale problems of SDEs with jumps. Qiao and Zhang [16] proved that, under non-Lipschitz conditions, for almost all sample points ω, the solutions to a certain SDE with jumps form a homeomorphism flow. Later, [13, 14] studied the exponential ergodicity and Euler–Maruyama approximation for SDEs with jumps and non-Lipschitz coefficients. Recently, [4] established the equivalence between SDEs with jumps and the corresponding non-local Fokker–Planck equations under only linear growth conditions.

In this paper, we study limit theorems for SDEs with jumps, i.e. convergence for solutions to a sequence of stochastic differential equations with jumps when the coefficients converge in some appropriate sense. More precisely, we fix a $T>0$ and consider the following sequence of SDEs with jumps:

$$\mathrm{d}X_t^n = b^n(t, X_t^n)\,\mathrm{d}t + \sigma^n(t, X_t^n)\,\mathrm{d}B_t + \gamma^n \int_{\mathbb{U}} g(t, X_{t-}^n, u)\,N(\mathrm{d}t, \mathrm{d}u), \qquad t \in [0, T], \tag{1}$$

where (B_t) is an m-dimensional Brownian motion and $N(\mathrm{d}t, \mathrm{d}u)$ is a Poisson random measure with intensity $\mathrm{d}t\,\nu(\mathrm{d}u)$. Here, ν is a finite measure defined on $(\mathbb{U}, \mathscr{U})$, where $(\mathbb{U}, \mathscr{U})$ is any measurable space. The coefficients $b^n\colon [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma^n\colon [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$ are Borel measurable functions, $\{\gamma^n\}$ is a sequence of real numbers, and $g\colon [0, T] \times \mathbb{R}^d \times \mathbb{U} \mapsto \mathbb{R}^d$ is Borel measurable. Then, when $b^n \to b$, $a^n \to a$, and $\gamma^n \to \gamma$ in some sense as $n \to \infty$, where $a^n(t, x) = (\sigma^n\sigma^{n*})(t, x)$, $a(t, x) = (\sigma\sigma^*)(t, x)$, and σ^* denotes the transpose of the matrix σ,

Received 20 March 2024; accepted 23 January 2025.

* Postal address: School of Mathematics, Southeast University Nanjing, Jiangsu 211189, China; Department of Mathematics, University of Illinois at Urbana-Champaign Urbana, IL 61801, USA. Email: hjqiaogean@seu.edu.cn

we will prove that the solutions of (1) converge to that of the following equation in some suitable sense:

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t + \gamma \int_{\mathbb{U}} g(t, X_{t-}, u)\,N(\mathrm{d}t, \mathrm{d}u), \qquad t \in [0, T], \tag{2}$$

where $b\colon [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$ are Borel measurable and γ is a real number.

This problem was initially proposed by Stroock and Varadhan in 1979 [21]. From then on, similar problems have been discussed by many experts in various formulations [1–3, 7, 9, 10, 15, 17, 18, 22, 23]. For the case of $g = 0$, [21] proved that if a and b are locally bounded measurable functions which are continuous in x for each $t \in [0, T]$, and for any $R > 0$,

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \sup_{x\in B_R} (\|a^n(t, x)\| + |b^n(t, x)|) < \infty,$$

$$\lim_{n\to\infty} \int_0^T \sup_{x\in B_R} (\|a^n(t, x) - a(t, x)\| + |b^n(t, x) - b(t, x)|)\,\mathrm{d}t = 0,$$

where $B_R := \{x \in \mathbb{R}^d,\ |x| \leqslant R\}$, a martingale solution of (1) converges weakly to a unique martingale solution of (2) in [21, Chapter 11] (see Definition 2 for the definition of a martingale solution). Then [10] showed that the convergence holds in the L^1 sense when b, b^n, σ, and σ^n are growing linearly in x, and

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \sup_{x\in B_R} (\|\sigma^n(t, x) - \sigma(t, x)\| + |b^n(t, x) - b(t, x)|) = 0.$$

Later, [17] required that b, b^n, σ, and σ^n are growing linearly in x and that a and a^n are non-degenerate, and established the convergence in the distributional sense. Recently, [3] obtained convergence in the distributional sense under bounded conditions by means of the superposition principles. We mention that in [3, 10, 17, 18, 21], boundedness, non-degeneracy, or uniform convergence of b, b^n, σ, and σ^n are required.

For the case treated here, [7] systematically studied the problem and obtained limit theorems of SDEs driven by càdlàg processes under Lipschitz conditions. The following SDEs were considered in [22]: for $d = 1$,

$$\mathrm{d}X^n_t = b^n(t, X^n_t)\,\mathrm{d}t + \sigma^n(t, X^n_{t-})\,\mathrm{d}M^n_t, \qquad \mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t,$$

where $\{M^n\}$ is a sequence of square-integrable martingales. It was proved that if $\{M^n\}$ has good convergence, b^n and σ^n are uniformly bounded and $b^n(t, x) \to b(t, x)$, $\sigma^n(t, x) \to \sigma(t, x)$, X^n converges to X in the distributional sense. Later, [10, 23] generalized the result in [22] to pure jump SDEs and SDEs with jumps, respectively. Recently, [2] established the convergence of Markov transition semigroups for SDEs with jumps when the coefficients are smooth enough. Note that the results in [1, 2, 7, 10, 22, 23] cannot be applied to (1) and (2).

In this paper, our first aim is to remove boundedness, non-degeneracy, and uniform convergence of b, b^n, σ, and σ^n, and establish a limit theorem under some pretty weak conditions. Thus, our result can cover some results in [3, 10, 17, 21]. Besides, [15] presented a limit theorem for SDEs driven by Lévy processes when the coefficients satisfy some weak conditions. However, no concrete and verifiable conditions are given there. Therefore, our second aim is to analyze some special cases and give some concrete and verifiable conditions.

Our motivation is twofold. First, this sort of theorem would be of interest because the study of the asymptotic behavior of stochastic processes satisfying an SDE with jumps is often reduced to this kind of limit theorem. Second, we offer some weak conditions of convergence for the martingale solutions to (1). In particular, we give concrete and verifiable conditions under which the martingale solutions to (1) converge to that of an SDE with pure jumps (Corollary 1), that of an SDE without jumps (Proposition 3), and that of an ordinary differential equation (Proposition 4). Thus, it is convenient to apply these limit theorems to approximation theory and statistics [7].

The paper is arranged as follows. In the next section we introduce some concepts such as weak solutions and martingale solutions of SDEs with jumps, weak solutions of Fokker–Planck equations, and their relationship. The main results are stated in Section 3, and the main theorem is proved in Section 4. In Section 5, we analyze some special cases and give some concrete and verifiable conditions.

The following convention will be used throughout the paper: C with or without indices will denote different positive constants whose values may change from one place to another.

2. Preliminaries

2.1. Notation

We introduce some notation to be used throughout. We use $|\cdot|$ and $\|\cdot\|$ for the norms of vectors and matrices, respectively. Let $\langle\cdot,\cdot\rangle$ be the scalar product in $\mathbb{R}^d$.

$C^2(\mathbb{R}^d)$ represents the space of continuous functions on $\mathbb{R}^d$ that have continuous partial derivatives of order up to two, and $C^2_{\mathrm{b}}(\mathbb{R}^d)$ is the subspace of $C^2(\mathbb{R}^d)$ consisting of functions whose derivatives up to order two are bounded. $C^2_{\mathrm{c}}(\mathbb{R}^d)$ is the collection of all functions in $C^2(\mathbb{R}^d)$ with compact supports, and $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ denotes the collection of all real-valued C^∞ functions with compact supports.

Let $\mathscr{B}(\mathbb{R}^d)$ be the Borel σ-field on $\mathbb{R}^d$. Let $\mathcal{P}(\mathbb{R}^d)$ be the space of all probability measures on $\mathscr{B}(\mathbb{R}^d)$, equipped with the topology of weak convergence. Let $\mathcal{P}_1(\mathbb{R}^d)$ be the collection of all the probability measures μ on $\mathscr{B}(\mathbb{R}^d)$ satisfying $\mu(|\cdot|) := \int_{\mathbb{R}^d}|x|\,\mu(\mathrm{d}x) < \infty$. Let $L^\infty([0,T],\mathcal{P}_1(\mathbb{R}^d))$ be the collection of all measurable families $(\mu_t)_{t\in[0,T]}$ of probability measures on $\mathscr{B}(\mathbb{R}^d)$ satisfying $\sup_{t\in[0,T]}\mu_t(|\cdot|) < \infty$.

2.2. Weak solutions and martingale solutions for SDEs with jumps

In this subsection we introduce the concepts of weak solutions and martingale solutions for SDEs with jumps, and point out their relationship.

First of all, consider the following SDE with jumps:

$$\mathrm{d}X_t = b(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)\,\mathrm{d}B_t + \int_{\mathbb{U}} f(t,X_{t-},u)\,N(\mathrm{d}t,\mathrm{d}u), \qquad t\in[0,T], \tag{3}$$

where $f\colon [0,T]\times\mathbb{R}^d\times\mathbb{U}\mapsto\mathbb{R}^d$ is Borel measurable. We recall the definition of weak solutions to (3). Although [8] is a good reference for the definition below, it does not deal with SDEs with jumps.

Definition 1. (*Weak solutions*) By a weak solution to (3) we mean a septuple $\{(\Omega,\mathscr{F},\mathbb{P};(\mathscr{F}_t)_{t\in[0,T]}),(B,N,X)\}$, where $(\Omega,\mathscr{F},\mathbb{P};(\mathscr{F}_t)_{t\in[0,T]})$ is a complete filtered probability space, (B_t) is an $(\mathscr{F}_t)$-adapted Brownian motion, $N(\mathrm{d}t,\mathrm{d}u)$ is an $(\mathscr{F}_t)$-adapted Poisson random measure, independent of (B_t), with intensity $\mathrm{d}t\,\nu(\mathrm{d}u)$, and (X_t) is an $(\mathscr{F}_t)$-adapted process such that,

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for all $t \in [0, T]$,

$$\mathbb{P}\left(\int_0^t \left(|b(s, X_s)| + \|\sigma\sigma^*(s, X_s)\| + \int_{\mathbb{U}} |f(s, X_{s-}, u)| \, \nu(\mathrm{d}u) \right) \mathrm{d}s < \infty \right) = 1$$

and $X_t = X_0 + \int_0^t b(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}B_s + \int_0^t \int_{\mathbb{U}} f(s, X_{s-}, u) \, N(\mathrm{d}s, \mathrm{d}u)$, $\mathbb{P}$-almost surely.

If two weak solutions to (3), $\{(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\in[0,T]}), (B, N, X^1)\}$ and $\{(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\in[0,T]}), (B, N, X^2)\}$ with $X_0^1 = X_0^2$, satisfy $X_t^1 = X_t^2$, $t \in [0, T]$, $\mathbb{P}$-almost surely, we say that pathwise uniqueness holds for (3).

If any two weak solutions to (3) with the same initial distribution have the same law, we say that uniqueness in law holds for (3).

It is known that pathwise uniqueness implies uniqueness in law for (3).

Let $D_T := D([0, T], \mathbb{R}^d)$ be the collection of càdlàg functions from $[0, T]$ to $\mathbb{R}^d$. A generic element in D_T is denoted by w. We equip D_T with the Skorokhod topology, and D_T is a Polish space. For any $t \in [0, T]$, set $e_t : D_T \mapsto \mathbb{R}^d$ such that $e_t(w) = w_t$, $w \in D_T$. Let $\mathcal{B}_t := \sigma\{w_s : s \in [0, t]\}$, $\bar{\mathcal{B}}_t := \mathrm{Cap}_{s>t}\mathcal{B}_s$, and $\mathcal{B} := \mathcal{B}_T$. For $\phi \in C_b^2(\mathbb{R}^d)$, set

$$(\mathscr{A}_t\phi)(x) := b_i(t, x)\partial_i\phi(x) + \frac{1}{2}a_{ij}(t, x)\partial_{ij}\phi(x), \quad (\mathscr{B}_t\phi)(x) := \int_{\mathbb{U}} [\phi(x + f(t, x, u)) - \phi(x)] \, \nu(\mathrm{d}u).$$

In the following, we define martingale solutions of (3) (cf. [8, 21]).

Definition 2. (*Martingale solutions*) For $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, a probability measure $\mathbb{P}$ on $(D_T, \mathcal{B})$ is called a martingale solution of (3) with the initial law μ_0 at time 0, if:

(i) $\mathbb{P} \circ e_0^{-1} = \mu_0$;

(ii) For any $\phi \in C_c^2(\mathbb{R}^d)$, $\mathcal{M}_t^\phi := \phi(w_t) - \phi(w_0) - \int_0^t (\mathscr{A}_s\phi + \mathscr{B}_s\phi)(w_s) \, \mathrm{d}s$ is a $(\bar{\mathcal{B}}_t)_{t\in[0,T]}$-adapted martingale under the probability measure $\mathbb{P}$.

The uniqueness of the martingale solutions to (3) means that, for any $s \in [0, T]$ and any $\mu_s \in \mathcal{P}(\mathbb{R}^d)$, if $\hat{\mathbb{P}}, \tilde{\mathbb{P}}$ are two martingale solutions to (3) with the initial law μ_s at the time s, then $\hat{\mathbb{P}} = \tilde{\mathbb{P}}$.

Next, we make the following assumptions:

$\mathbf{H}_{b,\sigma}$ There is a constant C_1 such that, for all $(t, x) \in [0, T] \times \mathbb{R}^d$, $|b(t, x)| + \|\sigma(t, x)\| \leqslant C_1(1 + |x|)$.

$\mathbf{H}_f$ There is a constant C_2 such that, for all $(t, x) \in [0, T] \times \mathbb{R}^d$, $\int_{\mathbb{U}} |f(t, x, u)|^2 \, \nu(\mathrm{d}u) \leqslant C_2(1 + |x|)^2$.

$\mathbf{H}_f'$ By the Hölder inequality, it is easy to see that $\int_{\mathbb{U}} |f(t, x, u)| \, \nu(\mathrm{d}u) \leqslant C(1 + |x|)$.

The relationship between martingale solutions and weak solutions is as follows.

Proposition 1. *Assume that* $\mathbf{H}_{b,\sigma}$ *and* $\mathbf{H}_f$ *hold.*

(i) For any $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, the existence of a weak solution $(X_t)_{t\in[0,T]}$ to (3) with $\mathcal{L}_{X_0} = \mu_0$ is equivalent to the existence of a martingale solution $\mathbb{P}$ to (3) with the initial law μ_0. Moreover, $\mathcal{L}_{X_t} = \mathbb{P} \circ e_t^{-1}$ for any $t \in [0, T]$.

(ii) The uniqueness of martingale solutions $\mathbb{P}$ to (3) is equivalent to the uniqueness in law of weak solutions $(X_t)_{t\in[0,T]}$ to (3).

Since the proof of Proposition 1 is direct, we omit it (cf. [24, Theorem 5.6]).

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2.3. Weak solutions of Fokker–Planck equations

In this subsection we introduce weak solutions of the Fokker–Planck equations (FPEs) and state some of their properties.

Consider the FPE associated with (3):

$$\partial_t \mu_t = (\mathscr{A}_t + \mathscr{B}_t)^* \mu_t, \tag{4}$$

where $(\mathscr{A}_t + \mathscr{B}_t)^*$ is the adjoint operator of $\mathscr{A}_t + \mathscr{B}_t$ in $C_{\mathrm{b}}^2(\mathbb{R}^d)$, and $(\mu_t)_{t\in[0,T]}$ is a family of probability measures on $\mathbb{R}^d$. Weak solutions of (4) are defined as follows.

Definition 3. A measurable family $(\mu_t)_{t\in[0,T]}$ of probability measures is called a weak solution of the non-local FPE (4) starting from μ_0 at time 0 if, for any $R > 0$ and $t \in [0, T]$,

$$\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{B_R}(x) \bigg(|b(s, x)| + \|a(s, x)\| + \int_{\mathbb{U}} |f(s, x, u)| \, \nu(\mathrm{d}u) \bigg) \, \mu_s(\mathrm{d}x) \, \mathrm{d}s < \infty, \tag{5}$$

where $\mathbf{1}_{B_R}$ is the indicator function of B_R (i.e. $\mathbf{1}_{B_R}(x) = 1$ for $x \in B_R$, and 0 for $x \notin B_R$) and, for all $\phi \in C_{\mathrm{c}}^2(\mathbb{R}^d)$ and $t \in [0, T]$, $\mu_t(\phi) = \mu_0(\phi) + \int_0^t \mu_s(\mathscr{A}_s\phi + \mathscr{B}_s\phi) \, \mathrm{d}s$, where $\mu_t(\phi) := \int_{\mathbb{R}^d} \phi(x) \, \mu_t(\mathrm{d}x)$. The uniqueness of weak solutions to (4) means that if, for any $s \in [0, T]$ and any $\mu_s \in \mathcal{P}(\mathbb{R}^d)$, $(\hat{\mu}_t)_{t\in[s,T]}$ and $(\tilde{\mu}_t)_{t\in[s,T]}$ are two weak solutions to (4) starting from μ_s at time s, $\hat{\mu}_t = \tilde{\mu}_t$ for any $t \in [s, T]$.

We claim that this definition makes sense. That is, under (5) we have $\int_0^t |\mu_s(\mathscr{A}_s\phi + \mathscr{B}_s\phi)| \, \mathrm{d}s < \infty$ for all $\phi \in C_{\mathrm{c}}^2(\mathbb{R}^d)$. Indeed, for any $\phi \in C_{\mathrm{c}}^2(\mathbb{R}^d)$, we assume that the support of ϕ is in some ball B_R. Then

$$\begin{aligned}
\int_0^t |\mu_s(\mathscr{A}_s\phi + \mathscr{B}_s\phi)| \, \mathrm{d}s &\leqslant \int_0^t \int_{\mathbb{R}^d} \bigg[|b_i(s, x)\partial_i\phi(x)| + \frac{1}{2} |(\sigma\sigma^*)_{ij}(s, x)\partial_{ij}\phi(x)| \\
&\qquad + \int_{\mathbb{U}} |\phi(x + f(s, x, u)) - \phi(x)| \, \nu(\mathrm{d}u) \bigg] \, \mu_s(\mathrm{d}x) \, \mathrm{d}s \\
&\leqslant \|\phi\|_{C_{\mathrm{c}}^2(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{B_R}(x) \bigg(|b(s, x)| + \|a(s, x)\| \\
&\qquad + \int_{\mathbb{U}} |f(s, x, u)| \, \nu(\mathrm{d}u) \bigg) \, \mu_s(\mathrm{d}x) \, \mathrm{d}s \\
&\quad + \|\phi\|_{C_{\mathrm{c}}^2(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{B_R^{\mathrm{c}}}(x) \nu(\mathbb{U}) \, \mu_s(\mathrm{d}x) \, \mathrm{d}s < \infty.
\end{aligned}$$

If a weak solution $(\mu_t)_{t\in[0,T]}$ of the non-local FPE (4) is absolutely continuous with respect to the Lebesgue measure, there exists a measurable family of non-negative functions $(v_t)_{t\in[0,T]}$ with $\int_{\mathbb{R}^d} v_t(x) \, \mathrm{d}x = 1$ such that $\mu_t(\mathrm{d}x) = v_t(x) \, \mathrm{d}x$ for any $t \in [0, T]$. Thus, $(v_t)_{t\in[0,T]}$ satisfies, in the distributional sense,

$$\partial_t v_t = -\partial_i(b_i v_t) + \partial_{ij}(a_{ij} v_t) + \mathscr{B}_t^* v_t. \tag{6}$$

Set

$$\mathscr{L}_+ := \bigg\{ v = (v_t)_{t\in[0,T]} \colon v_t \geqslant 0, \ \int_{\mathbb{R}^d} v_t(x) \, \mathrm{d}x = 1 \text{ for all } t \in [0, T], \\ \text{and } \sup_{t\in[0,T]} \bigg(\int_{\mathbb{R}^d} |x| v_t(x) \, \mathrm{d}x \bigg) < \infty \bigg\}.$$

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If there exists a $v \in \mathscr{L}_+$ satisfying (6) in the distributional sense, we say that (6) has a weak solution in $\mathscr{L}_+$. Later, we will assume that (6) has a unique weak solution in $\mathscr{L}_+$, i.e. for any $s \in [0, T]$ and any non-negative measurable function v_s with $\int_{\mathbb{R}^d} v_s(x)\,\mathrm{d}x = 1$ and $\int_{\mathbb{R}^d} |x| v_s(x)\mathrm{d}x < \infty$, if $\hat{v}$ and $\tilde{v}$ are two weak solutions of (6) in $\mathscr{L}_+$ starting from v_s at time s, then $\hat{v}_t(x) = \tilde{v}_t(x)$ for all $x \in \mathbb{R}^d$ and for any $t \in [s, T]$.

Proposition 2. *Assume that* $\mathbf{H}_{b,\sigma}$ *and* $\mathbf{H}'_f$ *hold. If, for any* $\mu_0(\mathrm{d}x) = \bar{v}_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$*, there exists a measurable family of non-negative functions* $(v_t)_{t\in[0,T]}$ *with* $\int_{\mathbb{R}^d} v_t(x)\,\mathrm{d}x = 1$ *and* $v_0(x) = \bar{v}_0(x)$ *such that* v *is a weak solution of the non-local FPE (6), then* $v \in \mathscr{L}_+$.

Since the proof of Proposition 2 is similar to that for [4, Remark 1.3(ii)], we omit it.

2.4. The superposition principle for (3) and (4)

It is well known that, for any $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, if $\mathbb{P}$ is a martingale solution to (3) with the initial law μ_0, by simple computation it holds that $(\mathbb{P} \circ e_t^{-1})$ is a weak solution of (4) starting from μ_0. The natural question is whether the converse result is right. The answer is affirmative. The following theorem describes in detail the relationship between martingale solutions to (3) and weak solutions to (4).

Theorem 1. ([19, Corollary 1.9]) *Suppose that* $\mathbf{H}_{b,\sigma}$ *and* $\mathbf{H}_f$ *hold.*

(i) For any $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$*, the existence of a martingale solution* $\mathbb{P}$ *to (3) with the initial law* μ_0 *is equivalent to the existence of a weak solution* $(\mu_t)_{t\in[0,T]}$ *to (4) starting from* μ_0*. Moreover,* $\mathbb{P} \circ e_t^{-1} = \mu_t$ *for any* $t \in [0, T]$.

(ii) The uniqueness of the martingale solutions $\mathbb{P}$ *to (3) is equivalent to the uniqueness of the weak solutions* $(\mu_t)_{t\in[0,T]}$ *to (4).*

Remark 1. Theorem 1 is usually called a superposition principle.

3. Main results

First of all, we take $f(t, x, u) = \gamma g(t, x, u)$ for $\gamma \in \mathbb{R}$. So, (3) and FPE (6) become

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t + \gamma \int_{\mathbb{U}} g(t, X_{t-}, u)\,N(\mathrm{d}t, \mathrm{d}u), \quad t \in [0, T], \tag{7}$$

$$\partial_t v_t = -\partial_i(b_i v_t) + \partial_{ij}(a_{ij} v_t) + \mathscr{B}_t^* v_t, \tag{8}$$

where $(\mathscr{B}_t\phi)(x) := \int_{\mathbb{U}} [\phi(x + \gamma g(t, x, u)) - \phi(x)]\,\nu(\mathrm{d}u)$ for $\phi \in C_b^2(\mathbb{R}^d)$. Consider the following sequence of SDEs with jumps: for any $n \in \mathbb{N}$,

$$\mathrm{d}X_t^n = b^n(t, X_t^n)\,\mathrm{d}t + \sigma^n(t, X_t^n)\,\mathrm{d}B_t + \gamma^n \int_{\mathbb{U}} g(t, X_{t-}^n, u)\,N(\mathrm{d}t, \mathrm{d}u), \quad t \in [0, T], \tag{9}$$

where $b^n \colon [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma^n \colon [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$ are Borel measurable functions, and $\{\gamma^n\}$ is a real sequence. We study the relationship between martingale solutions of (7) and of (9) when $b^n \to b$, $a^n \to a$, $\gamma^n \to \gamma$ in some sense, where $a^n := \sigma^n \sigma^{n*}$.

The main result in this section is the following theorem.

Theorem 2. *Suppose that* b^n*,* b*,* σ^n*, and* σ *satisfy* $\mathbf{H}_{b,\sigma}$ *uniformly,* g *satisfies* $\mathbf{H}_f$*, and (7) has a unique martingale solution. For any* $\mu_0(\mathrm{d}x) = v_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$*, let* $\mathbb{P}^n$*,* $\mathbb{P}$ *be martingale solutions of (9) and (7) with the initial law* μ_0*, respectively. Assume that*

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(i) $b^n \to b$, $a^n \to a$ *in* $L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^d)$, $\gamma^n \to \gamma$ *as* $n \to \infty$*;*

(ii) $\mathbb{P}^n \circ e_t^{-1}$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$ *and* $v_t^n(x)$ *denotes the density, i.e.* $v_t^n(x) := (\mathbb{P}^n \circ e_t^{-1})(\mathrm{d}x)/\mathrm{d}x$ *for any* $t \in [0, T]$*, and* $\sup_{x \in \mathbb{R}^d} |v_t^n(x)| \leqslant C_T$.

Then $\mathbb{P}^n \to \mathbb{P}$ *in* $\mathcal{P}(D_T)$.

The proof of Theorem 2 is given in the next section.

Remark 2. Here we recall that the uniqueness of martingale solutions for (9) is not needed. Also, we mention that condition (ii) cannot be weakened, otherwise $\{\mathbb{P}^n\}$ does not converge to $\mathbb{P}$.

Remark 3. If $\gamma^n = \gamma = 0$, b^n, b, σ^n, and σ are locally bounded and Theorem 2 reduces to [21, Theorem 11.1.4]. Therefore, Theorem 2 is more general.

If (8) has a unique weak solution in $\mathscr{L}_+$, (7) has a unique martingale solution by Theorem 1. Thus, our result can cover [3, Theorem 3.7].

Next, we give an example to explain that b^n, σ^n usually happen.

Example 1. Assume that $d = m$, $b\colon [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \mapsto \mathbb{S}_+(\mathbb{R}^d)$ are continuous and satisfy $\mathbf{H}_{b,\sigma}$, $\gamma \in \mathbb{R}$, g satisfies $\mathbf{H}_f$ and (7) has a unique martingale solution. Let $\mathbb{P}$ be the unique martingale solution of (7) with the initial law $\mu_0 = v_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$. Set

$$
\begin{aligned}
b_i^n(t, x) &:= \int_{\mathbb{R}^d} \varphi_n(x - y) b_i(t, y)\,\mathrm{d}y, \\
a_{ij}^n(t, x) &:= \int_{\mathbb{R}^d} \varphi_n(x - y) a_{ij}(t, y)\,\mathrm{d}y, \qquad i, j = 1, 2, \ldots, d, \\
\sigma^n(t, x) &:= \sqrt{2a^n(t, x)}, \qquad a^n(t, x) = (a_{ij}^n(t, x)), \\
\gamma^n &:= \frac{n}{n+1}\gamma,
\end{aligned}
$$

where $\varphi \in C_c^\infty(\mathbb{R}^d)$ is a non-negative mollifier with the support in B_1 and $\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x = 1$, $\varphi_n(x) = n^d \varphi(nx)$. We also assume that $\mathbb{P}^n$ is a martingale solution of the corresponding (9) with the initial law μ_0, $\mathbb{P}^n \circ e_t^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, and $v_t^n(x)$ denotes the density, i.e. $v_t^n(x) := (\mathbb{P}^n \circ e_t^{-1})(\mathrm{d}x)/\mathrm{d}x$ for any $t \in [0, T]$, and $\sup_{x \in \mathbb{R}^d} |v_t^n(x)| \leqslant C_T$. So, b^n, σ^n, γ^n, and a^n satisfy the conditions of Theorem 2. Therefore, $\mathbb{P}^n \to \mathbb{P}$ in $\mathcal{P}(D_T)$.

4. Proof of Theorem 2

The proof is divided into two parts. In the first part we prove that $\{\mathbb{P}^n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(D_T)$. Then the convergence for a subsequence of $\{\mathbb{P}^n\}_{n \in \mathbb{N}}$ to $\mathbb{P}$ is established in the second part.

Proof of Theorem 2. Step 1. We prove that $\{\mathbb{P}^n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(D_T)$.

By Aldous' tightness criterion (see also [7, Theorem 4.5, p. 356]), it is sufficient to check that

(i′) $\lim_{K \to \infty} \sup_n \mathbb{P}^n(\sup_{t \in [0,T]} |w_t| > K) = 0$;

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(ii′) For any stopping time τ, $\lim_{\theta\to 0}\sup_n \sup_{0\leqslant\tau<\tau+\theta\leqslant T}\mathbb{P}^n(|w_{\tau+\theta}-w_\tau|\geqslant N)=0$ for all $N>0$.

Since, for any $n\in\mathbb{N}$, $\mathbb{P}^n$ is a martingale solution of (9) with the initial law μ_0, Proposition 1 yields that there exists a weak solution $\{(\hat{\Omega}^n, \hat{\mathscr{F}}^n, \hat{\mathbb{P}}^n; (\hat{\mathscr{F}}^n_t)_{t\in[0,T]}), (\hat{B}^n, \hat{N}^n, \hat{X}^n)\}$ of (9) with $\mathcal{L}_{\hat{X}^n_t}=\mathbb{P}^n\circ e_t^{-1}$ and $\mathcal{L}_{\hat{X}^n_0}=\mu_0$. So, by Definition 1, for any $t\in[0,T]$,

$$\hat{X}^n_t=\hat{X}^n_0+\int_0^t b^n(s,\hat{X}^n_s)\,\mathrm{d}s+\int_0^t\sigma^n(s,\hat{X}^n_s)\,\mathrm{d}\hat{B}^n_s+\int_0^t\int_{\mathbb{U}}\gamma^n g(s,\hat{X}^n_s,u)\,\hat{N}^n(\mathrm{d}s,\mathrm{d}u). \tag{10}$$

Moreover, the Burkholder–Davis–Gundy inequality implies that

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\sup_{s\in[0,t]}|\hat{X}^n_s|\bigg) &\leqslant \mathbb{E}^{\hat{\mathbb{P}}^n}|\hat{X}^n_0|+\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_0^t|b^n(r,\hat{X}^n_r)|\,\mathrm{d}r\bigg)+\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\sup_{s\in[0,t]}\bigg|\int_0^s\sigma^n(r,\hat{X}^n_r)\,\mathrm{d}\hat{B}^n_r\bigg|\bigg)\\
&\quad+\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_0^t\int_{\mathbb{U}}|\gamma^n||g(r,\hat{X}^n_r,u)|\,\hat{N}^n(\mathrm{d}r,\mathrm{d}u)\bigg)\\
&\leqslant \mu_0(|\cdot|)+\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_0^t|b^n(r,\hat{X}^n_r)|\,\mathrm{d}r\bigg)+C\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_0^t\|\sigma^n(r,\hat{X}^n_r)\|^2\,\mathrm{d}r\bigg)^{1/2}\\
&\quad+\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_0^t\int_{\mathbb{U}}|\gamma^n||g(r,\hat{X}^n_r,u)|\,\nu(\mathrm{d}u)\,\mathrm{d}r\bigg)\\
&\leqslant \mu_0(|\cdot|)+\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_0^t C(1+|\hat{X}^n_r|)\,\mathrm{d}r\bigg)+C\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_0^t C(1+|\hat{X}^n_r|)^2\,\mathrm{d}r\bigg)^{1/2},
\end{aligned}$$

and $\mathbb{E}^{\hat{\mathbb{P}}^n}\big(\sup_{s\in[0,t]}(1+|\hat{X}^n_s|)\big)\leqslant 1+\mu_0(|\cdot|)+C(t+t^{1/2})\mathbb{E}^{\hat{\mathbb{P}}^n}\big(\sup_{s\in[0,t]}(1+|\hat{X}^n_s|)\big)$. By taking t_0 with $C(t_0+t_0^{1/2})<\frac{1}{2}$, we obtain that $\mathbb{E}^{\hat{\mathbb{P}}^n}\big(\sup_{s\in[0,t_0]}|\hat{X}^n_s|\big)\leqslant 2\mu_0(|\cdot|)+1$. On $[t_0,2t_0], [2t_0,3t_0],\ldots,[[T/t_0]t_0,T]$, in the same way we deduce that

$$\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\sup_{t\in[0,T]}|\hat{X}^n_t|\bigg)\leqslant 2^{[T/t_0]+1}\mu_0(|\cdot|)+2^{[T/t_0]+1}-1, \tag{11}$$

where $[T/t_0]$ denotes the largest integer no more than T/t_0. Thus, it follows from (11) that $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ satisfies (i′).

Next, we go back to (10). For any stopping time τ and $\theta>0$ with $0\leqslant\tau<\tau+\theta\leqslant T$,

$$\hat{X}^n_{\tau+\theta}-\hat{X}^n_\tau=\int_\tau^{\tau+\theta}b^n(s,\hat{X}^n_s)\,\mathrm{d}s+\int_\tau^{\tau+\theta}\sigma^n(s,\hat{X}^n_s)\,\mathrm{d}\hat{B}^n_s+\int_\tau^{\tau+\theta}\int_{\mathbb{U}}\gamma^n g(s,\hat{X}^n_s,u)\,\hat{N}^n(\mathrm{d}s,\mathrm{d}u).$$

By similar deduction to previously, we have

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{P}}^n}|\hat{X}^n_{\tau+\theta}-\hat{X}^n_\tau| &\leqslant \mathbb{E}^{\hat{\mathbb{P}}^n}\int_\tau^{\tau+\theta}C(1+|\hat{X}^n_s|)\,\mathrm{d}s+C\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\int_\tau^{\tau+\theta}C(1+|\hat{X}^n_s|)^2\,\mathrm{d}s\bigg)^{1/2}\\
&\leqslant C\mathbb{E}^{\hat{\mathbb{P}}^n}\bigg(\sup_{s\in[0,T]}(1+|\hat{X}^n_s|)\bigg)(\theta+\theta^{1/2})\\
&\leqslant C(2^{[T/t_0]+1}\mu_0(|\cdot|)+2^{[T/t_0]+1})(\theta+\theta^{1/2}),
\end{aligned}$$

where the last inequality is based on (11). By some elementary computations, $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ satisfies (ii′). Thus, $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ is relatively weakly compact. That is, there exists a weak convergence subsequence still denoted as $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$.

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Step 2. We show that a limit point of $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ is $\mathbb{P}$.

Assume that a limit point of $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ is $\bar{\mathbb{P}}$. Since $\mathbb{P}$ is the unique martingale solution of (7) with the initial law μ_0, we only need to prove that $\bar{\mathbb{P}}$ is also a martingale solution of (7) with the initial law μ_0. That is, it is sufficient to check that for $0 \leqslant s < t \leqslant T$ and a bounded continuous $\bar{\mathcal{B}}_s$-measurable functional $\chi_s\colon D_T \mapsto \mathbb{R}$,

$$\int_{D_T}\bigg[\phi(w_t)-\phi(w_s)-\int_s^t(\mathscr{A}_r\phi+\mathscr{B}_r\phi)(w_r)\,\mathrm{d}r\bigg]\chi_s(w)\,\bar{\mathbb{P}}(\mathrm{d}w)=0 \quad \text{for all } \phi\in C_c^2(\mathbb{R}^d). \tag{12}$$

Next, note that $\mathbb{P}^n\circ e_t^{-1}\to\bar{\mathbb{P}}\circ e_t^{-1}$ in $\mathcal{P}(\mathbb{R}^d)$ and $\mathbb{P}^n\circ e_0^{-1}=\mu_0=\bar{\mathbb{P}}\circ e_0^{-1}$. Thus, by (ii), there exists a $\bar{\nu}=(\bar{\nu}_t)_{t\in[0,T]}$ with $\bar{\nu}_t(x)\geqslant 0$ for all $x\in\mathbb{R}^d$ and $\int_{\mathbb{R}^d}\bar{\nu}_t(x)\,\mathrm{d}x=1$ for any $t\in[0,T]$ such that $\bar{\mathbb{P}}\circ e_t^{-1}(\mathrm{d}x)=\bar{\nu}_t(x)\,\mathrm{d}x$ and $\lim_{n\to\infty}\int_{\mathbb{R}^d}\psi(x)\nu_t^n(x)\,\mathrm{d}x=\int_{\mathbb{R}^d}\psi(x)\bar{\nu}_t(x)\,\mathrm{d}x$ for any $\psi\in C_b(\mathbb{R}^d)$ and $\bar{\nu}_0(x)=\nu_0(x)$. Also, by (11), $\sup_n\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|x|\nu_t^n(x)\,\mathrm{d}x\leqslant C$. By suitable approximation, we have $\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|x|\bar{\nu}_t(x)\,\mathrm{d}x<\infty$. In the following, set $\nu_{t,x}(\mathrm{d}z):=\nu(\mathrm{d}g^{-1}(t,x,\cdot)(z))$, and $\mathscr{B}_t\phi(x)=\int_{\mathbb{R}^d}[\phi(x+\gamma z)-\phi(x)]\,\nu_{t,x}(\mathrm{d}z)$. Thus, based on [4, Lemma 4.1] and our Lemma 1, we know that, for any $\varepsilon>0$ and the coefficients b, a, there exist $\tilde{b}\colon[0,T]\times\mathbb{R}^d\mapsto\mathbb{R}^d$, $\tilde{a}\colon[0,T]\times\mathbb{R}^d\mapsto\mathbb{S}_+(\mathbb{R}^d)$, where $\mathbb{S}_+(\mathbb{R}^d)$ is the set of non-negative symmetric definite $d\times d$ real matrices, and a family of measures $\tilde{\nu}_{\cdot,\cdot}$ such that

- (i″) $\tilde{b}$, $\tilde{a}$ are continuous and compactly supported;
- (ii″) for any $\phi\in C_c^2(\mathbb{R}^d)$, $(t,x)\mapsto\tilde{\mathscr{B}}_t\phi(x)$ is continuous, where $\tilde{\mathscr{B}}_t\phi(x):=\int_{\mathbb{R}^d}[\phi(x+\gamma z)-\phi(x)]\,\tilde{\nu}_{t,x}(\mathrm{d}z)$, and $\sup_{t\in[0,T],x\in\mathbb{R}^d}|\tilde{\mathscr{B}}_t\phi(x)|<\infty$;
- (iii″) we have

$$\int_0^T\int_{\mathbb{R}^d}\bigg(|b(t,x)-\tilde{b}(t,x)|+\frac{\|a(t,x)-\tilde{a}(t,x)\|}{1+|x|}+|\mathscr{B}_t\phi(x)-\tilde{\mathscr{B}}_t\phi(x)|\bigg)\bar{\nu}_t(x)\,\mathrm{d}x\,\mathrm{d}t<\varepsilon.$$

Also, the operators with respect to $\tilde{b}$, $\tilde{a}$, and $\tilde{\nu}_{\cdot,\cdot}$ are denoted as $\tilde{\mathscr{A}}_t+\tilde{\mathscr{B}}_t$.

Now, we treat (12). Inserting $\tilde{\mathscr{A}}_t+\tilde{\mathscr{B}}_t$, we get

$$\begin{aligned}
&\bigg|\int_{D_T}\bigg[\phi(w_t)-\phi(w_s)-\int_s^t(\mathscr{A}_r\phi+\mathscr{B}_r\phi)(w_r)\,\mathrm{d}r\bigg]\chi_s(w)\,\bar{\mathbb{P}}(\mathrm{d}w)\bigg|\\
&\quad\leqslant\bigg|\int_{D_T}\bigg[\phi(w_t)-\phi(w_s)-\int_s^t(\tilde{\mathscr{A}}_r\phi+\tilde{\mathscr{B}}_r\phi)(w_r)\,\mathrm{d}r\bigg]\chi_s(w)\,\bar{\mathbb{P}}(\mathrm{d}w)\bigg|\\
&\qquad+\bigg|\int_{D_T}\bigg[\int_s^t((\tilde{\mathscr{A}}_r\phi+\tilde{\mathscr{B}}_r\phi)(w_r)-(\mathscr{A}_r\phi+\mathscr{B}_r\phi)(w_r))\,\mathrm{d}r\bigg]\chi_s(w)\,\bar{\mathbb{P}}(\mathrm{d}w)\bigg|\\
&\quad=:I_1+I_2.
\end{aligned}\tag{13}$$

To deal with I_1, we recall that $\mathbb{P}^n$ is a martingale solution of (9) with the initial law μ_0, which implies that

$$\int_{D_T}\bigg[\phi(w_t)-\phi(w_s)-\int_s^t(\mathscr{A}_r^n\phi+\mathscr{B}_r^n\phi)(w_r)\,\mathrm{d}r\bigg]\chi_s(w)\,\mathbb{P}^n(\mathrm{d}w)=0,$$

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where $\mathscr{A}_r^n + \mathscr{B}_r^n$ denotes the generator of (9). So,

$$
\begin{aligned}
&\left| \int_{D_T} \left[\phi(w_t) - \phi(w_s) - \int_s^t (\tilde{\mathscr{A}}_r\phi + \tilde{\mathscr{B}}_r\phi)(w_r)\,\mathrm{d}r \right] \chi_s(w)\,\mathbb{P}^n(\mathrm{d}w) \right| \\
&\quad = \left| \int_{D_T} \left[\int_s^t ((\mathscr{A}_r^n\phi + \mathscr{B}_r^n\phi)(w_r) - (\tilde{\mathscr{A}}_r\phi + \tilde{\mathscr{B}}_r\phi)(w_r))\,\mathrm{d}r \right] \chi_s(w)\,\mathbb{P}^n(\mathrm{d}w) \right| \\
&\quad \leqslant C \int_{D_T} \int_s^t |(\mathscr{A}_r^n\phi + \mathscr{B}_r^n\phi)(w_r) - (\tilde{\mathscr{A}}_r\phi + \tilde{\mathscr{B}}_r\phi)(w_r)|\,\mathrm{d}r\,\mathbb{P}^n(\mathrm{d}w) \\
&\quad \leqslant C \int_s^t \int_{\mathbb{R}^d} \bigg[|(b_i^n(r,x) - \tilde{b}_i(r,x))\partial_i\phi(x)| + \frac{1}{2}|(a_{ij}^n(r,x) - \tilde{a}_{ij}(r,x))\partial_{ij}\phi(x)| \\
&\qquad\qquad + |\mathscr{B}_r^n\phi(x) - \mathscr{B}_r\phi(x)| + |\mathscr{B}_r\phi(x) - \tilde{\mathscr{B}}_r\phi(x)| \bigg] v_r^n(x)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad \leqslant C \int_s^t \int_{\mathbb{R}^d} \bigg[|(b_i^n(r,x) - \tilde{b}_i(r,x))\partial_i\phi(x)| + \frac{1}{2}|(a_{ij}^n(r,x) - \tilde{a}_{ij}(r,x))\partial_{ij}\phi(x)| \\
&\qquad\qquad + \int_{\mathbb{R}^d} |\phi(x+\gamma^n z) - \phi(x+\gamma z)|\,\nu_{r,x}(\mathrm{d}z) + |\mathscr{B}_r\phi(x) - \tilde{\mathscr{B}}_r\phi(x)| \bigg] v_r^n(x)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad \leqslant C \int_s^t \int_{\mathbb{R}^d} \bigg[|(b_i^n(r,x) - \tilde{b}_i(r,x))\partial_i\phi(x)| + \frac{1}{2}|(a_{ij}^n(r,x) - \tilde{a}_{ij}(r,x))\partial_{ij}\phi(x)| \\
&\qquad\qquad + \|\phi\|_{C_c^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} \mathbf{1}_{|x+\gamma z|\leqslant M} |\gamma^n - \gamma||z|\,\nu_{r,x}(\mathrm{d}z) \\
&\qquad\qquad + \int_{\mathbb{R}^d} \mathbf{1}_{|x+\gamma z|>M} |\phi(x+\gamma^n z)|\,\nu_{r,x}(\mathrm{d}z) + |\mathscr{B}_r\phi(x) - \tilde{\mathscr{B}}_r\phi(x)| \bigg] v_r^n(x)\,\mathrm{d}x\,\mathrm{d}r,
\end{aligned} \tag{14}
$$

where the fact that $\mathrm{supp}(\phi) \subset B_M$ for $M > 0$ is applied in the last inequality. As $n \to \infty$, based on (i), (i″), (ii″), and $\lim_{n\to\infty} \int_{\mathbb{R}^d} \psi(x) v_t^n(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\bar{v}_t(x)\,\mathrm{d}x$ for any $\psi \in C_{\mathrm{b}}(\mathbb{R}^d)$, (14) yields that

$$
\begin{aligned}
I_1 &\leqslant C \int_s^t \int_{\mathbb{R}^d} \bigg[|(b_i(r,x) - \tilde{b}_i(r,x))\partial_i\phi(x)| + \frac{1}{2}|(a_{ij}(r,x) - \tilde{a}_{ij}(r,x))\partial_{ij}\phi(x)| \\
&\qquad\qquad + |\mathscr{B}_r\phi(x) - \tilde{\mathscr{B}}_r\phi(x)| \bigg] \bar{v}_r(x)\,\mathrm{d}x\,\mathrm{d}r \\
&\leqslant C \int_s^t \int_{\mathbb{R}^d} \bigg[|b(r,x) - \tilde{b}(r,x)| + \frac{1}{2}\|a(r,x) - \tilde{a}(r,x)\| \mathbf{1}_{|x|\leqslant M} \\
&\qquad\qquad + |\mathscr{B}_r\phi(x) - \tilde{\mathscr{B}}_r\phi(x)| \bigg] \bar{v}_r(x)\,\mathrm{d}x\,\mathrm{d}r \\
&\leqslant C \int_s^t \int_{\mathbb{R}^d} \bigg[|b(r,x) - \tilde{b}(r,x)| + \frac{\|a(r,x) - \tilde{a}(r,x)\|}{1+|x|} + |\mathscr{B}_r\phi(x) - \tilde{\mathscr{B}}_r\phi(x)| \bigg] \bar{v}_r(x)\,\mathrm{d}x\,\mathrm{d}r \\
&< C\varepsilon,
\end{aligned} \tag{15}
$$

where we used $\mathbf{1}_{|x|\leqslant M} \leqslant (1+M)/(1+|x|)$ and (iii′) in the third and fourth inequalities, respectively.

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We now treat I_2. By similar deduction to (15), we can obtain

$$I_2 \leqslant C \int_{D_T} \int_s^t |(\tilde{\mathscr{A}}_r\phi + \tilde{\mathscr{B}}_r\phi)(w_r) - (\mathscr{A}_r\phi + \mathscr{B}_r\phi)(w_r)| \,\mathrm{d}r\, \bar{\mathbb{P}}(\mathrm{d}w)$$
$$\leqslant C \int_s^t \int_{\mathbb{R}^d} \bigg[|(b_i(r,x) - \tilde{b}_i(r,x))\partial_i\phi(x)| + \frac{1}{2}|(a_{ij}(r,x) - \tilde{a}_{ij}(r,x))\partial_{ij}\phi(x)|$$
$$+ |\mathscr{B}_r\phi(x) - \tilde{\mathscr{B}}_r\phi(x)|\bigg]\bar{\nu}_r(x)\,\mathrm{d}x\,\mathrm{d}r < C\varepsilon. \tag{16}$$

Combining (15) and (16) with (13), we get that

$$\left| \int_{D_T} \left[\phi(w_t) - \phi(w_s) - \int_s^t (\mathscr{A}_r\phi + \mathscr{B}_r\phi)(w_r)\,\mathrm{d}r\right]\chi_s(w)\,\bar{\mathbb{P}}(\mathrm{d}w)\right| < C\varepsilon.$$

Letting $\varepsilon \to 0$, we finally have (12). The proof is complete. □

Lemma 1. *For any $\varepsilon > 0$, there is a family of measures $\tilde{\nu}_{\cdot,\cdot}$ such that, for any $\phi \in C_c^2(\mathbb{R}^d)$,*

(i) $(t,x) \mapsto \tilde{\mathscr{B}}_t\phi(x)$ is continuous and $\sup_{t\in[0,T],x\in\mathbb{R}^d} |\tilde{\mathscr{B}}_t\phi(x)| < \infty$;

(ii) $\int_0^T \int_{\mathbb{R}^d} |\mathscr{B}_t\phi(x) - \tilde{\mathscr{B}}_t\phi(x)|\bar{\nu}_t(x)\,\mathrm{d}x\,\mathrm{d}t < \varepsilon$.

Proof. The method comes from [19, Lemma 3.8]. By [6, Lemma 14.50, p. 469], there is a measurable function $h_{t,x}(\theta)\colon [0,T] \times \mathbb{R}^d \times [0,\infty) \mapsto \mathbb{R}^d \cup \{\infty\}$ such that, for $t \in [0,T]$ and $x \in \mathbb{R}^d$, $\nu_{t,x}(A) = \int_0^\infty \mathbf{1}_A(h_{t,x}(\theta))\,\mathrm{d}\theta$ for all $A \in \mathscr{B}(\mathbb{R}^d)$. So, for any $\phi \in C_c^2(\mathbb{R}^d)$,

$$\mathscr{B}_t\phi(x) = \int_{\mathbb{R}^d} [\phi(x+\gamma z) - \phi(x)]\,\nu_{t,x}(\mathrm{d}z) = \int_0^\infty [\phi(x + \gamma h_{t,x}(\theta)) - \phi(x)]\,\mathrm{d}\theta.$$

Next, by the theory of functional analysis, we know that there exists a sequence of measurable functions $\{h^n_{t,x}(\theta), n \in \mathbb{N}\}$ such that, for any $\theta \geqslant 0$ and $n \in \mathbb{N}$, $(t,x) \mapsto h^n_{t,x}(\theta)$ is continuous with compact support, $|h^n_{t,x}(\theta)| \leqslant |h_{t,x}(\theta)|$, and

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^d} \int_0^\infty \big(|h^n_{t,x}(\theta) - h_{t,x}(\theta)|^2 \wedge 1\big)\bar{\nu}_t(x)\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Thus, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \big(|h^N_{t,x}(\theta) - h_{t,x}(\theta)|^2 \wedge 1\big)\bar{\nu}_t(x)\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t < \varepsilon. \tag{17}$$

Now, for $t \in [0,T]$ and $x \in \mathbb{R}^d$, set $\tilde{\nu}_{t,x}(A) := \int_0^\infty \mathbf{1}_A\big(h^N_{t,x}(\theta)\big)\,\mathrm{d}\theta$ for all $A \in \mathscr{B}(\mathbb{R}^d)$, and $(t,x) \mapsto \tilde{\nu}_{t,x}(A)$ is continuous. So, for any $\phi \in C_c^2(\mathbb{R}^d)$,

$$(t,x) \mapsto \tilde{\mathscr{B}}_t\phi(x) = \int_{\mathbb{R}^d} [\phi(x+\gamma z) - \phi(x)]\,\tilde{\nu}_{t,x}(\mathrm{d}z)$$

is continuous. Also, note that

$$|\tilde{\mathscr{B}}_t\phi(x)| \leqslant \int_0^\infty |\phi\big(x + \gamma h^N_{t,x}(\theta)\big) - \phi(x)|\,\mathrm{d}\theta \leqslant C \int_0^\infty \big(|\gamma h^N_{t,x}(\theta)| \wedge 1\big)\,\mathrm{d}\theta.$$

Since $h^N_{t,x}(\theta)$ has a compact support in (t,x), $\sup_{t\in[0,T],x\in\mathbb{R}^d}|\tilde{\mathscr{B}}_t\phi(x)|<\infty$. Thus, (i) is proved.

For (ii), we know that

$$
\begin{aligned}
&|\mathscr{B}_t\phi(x)-\tilde{\mathscr{B}}_t\phi(x)|\\
&\leqslant \int_0^\infty |\phi(x+\gamma h_{t,x}(\theta))-\phi\big(x+\gamma h^N_{t,x}(\theta)\big)|\,\mathrm{d}\theta\\
&\leqslant C\int_0^\infty \big(\mathbf{1}_{B_l}(h_{t,x}(\theta))\mathbf{1}_{B_{R+|\gamma|l}}(x)+\mathbf{1}_{B^{\mathrm{c}}_{l\vee((|x|-R)/|\gamma|)}}(h_{t,x}(\theta))\big)\big(|h^N_{t,x}(\theta)-h_{t,x}(\theta)|\wedge 1\big)\,\mathrm{d}\theta\\
&\leqslant C\bigg(\int_0^\infty \big(\mathbf{1}_{B_l}(h_{t,x}(\theta))\mathbf{1}_{B_{R+|\gamma|l}}(x)+\mathbf{1}_{B^{\mathrm{c}}_{l\vee((|x|-R)/|\gamma|)}}(h_{t,x}(\theta))\big)\,\mathrm{d}\theta\bigg)^{1/2}\\
&\quad\times\bigg(\int_0^\infty \big(|h^N_{t,x}(\theta)-h_{t,x}(\theta)|^2\wedge 1\big)\,\mathrm{d}\theta\bigg)^{1/2}\\
&\leqslant C\big(\mathbf{1}_{B_{R+|\gamma|l}}(x)\nu_{t,x}(B_l)+\nu_{t,x}\big(B^{\mathrm{c}}_{l\vee((|x|-R)/|\gamma|)}\big)\big)^{1/2}\bigg(\int_0^\infty \big(|h^N_{t,x}(\theta)-h_{t,x}(\theta)|^2\wedge 1\big)\,\mathrm{d}\theta\bigg)^{1/2}\\
&\leqslant C\bigg(\int_0^\infty \big(|h^N_{t,x}(\theta)-h_{t,x}(\theta)|^2\wedge 1\big)\,\mathrm{d}\theta\bigg)^{1/2},
\end{aligned}
$$

where $l>0$ is a constant, and $\mathrm{supp}(\phi)\subset B_R$ and $\nu_{t,x}(\mathbb{R}^d)<\infty$ are used in the second and fifth inequalities, respectively. Therefore, the Hölder inequality implies that

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}|\mathscr{B}_t\phi(x)-\tilde{\mathscr{B}}_t\phi(x)|\bar{\nu}_t(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\int_0^T\int_{\mathbb{R}^d}\bigg(\int_0^\infty \big(|h^N_{t,x}(\theta)-h_{t,x}(\theta)|^2\wedge 1\big)\,\mathrm{d}\theta\bigg)^{1/2}\bar{\nu}_t(x)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant CT^{1/2}\bigg(\int_0^T\int_{\mathbb{R}^d}\int_0^\infty \big(|h^N_{t,x}(\theta)-h_{t,x}(\theta)|^2\wedge 1\big)\,\mathrm{d}\theta\,\bar{\nu}_t(x)\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/2}\leqslant CT^{1/2}\varepsilon^{1/2},
\end{aligned}
$$

where (17) is used in the last inequality. The proof is complete. □

5. Special cases

In this section we analyze some special cases for (7) and (9) and give some concrete and verifiable conditions.

5.1. $\gamma\neq 0$, $g(t,x,u)=u$

In this subsection we take $\mathbb{U}\in\mathscr{B}(\mathbb{R}^d)$ and $g(t,x,u)=u$, and assume that, for any $p\geqslant 1$, $\int_{\mathbb{U}}|u|^2(1+|u|)^p\,\nu(\mathrm{d}u)<\infty$. Thus, (7) and (9) become

$$
\mathrm{d}X_t=b(t,X_t)\,\mathrm{d}t+\sigma(t,X_t)\,\mathrm{d}B_t+\gamma\int_{\mathbb{U}}u\,N(\mathrm{d}t,\mathrm{d}u),\qquad t\in[0,T],\tag{18}
$$

$$
\mathrm{d}X^n_t=b^n(t,X^n_t)\,\mathrm{d}t+\sigma^n(t,X^n_t)\,\mathrm{d}B_t+\gamma^n\int_{\mathbb{U}}u\,N(\mathrm{d}t,\mathrm{d}u),\qquad t\in[0,T].\tag{19}
$$

The following theorem characterizes the relationship between martingale solutions of (18) and of (19).

Theorem 3. *Suppose that b^n, b, σ^n, and σ satisfy $\mathbf{H}_{b,\sigma}$ uniformly and, for some $q>1$, $|\nabla b|\in L^1([0,T],L^q_{\mathrm{loc}}(\mathbb{R}^d))$, $(\partial_i b_i)^-\in L^1([0,T],L^\infty(\mathbb{R}^d))$, and $\|\nabla\sigma\|^2\in L^1([0,T],L^\infty(\mathbb{R}^d))$.*

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For $\mu_0(\mathrm{d}x) = v_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} v_0(x)^r (1+|x|^2)^{(r-1)d}\,\mathrm{d}x < \infty$ *for some* $r > 1$, *let* $\mathbb{P}^n, \mathbb{P}$ *be martingale solutions of (19) and (18) with the initial law* μ_0, *respectively. Assume that*

(i) $b^n \to b$, $a^n \to a$ *in* $L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^d)$, $\gamma^n \to \gamma$ *as* $n \to \infty$;

(ii) $\mathbb{P}^n \circ e_t^{-1}$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$ *and* $v_t^n(x)$ *denotes the density, i.e.* $v_t^n(x) := (\mathbb{P}^n \circ e_t^{-1})(\mathrm{d}x)/\mathrm{d}x$ *for any* $t \in [0, T]$, *and* $\sup_{x\in\mathbb{R}^d} |v_t^n(x)| \leqslant C_T$.

Then $\mathbb{P}^n \to \mathbb{P}$ *in* $\mathcal{P}(D_T)$.

Proof. By Theorem 2, we only need to prove that (18) has a unique martingale solution.

First of all, we can take a complete filtered probability space $\big(\check{\Omega}, \check{\mathscr{F}}, \check{\mathbb{P}}; (\check{\mathscr{F}}_t)_{t\in[0,T]}\big)$, an $(\check{\mathscr{F}}_t)$-adapted Brownian motion $(\check{B}_t)$ and an $(\check{\mathscr{F}}_t)$-adapted Poisson random measure $\check{N}(\mathrm{d}t, \mathrm{d}u)$ independent of $(\check{B}_t)$ with intensity measure $\mathrm{d}t\,\nu(\mathrm{d}u)$. We consider the equation

$$\check{X}_t(x) = x + \int_0^t \left(b(s, \check{X}_s(x)) - \int_{\mathbb{U}} \gamma u\,\nu(\mathrm{d}u)\right)\mathrm{d}s + \int_0^t \sigma(s, \check{X}_s(x))\,\mathrm{d}\check{B}_s + \int_0^t \int_{\mathbb{U}} \gamma u\,\tilde{\check{N}}(\mathrm{d}s\,\mathrm{d}u), \quad x \in \mathbb{R}^d,\ t \in [0, T], \tag{20}$$

where $\tilde{\check{N}}(\mathrm{d}s\,\mathrm{d}u) := \check{N}(\mathrm{d}s\,\mathrm{d}u) - \nu(\mathrm{d}u)\,\mathrm{d}s$ is the compensated martingale measure of $\check{N}(\mathrm{d}s\,\mathrm{d}u)$. By [24, Theorem 4.2], there exists a weak solution $\check{X}_t(x)$ of (20) satisfying

$$\sup_{t\in[0,T]} \check{\mathbb{E}}\left(\int_{\mathbb{R}^d} |\psi(\check{X}_t(x))|^{r^*}\,\rho(\mathrm{d}x)\right) \leqslant C\|\psi\|^{r^*}_{L^{r^*}_\rho} \quad \text{for all } \psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d), \tag{21}$$

where C is independent of ψ, $\check{\mathbb{E}}$ denotes the expectation under the probability measure $\check{\mathbb{P}}$, and

$$\frac{1}{r} + \frac{1}{r^*} = 1, \qquad \rho(\mathrm{d}x) = \frac{1}{(1+|x|^2)^d}\,\mathrm{d}x, \qquad \|\psi\|^{r^*}_{L^{r^*}_\rho} := \int_{\mathbb{R}^d} |\psi(x)|^{r^*}\,\rho(\mathrm{d}x).$$

Next, we choose an $\check{\mathscr{F}}_0$-measurable d-dimensional random vector $\check{X}_0$ such that $\check{\mathbb{P}} \circ \check{X}_0^{-1} = \mu_0$. Thus, $\bar{X}_\cdot := \check{X}_\cdot(\check{X}_0)$ solves the equation

$$\bar{X}_t = \check{X}_0 + \int_0^t \left(b(s, \bar{X}_s) - \int_{\mathbb{U}} \gamma u\,\nu(\mathrm{d}u)\right)\mathrm{d}s + \int_0^t \sigma(s, \bar{X}_s)\,\mathrm{d}\check{B}_s + \int_0^t \int_{\mathbb{U}} \gamma u\,\tilde{\check{N}}(\mathrm{d}s\,\mathrm{d}u). \tag{22}$$

Moreover, it follows from the Hölder and Jensen inequalities that, for any $\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \psi(x)\,\mathcal{L}_{\bar{X}_t}(\mathrm{d}x)\right| &= |\check{\mathbb{E}}\psi(\bar{X}_t)| \\
&= |\check{\mathbb{E}}[\check{\mathbb{E}}[\psi(\check{X}_t(x)) \mid x = \check{X}_0]]| \\
&= \left|\int_{\mathbb{R}^d} \check{\mathbb{E}}[\psi(\check{X}_t(x))]v_0(x)\,\mathrm{d}x\right| \\
&\leqslant \left(\int_{\mathbb{R}^d} |\check{\mathbb{E}}[\psi(\check{X}_t(x))]|^{r^*}\,\rho(\mathrm{d}x)\right)^{1/r^*} \left(\int_{\mathbb{R}^d} v_0(x)^r(1+|x|^2)^{rd}\,\rho(\mathrm{d}x)\right)^{1/r} \\
&\leqslant \left(\check{\mathbb{E}}\int_{\mathbb{R}^d} |\psi(\check{X}_t(x))|^{r^*}\,\rho(\mathrm{d}x)\right)^{1/r^*} \left(\int_{\mathbb{R}^d} v_0(x)^r(1+|x|^2)^{(r-1)d}\,\mathrm{d}x\right)^{1/r} \\
&\leqslant C\|\psi\|_{L^{r^*}_\rho},
\end{aligned}$$

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where the last inequality is based on (21). By the theory of functional analysis, we know that there exists a $v_t(\cdot)(1+|\cdot|^2)^d \in L^r_\rho$ such that

$$\int_{\mathbb{R}^d} \psi(x)v_t(x)(1+|x|^2)^d \,\rho(\mathrm{d}x) = \int_{\mathbb{R}^d} \psi(x)v_t(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\,\mathcal{L}_{\bar{X}_t}(\mathrm{d}x).$$

So, [8, Problem 4.25, p. 325] gives $\mathcal{L}_{\bar{X}_t}(\mathrm{d}x) = v_t(x)\,\mathrm{d}x$ for any $t \in [0, T]$. By Proposition 1 and Theorem 1, v solves the following FPE in the distribution sense:

$$\partial_t v_t = -\partial_i(b_i v_t) + \partial_{ij}(a_{ij}v_t) + \int_{\mathbb{U}} [v_t(\cdot - \gamma u) - v_t(\cdot)]\,\nu(\mathrm{d}u).$$

Moreover, using $\mu_0(\mathrm{d}x) = v_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$ and Proposition 2, we conclude that $v \in \hat{\mathscr{L}}_+$, where

$$\hat{\mathscr{L}}_+ := \Bigg\{ v = (v_t)_{t\in[0,T]} \colon v_t \geqslant 0, \int_{\mathbb{R}^d} v_t(x)\,\mathrm{d}x = 1 \text{ for any } t \in [0, T], \text{ and}$$
$$\sup_{t\in[0,T]} \left(\int_{\mathbb{R}^d} v_t(x)^r(1+|x|^2)^{(r-1)d}\,\mathrm{d}x \right) < \infty, \sup_{t\in[0,T]} \left(\int_{\mathbb{R}^d} |x|v_t(x)\,\mathrm{d}x \right) < \infty \Bigg\}.$$

Finally, note that, by [24, Theorem 4.2], (22) has a pathwise unique weak solution with the initial distribution μ_0 at time 0. So, for any time $s \in [0, T]$ and $\mu_s(\mathrm{d}x) := v_s(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} v_s(x)^r(1+|x|^2)^{(r-1)d}\,\mathrm{d}x < \infty$, by the same deduction as in [24, Theorem 4.2] we obtain that (22) has a pathwise unique weak solution with the initial distribution μ_s at time s. From this and Proposition 1, we know that (18) has a unique martingale solution. The proof is complete. □

Here we recall that σ in Theorem 3 can be degenerate. Let $\sigma = 0$; (18) becomes

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \gamma \int_{\mathbb{U}} u\,N(\mathrm{d}t, \mathrm{d}u), \qquad t \in [0, T]. \tag{23}$$

We immediately have the following result.

Corollary 1. *Suppose that b^n, b, and σ^n satisfy $\mathbf{H}_{b,\sigma}$ uniformly and, for some $q > 1$, $|\nabla b| \in L^1([0, T], L^q_{\mathrm{loc}}(\mathbb{R}^d))$, $(\partial_i b_i)^- \in L^1([0, T], L^\infty(\mathbb{R}^d))$. For any $\mu_0(\mathrm{d}x) = v_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} v_0(x)^r(1+|x|^2)^{(r-1)d}\,\mathrm{d}x < \infty$ for some $r > 1$, let $\mathbb{P}^n$, $\mathbb{P}$ be martingale solutions of (19) and (23) with the initial law μ_0, respectively. Assume that*

(i) $b^n \to b$, $a^n \to 0$ in $L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^d)$, $\gamma^n \to \gamma$ as $n \to \infty$;

(ii) $\mathbb{P}^n \circ e_t^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and $v^n_t(x)$ denotes the density, i.e. $v^n_t(x) := (\mathbb{P}^n \circ e_t^{-1})(\mathrm{d}x)/\mathrm{d}x$ for any $t \in [0, T]$, and $\sup_{x\in\mathbb{R}^d} |v^n_t(x)| \leqslant C_T$.

Then $\mathbb{P}^n \to \mathbb{P}$ in $\mathcal{P}(D_T)$.

Remark 4. This corollary means that SDEs with jumps can converge to SDEs with pure jumps in some sense.

5.2. $\sigma \neq 0$, $\gamma = 0$

In this subsection we take $\sigma \neq 0$, $\gamma = 0$, and require that σ^n, σ are independent of the space variable x. Thus, (7)–(9) become

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, \qquad t \in [0, T], \tag{24}$$

$$\partial_t \nu_t = -\partial_i(b_i \nu_t) + \partial_{ij}(a_{ij}\nu_t), \tag{25}$$

$$\mathrm{d}X_t^n = b^n(t, X_t^n)\,\mathrm{d}t + \sigma^n(t)\,\mathrm{d}B_t + \gamma^n \int_{\mathbb{U}} g(t, x, u)\, N(\mathrm{d}t, \mathrm{d}u), \qquad t \in [0, T]. \tag{26}$$

The following proposition describes the relationship between martingale solutions of (24) and of (26).

Proposition 3. *Suppose that b^n, b, σ^n, σ, and $\{\gamma^n\}$ are uniformly bounded, g satisfies $\mathbf{H}_f$, and $b \in L^1([0, T], BV_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d))$, $\partial_i b_i \in L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^d)$, $(\partial_i b_i)^- \in L^1([0, T], L^\infty(\mathbb{R}^d))$. For any $\mu_0(\mathrm{d}x) = \nu_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$ with $\|\nu_0\|_\infty < \infty$, let $\mathbb{P}^n, \mathbb{P}$ be martingale solutions of (26) and (24) with the initial law μ_0, respectively. Assume that*

(i) $b^n \to b$ in $L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^d)$, $a^n \to a$ in $L^1_{\mathrm{loc}}([0, T])$, $\gamma^n \to 0$ as $n \to \infty$;

(ii) $\mathbb{P}^n \circ e_t^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and $\nu_t^n(x)$ denotes the density, i.e. $\nu_t^n(x) := (\mathbb{P}^n \circ e_t^{-1})(\mathrm{d}x)/\mathrm{d}x$ for any $t \in [0, T]$, and $\sup_{x \in \mathbb{R}^d} |\nu_t^n(x)| \leqslant C_T$.

Then $\mathbb{P}^n \to \mathbb{P}$ in $\mathcal{P}(D_T)$.

Proof. By [3, Theorem 4.12] and Proposition 2, (25) has a unique weak solution in

$$\tilde{\mathscr{L}}_+ := \left\{ \nu = (\nu_t)_{t\in[0,T]} \colon \nu_t \geqslant 0, \int_{\mathbb{R}^d} \nu_t(x)\,\mathrm{d}x = 1 \text{ for any } t \in [0, T], \text{ and } \sup_{t\in[0,T]} \|\nu_t(\cdot)\|_\infty < \infty, \sup_{t\in[0,T]} \int_{\mathbb{R}^d} |x|\nu_t(x)\,\mathrm{d}x < \infty \right\},$$

which together with Theorem 1 implies that (24) has a unique martingale solution. Thus, the remaining proof is similar to that of Theorem 2, and is omitted it to save space. □

Remark 5. This proposition means that SDEs with jumps can converge to SDEs without jumps in some sense. Note that the result in [5] can be regarded as a discrete version of Proposition 3.

5.3. $\sigma = 0$, $\gamma = 0$

In this subsection we take $\sigma = 0$, $\gamma = 0$, and require that σ^n are independent of the space variable x. Thus, (7) and (9) become

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t, \qquad t \in [0, T], \tag{27}$$

$$\mathrm{d}X_t^n = b^n(t, X_t^n)\,\mathrm{d}t + \sigma^n(t)\,\mathrm{d}B_t + \gamma^n \int_{\mathbb{U}} g(t, x, u)\, N(\mathrm{d}t, \mathrm{d}u), \qquad t \in [0, T]. \tag{28}$$

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That is, (27) becomes an ordinary differential equation. Then a martingale solution of (27) is a measure on $C([0, T], \mathbb{R}^d)$ concentrated on integral curves of b [3, Lemma 3.8]. So, the following proposition presents the relationship between martingale solutions of (28) and of (27).

Proposition 4. *Suppose that b^n, b, σ^n, and $\{\gamma^n\}$ are uniformly bounded, g satisfies $\mathbf{H}_f$, and that $b \in L^1([0, T], BV_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d))$, $\partial_i b_i \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^d)$, and $(\partial_i b_i)^- \in L^1([0, T], L^\infty(\mathbb{R}^d))$. Let $\mu_0(\mathrm{d}x) = v_0(x)\,\mathrm{d}x \in \mathcal{P}_1(\mathbb{R}^d)$ with $\|v_0\|_\infty < \infty$, and $\mathbb{P}^n$, $\mathbb{P}$ be martingale solutions of (28) and (27) with the initial law μ_0, respectively. Assume that*

(i) $b^n \to b$, $a^n \to 0$ *in* $L^1_{\text{loc}}([0, T] \times \mathbb{R}^d)$, $\gamma^n \to 0$ *as* $n \to \infty$;

(ii) $\mathbb{P}^n \circ e_t^{-1}$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$ *and* $v_t^n(x)$ *denotes the density, i.e.* $v_t^n(x) := (\mathbb{P}^n \circ e_t^{-1})(\mathrm{d}x)/\mathrm{d}x$ *for any* $t \in [0, T]$, *and* $\sup_{x\in\mathbb{R}^d} |v_t^n(x)| \leqslant C_T$.

Then $\mathbb{P}^n \to \mathbb{P}$ *in* $\mathcal{P}(D_T)$.

Since the proof is similar to that of Proposition 3, we omit it.

Remark 6. This proposition means that SDEs with jumps can converge to ordinary differential equations in some sense. Moreover, if we take $\gamma^n = 0$, $b^n = b$, and

$$\sigma^n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{d\times m},$$

Proposition 4 is just right [3, Corollary 3.9].

If $\sigma^n = 0$ and b^n, b, g are independent of t, (28) and (27) fall into the framework in [11, 12]. Comparing Proposition 4 with [12, Theorem 2.11], we find that our conditions are weaker and then our result is weaker.

Acknowledgements

The author is very grateful to Professor Renming Song for valuable discussions. The author would also like to thank the anonymous referee for giving useful suggestions to improve this paper.

Funding information

This work was partly supported by NSF of China (Nos. 11001051, 11371352, 12071071) and the China Scholarship Council under Grant No. 201906095034.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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