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POSITIVE SOLUTIONS FOR A CLASS OF $p(x)$-LAPLACIAN PROBLEMS

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Abstract. We consider the system

$$\begin{cases} -\Delta_{p(x)}u = \lambda_1 f(v) + \mu_1 h(u) & \text{in } \Omega \\ -\Delta_{q(x)}v = \lambda_2 g(u) + \mu_2 \gamma(v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

where $p(x), q(x) \in C^1(R^N)$ are radial symmetric functions such that $\sup|\nabla p(x)| < \infty$, $\sup|\nabla q(x)| < \infty$ and $1 < \inf p(x) \le \sup p(x) < \infty$, $1 < \inf q(x) \le \sup q(x) < \infty$, where $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$, $-\Delta_{q(x)}v = -\text{div}(|\nabla v|^{q(x)-2}\nabla v)$, respectively are called $p(x)$-Laplacian and $q(x)$-Laplacian, λ_1, λ_2, μ_1 and μ_2 are positive parameters and $\Omega = B(0, R) \subset R^N$ is a bounded radial symmetric domain, where R is sufficiently large. We prove the existence of a positive solution when

$$\lim_{u\to+\infty} \frac{f(M(g(u))^{\frac{1}{q^- -1}})}{u^{p^- -1}} = 0,$$

for every $M > 0$, $\lim_{u\to+\infty} \frac{h(u)}{u^{p^- -1}} = 0$ and $\lim_{u\to+\infty} \frac{\gamma(u)}{u^{q^- -1}} = 0$. In particular, we do not assume any sign conditions on $f(0)$, $g(0)$, $h(0)$ or $\gamma(0)$.

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1. Introduction. The study of differential equations and variational problems with non-standard $p(x)$-growth conditions has been a new and interesting topic. Many results have been obtained on this kind of problem, for example, [**3**–**8**, **10**, **11**, **13**]. In [**6**, **7**] Fan and Zhao give the regularity of weak solutions for differential equations with non-standard $p(x)$-growth conditions. Zhang in [**12**] investigated the existence of positive solutions of the system

$$\begin{cases} -\Delta_{p(x)}u = f(v) & \text{in } \Omega \\ -\Delta_{q(x)}v = g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $p(x) \in C^1(R^N)$ is a function and $\Omega \subset R^N$ is a bounded domain. The operator $-\Delta_{p(x)}u = -\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Especially, if $p(x) \equiv p$ (a constant), (1) is the well-known p-Laplacian systems. There are many papers on the existence of solutions for p-Laplacian elliptic systems, for example, [**1–9**].

In [**9**] the authors consider the existence of positive weak solutions for the following p-Laplacian problems:

$$\begin{cases} -\Delta_p u = f(v) & \text{in } \Omega \\ -\Delta_p v = g(u) & \text{in } \Omega. \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

The first eigenfunction is used for constructing the subsolution of p-Laplacian problems successfully. On the condition of

$$\lim_{u \to +\infty} \frac{f(M(g(u))^{\frac{1}{p-1}})}{u^{p-1}} = 0, \quad \forall M > 0,$$

the authors show the existence of positive solutions for problem (2).

In this paper, we mainly consider the existence of positive solutions of the system

$$\begin{cases} -\Delta_{p(x)} u = \lambda_1 f(v) + \mu_1 h(u) & \text{in } \Omega \\ -\Delta_{q(x)} v = \lambda_2 g(u) + \mu_2 \gamma(v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where $p(x), q(x) \in C^1(R^N)$ are functions, λ_1, λ_2, μ_1 and μ_2 are positive parameters and $\Omega \subset R^N$ is a bounded domain.

In order to deal with $p(x)$-Laplacian problems, we need some theories on spaces $L^{p(x)}(\Omega)$, and $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [**8**]). If $\Omega \subset R^N$ is an open domain, then

$$C_+(\Omega) = \{h \mid h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega\},$$
$$h^+ = \sup_{x \in \Omega} h(x), \quad h^- = \inf_{x \in \Omega} h(x), \qquad \text{for any } h \in C(\Omega),$$
$$L^{p(x)}(\Omega) = \{u|u \text{ is a measurable real-valued function}, \textstyle\int_\Omega |u|^{p(x)}dx < \infty\}.$$

Throughout the paper, we will assume that $p, q \in C_+(\Omega)$ and

$$1 < \inf_{x \in R^N} p(x) \le \sup_{x \in R^N} p(x) < N,$$
$$1 < \inf_{x \in R^N} q(x) \le \sup_{x \in R^N} q(x) < N.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \middle| \textstyle\int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\},$$

and $(L^{p(x)}(\Omega), |.|_{p(x)})$ becomes a Banach space, which we call generalised Lebesgue space.

The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a separable, reflexive and uniformly convex Banach space (see [**8**, Theorem 1.10, 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \middle| |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces (see [**8**, Theorem 2.1]). We define that if

$$(L(u), v) = \int_{R^N} |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx, \qquad \forall u, v \in W^{1,p(x)}(\Omega),$$

then $L : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ is a continuous, bounded and strictly monotone operator and is also a homeomorphism (see [**4**, Theorem 3.11]). If $u, v \in W_0^{1,p(x)}(\Omega)$, (u, v) is called a weak solution of (3) which satisfies

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla \xi\, dx = \int_\Omega (\lambda_1 f(v) + \mu_1 h(u))\xi\, dx, \qquad \forall \xi \in W_0^{1,p(x)}(\Omega),$$
$$\int_\Omega |\nabla v|^{q(x)-2}\nabla v \nabla \xi\, dx = \int_\Omega (\lambda_2 g(u) + \mu_2 \gamma(v))\xi\, dx, \qquad \forall \xi \in W_0^{1,p(x)}(\Omega).$$

We make the following assumptions

(H.1) $p(x), q(x) \in C^1(R^N)$ are radial symmetric and $\sup|\nabla p(x)| < \infty$, $\sup|\nabla q(x)| < \infty$.
(H.2) $\Omega = B(0, R) = \{x||x| < R\}$ is a ball, where $R > 0$ is a sufficiently large constant.
(H.3) $f, g, h, \gamma : [0, \infty) \to R$ are C^1, monotone functions such that

$$\lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} g(u) = \lim_{u\to+\infty} h(u) = \lim_{u\to+\infty} \gamma(u) = +\infty.$$

(H.4) $\lim_{u\to+\infty} \frac{f(M(g(u))^{\frac{1}{q^- -1}})}{u^{p^- -1}} = 0$, for every $M > 0$.

(H.5) $\lim_{u\to+\infty} \frac{h(u)}{u^{p^- -1}} = \lim_{u\to+\infty} \frac{\gamma(u)}{u^{q^- -1}} = 0$.

We shall establish the following theorem.

2. Main results.

THEOREM 1. *If (H.1)–(H.5) hold, then* (3) *has a positive solution.*

Proof. We shall establish Theorem 1 by constructing a positive subsolution (ϕ_1, ϕ_2) and supersolution (z_1, z_2) of (3), such that $\phi_1 \le z_1$ and $\phi_2 \le z_2$. That is, (ϕ_1, ϕ_2) and (z_1, z_2) satisfy

$$\int_\Omega |\nabla \phi_1|^{p(x)-2}\nabla\phi_1 \cdot \nabla\xi\, dx \le \lambda_1 \int_\Omega f(\phi_2)\,\xi\, dx + \mu_1 \int_\Omega h(\phi_1)\,\xi\, dx,$$
$$\int_\Omega |\nabla \phi_2|^{q(x)-2}\nabla\phi_1 \cdot \nabla\xi\, dx \le \lambda_2 \int_\Omega g(\phi_1)\,\xi\, dx + \mu_2 \int_\Omega \gamma(\phi_2)\,\xi\, dx,$$

and

$$\int_\Omega |\nabla \phi_1|^{p(x)-2}\nabla\phi_1 \cdot \nabla\xi\, dx \ge \lambda_1 \int_\Omega f(\phi_2)\,\xi\, dx + \mu_1 \int_\Omega h(\phi_1)\,\xi\, dx,$$
$$\int_\Omega |\nabla \phi_2|^{q(x)-2}\nabla\phi_1 \cdot \nabla\xi\, dx \ge \lambda_2 \int_\Omega g(\phi_1)\,\xi\, dx + \mu_2 \int_\Omega \gamma(\phi_2)\,\xi\, dx,$$

for all $\xi \in W_0^{1,p(x)}(\Omega)$ with $\xi \ge 0$. Then (3) has a positive solution.

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Step 1. We construct a subsolution of (3).
Denote

$$a_1 = \frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}, \quad R_1 = \frac{R - a_1}{2},$$
$$a_2 = \frac{\inf q(x) - 1}{4(\sup|\nabla q(x)| + 1)}, \quad R_2 = \frac{R - a_2}{2},$$

and let $k_0 > 0$ such that $f(u), g(u), h(u), \gamma(u) \geq -k_0$ for all $u \geq 0$, and let

$$\phi_1(r) = \begin{cases} e^{-k(r-R)} - 1, \quad 2R_1 < r \leq R, \\ e^{a_1 k} - 1 + \int_r^{2R_1} (ke^{a_1 k})^{\frac{p(2R_1)-1}{p(r)-1}} \left[\frac{(2R_1)^{N-1}}{r^{N-1}} \sin\left(\varepsilon_1\left(r - 2R_1\right) + \frac{\pi}{2}\right) k_0(\lambda_1 + \mu_1)\right]^{\frac{1}{p(r)-1}} dr, \\ 2R_1 - \frac{\pi}{2\varepsilon_1} < r \leq 2R_1, \\ e^{a_1 k} - 1 + \int_{2R_1 - \frac{\pi}{2\varepsilon_1}}^{2R_1} (ke^{a_1 k})^{\frac{p(2R_1)-1}{p(r)-1}} \left[\frac{(2R_1)^{N-1}}{r^{N-1}} \sin\left(\varepsilon_1(r - 2R_1) + \frac{\pi}{2}\right) k_0(\lambda_1 + \mu_1)\right] \\ ^{\frac{1}{p(r)-1}} dr, \, r \leq 2R_1 - \frac{\pi}{2\varepsilon_1}, \end{cases}$$

where R_1 is sufficiently large and ε_1 is a small positive constant which satisfies

$$R_1 \leq 2R_1 - \frac{\pi}{2\varepsilon_1},$$

and let

$$\phi_2(r) = \begin{cases} e^{-k(r-R)} - 1, \quad 2R_2 < r \leq R, \\ e^{a_2 k} - 1 + \int_r^{2R_2} (ke^{a_2 k})^{\frac{q(2R_2)-1}{q(r)-1}} \left[\frac{(2R_2)^{N-1}}{r^{N-1}} \sin\left(\varepsilon_2(r - 2R_2) + \frac{\pi}{2}\right) k_0(\lambda_2 + \mu_2)\right]^{\frac{1}{q(r)-1}} dr, \\ 2R_2 - \frac{\pi}{2\varepsilon_2} < r \leq 2R_2, \\ e^{a_2 k} - 1 + \int_{2R_2 - \frac{\pi}{2\varepsilon_2}}^{2R_2} (ke^{a_2 k})^{\frac{q(2R_2)-1}{q(r)-1}} \left[\frac{(2R_2)^{N-1}}{r^{N-1}} \sin\left(\varepsilon_2(r - 2R_2) + \frac{\pi}{2}\right) k_0(\lambda_2 + \mu_2)\right] \\ ^{\frac{1}{q(r)-1}} dr, \, r \leq 2R_2 - \frac{\pi}{2\varepsilon_2}, \end{cases}$$

where R_2 is sufficiently large and ε_2 is a small positive constant which satisfies

$$R_2 \leq 2R_2 - \frac{\pi}{2\varepsilon_2}.$$

In the following, we will prove that (ϕ_1, ϕ_2) is a subsolution of (3).
Since

$$\phi_1'(r) = \begin{cases} e^{-k(r-R)} - 1, \quad 2R_1 < r \leq R, \\ -(ke^{a_1 k})^{\frac{p(2R_1)-1}{p(r)-1}} \left[\frac{(2R_1)^{N-1}}{r^{N-1}} \sin\left(\varepsilon_1(r - 2R_1) + \frac{\pi}{2}\right) k_0(\lambda_1 + \mu_1)\right]^{\frac{1}{p(r)-1}} dr, \\ 2R_1 - \frac{\pi}{2\varepsilon_1} < r \leq 2R_1, \\ 0, \quad 0 \leq r \leq 2R_1 - \frac{\pi}{2\varepsilon_1}, \end{cases}$$

and

$$\phi_2'(r) = \begin{cases} e^{-k(r-R)} - 1, \qquad 2R_2 < r \leq R, \\ -(ke^{a_2 k})^{\frac{q(2R_2)-1}{q(r)-1}} \left[\frac{(2R_2)^{N-1}}{r^{N-1}} \sin\left(\varepsilon_2(r - 2R_2) + \frac{\pi}{2}\right) k_0(\lambda_2 + \mu_2)\right]^{\frac{1}{q(r)-1}} dr, \\ 2R_2 - \frac{\pi}{2\varepsilon_2} < r \leq 2R_2, \\ 0, \quad 0 \leq r \leq 2R_2 - \frac{\pi}{2\varepsilon_2}, \end{cases}$$

it is easy to see that $\phi_1, \phi_2 \geq 0$ are decreasing and $\phi_1, \phi_2 \in C^1([0, R])$, $\phi_1(x) = \phi_1(|x|) \in C^1(\bar{\Omega})$ and $\phi_2(x) = \phi_2(|x|) \in C^1(\bar{\Omega})$.

Let $r = |x|$. By computation,

$$-\Delta_{p(x)}\phi_1 = -\mathrm{div}(|\nabla\phi_1(x)|^{p(x)-2}\nabla\phi_1(x)) = -(r^{N-1}|\phi_1'(r)|^{p(r)-2}\phi_1'(r))'/r^{N-1},$$

so then

$$-\Delta_{p(x)}\phi_1 = \begin{cases} \left(ke^{-k(r-R)}\right)^{p(r)-1}\left[-k(p(r)-1) + p'(r)\ln k - kp'(r)(r-R) + \frac{N-1}{r}\right], \\ 2R_1 < r \leq R, \\ \varepsilon_1\left(\frac{2R_1}{r}\right)^{N-1}(ke^{a_1k})^{p(2R_1)-1}\cos\left(\varepsilon_1(r-2R_1) + \frac{\pi}{2}\right)(\lambda_1+\mu_1), \\ 2R_1 - \frac{\pi}{2\varepsilon_1} < r \leq 2R_1, 0, \quad 0 \leq r \leq 2R_1 - \frac{\pi}{2\varepsilon_1}. \end{cases}$$

If k is sufficiently large, when $2R_1 < r \leq R$, then we have

$$-\Delta_{p(x)}\phi_1 \leq -k\left[\inf p(x) - 1 - \sup|\nabla p(x)|\left(\frac{\ln k}{k} + R - r\right) + \frac{N-1}{kr}\right] \leq -ka_1.$$

As a_1 is a constant dependent only on $p(x)$, if k is big enough, such that

$$-ka_1 < -(\lambda_1+\mu_1)k_0,$$

then we have

$$-\Delta_{p(x)}\phi_1 \leq -(\lambda_1+\mu_1)k_0 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \quad 2R_1 < |x| \leq R. \tag{4}$$

If k is sufficiently large, then

$$f(e^{a_2k}-1) \geq 1,\ h(e^{a_1k}-1) \geq 1,\ g(e^{a_1k}-1) \geq 1,\ \gamma(e^{a_2k}-1) \geq 1,$$

where k is dependent on $f,\ h,\ g,\ \gamma$ and $p,\ q$ and independent on R. Since

$$\begin{aligned} -\Delta_{p(x)}\phi_1 &= \varepsilon_1\left(\frac{2R_1}{r}\right)^{N-1}(ke^{a_1k})^{p(2R_1)-1}\cos\left(\varepsilon_1(r-2R_1)+\frac{\pi}{2}\right)(\lambda_1+\mu_1) \\ &\leq \varepsilon_1(\lambda_1+\mu_1)2^N k^{p^+}e^{a_1kp^+},\ \ 2R_1 - \frac{\pi}{2\varepsilon_1} < |x| \leq 2R_1, \end{aligned}$$

let

$$\varepsilon_1 = 2^{-N}k^{-p^+}e^{-a_1kp^+}.$$

Then we have

$$-\Delta_{p(x)}\phi_1 \leq \lambda_1+\mu_1 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \qquad 2R_1 - \frac{\pi}{2\varepsilon_1} < |x| \leq 2R_1. \tag{5}$$

Obviously,

$$-\Delta_{p(x)}\phi_1 = 0 \leq \lambda_1+\mu_1 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \quad |x| \leq 2R_1 - \frac{\pi}{2\varepsilon_1}. \tag{6}$$

Since $\phi_1(x) \in C^1(\Omega)$, combining (4), (5) and (6), we have

$$-\Delta_{p(x)}\phi_1 \leq \lambda_1 f(\phi_2) + \mu_1 h(\phi_1), \qquad \text{for} \quad \text{a.e. } x \in \Omega.$$

Similarly we have

$$-\Delta_{q(x)}\phi_2 \le \lambda_2 g(\phi_1) + \mu_2\gamma(\phi_2), \qquad \text{for} \quad \text{a.e. } x \in \Omega.$$

Since $\phi_1(x),\ \phi_2(x) \in C^1(\bar{\Omega})$, it is easy to see that (ϕ_1, ϕ_2) is a subsolution of (3).
Step 2. We construct a supersolution of (3).
Let z_1 be a radial solution of

$$-\Delta_{p(x)}z_1(x) = (\lambda_1 + \mu_1)\mu, \quad \text{in} \quad \Omega, \qquad z_1 = 0 \quad \text{on} \quad \partial\Omega.$$

We denote that if $z_1 = z_1(r) = z_1(|x|)$, then z_1 satisfies

$$-(r^{N-1}|z_1'|^{p(r)-2}z_1')' = r^{N-1}(\lambda_1 + \mu_1)\mu, \quad z_1(R) = 0,\ z_1'(0) = 0,$$

and so

$$z_1' = -\left|\frac{r(\lambda_1 + \mu_1)\mu}{N}\right|^{\frac{1}{p(r)-1}} \tag{7}$$

and

$$z_1 = \int_r^R \left|\frac{r(\lambda_1 + \mu_1)\mu}{N}\right|^{\frac{1}{p(r)-1}} dr.$$

We denote that if $\beta = \beta((\lambda_1 + \mu_1)\mu) = \max_{0\le r\le R} z_1(r)$, then

$$\beta((\lambda_1 + \mu_1)\mu) = \int_0^R \left|\frac{r(\lambda_1 + \mu_1)\mu}{N}\right|^{\frac{1}{p(r)-1}} dr = ((\lambda_1 + \mu_1)\mu)^{\frac{1}{p(t)-1}} \int_0^R \left|\frac{r}{N}\right|^{\frac{1}{p(r)-1}} dr,$$

where $t \in [0, 1]$. Since $\int_0^R |\frac{r}{N}|^{\frac{1}{p(r)-1}} dr$ is a constant, then there exists a positive constant $C \ge 1$ such that

$$\frac{1}{C}((\lambda_1 + \mu_1)\mu)^{\frac{1}{p^+-1}} \le \beta((\lambda_1 + \mu_1)\mu) = \max_{0\le r\le R} z_1(r) \le C((\lambda_1 + \mu_1)\mu)^{\frac{1}{p^--1}}. \tag{8}$$

We consider

$$\begin{cases} -\Delta_{p(x)}z_1 = (\lambda_1 + \mu_1)\mu & \text{in } \Omega \\ -\Delta_{q(x)}z_2 = (\lambda_2 + \mu_2)g(\beta((\lambda_1 + \mu_1)\mu)) & \text{in } \Omega, \\ z_1 = z_2 = 0 & \text{on } \partial\Omega \end{cases}$$

and then we shall prove that (z_1, z_2) is a supersolution for (3).

For $\xi \in W^{1,p(x)}(\Omega)$ with $\xi \ge 0$ it is easy to see that

$$\begin{aligned} \int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla\xi \, dx &= \int_\Omega (\lambda_2 + \mu_2)g(\beta((\lambda_1 + \mu_1)\mu))\xi \, dx \\ &\ge \int_\Omega \lambda_2 g(z_1)\xi dx + \int_\Omega \mu_2 g(\beta((\lambda_1 + \mu_1)\mu))\xi \, dx. \end{aligned}$$

By (H.5) for μ large enough, we have

$$g(\beta((\lambda_1 + \mu_1)\mu)) \ge \gamma\left([(\lambda_2 + \mu_2)(g(\beta((\lambda_1 + \mu_1)\mu)))]^{\frac{1}{q^--1}}\right) \ge \gamma(z_2).$$

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Hence

$$\int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla \xi \, dx \geq \int_\Omega \lambda_2 g(z_1)\xi \, dx + \int_\Omega \mu_2 \gamma(z_2)\xi \, dx. \tag{9}$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla \xi \, dx = \int_\Omega (\lambda_1 + \mu_1)\mu \xi \, dx.$$

Similar to (8), we have

$$\max_{0\leq r\leq R} z_2(r) \leq C[(\lambda_2 + \mu_2)g(\beta((\lambda_1 + \mu_1)\mu))]^{\frac{1}{(q^- -1)}}.$$

By (H.4) and (H.5), when μ is sufficiently large, according to (8), we have

$$\begin{aligned}(\lambda_1 + \mu_1)\mu &\geq \left[\frac{1}{C}\beta((\lambda_1 + \mu_1)\mu)\right]^{p^- -1} \\ &\geq \lambda_1 f\left[C([(\lambda_2 + \mu_2)(g(\beta((\lambda_1 + \mu_1)\mu)))]^{\frac{1}{(q^- -1)}}\right] + \mu_1 h(\beta((\lambda_1 + \mu_1)\mu) \\ &\geq \lambda_1 f(z_2) + \mu_1 h(z_1),\end{aligned}$$

and so

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla \xi \, dx \geq \int_\Omega \lambda_1 f(z_2)\xi \, dx + \int_\Omega \mu_1 h(z_1)\xi \, dx \tag{10}$$

According to (9) and (10), we can conclude that (z_1, z_2) is a supersolution of (3). Let μ be sufficiently large; then from (7) and the definition of (ϕ_1, ϕ_2), it is easy to see that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$. This completes the proof. □

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