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MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

A meeting of the Association for Symbolic Logic was held at the University of California, Los Angeles, on Friday and Saturday, the 22nd and 23rd of March, 1968, in conjunction with a meeting of the American Mathematical Society. Two invited addresses were presented, one by Professor Donald A. Martin entitled *Large cardinals and infinite games* and one by Professor Jack Silver entitled *Some consistency results in model theory*. In addition, eleven papers were delivered, and six were presented by title; the last six abstracts below are those presented by title.

YIANNIS N. MOSCHOVAKIS
PROGRAM CHAIRMAN

PETER ACZEL. *The lattice of recursive density types.*

If $n \in \omega = \{0, 1, 2, \cdots\}$ and $\alpha \subseteq \omega$, let $\alpha[n] = \text{card}(\{i \in \alpha \mid i < n\})$. If $\alpha, \beta \subseteq \omega$ let $\alpha \preccurlyeq \beta$ if there is a recursive function f such that $\forall n \alpha[n] \leq \beta[f(n)]$. Let $D(\alpha) = \{\beta \subseteq \omega \mid \alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha\}$ and $\Delta = \{D(\alpha) \mid \alpha \subseteq \omega\}$. The set Δ of recursive density types was first introduced by Medvedev and has been investigated by Pavlova and more recently by Rice and Gonshor. The relation $\preccurlyeq$ induces a partial ordering $\leq$ on Δ such that $\langle \Delta, \leq \rangle$ is a distributive lattice with last element $\infty = D(\omega)$ and has (an isomorphic copy of) $\langle \omega, \leq \rangle$ as initial segment.

Using Rice's characterisation of the hyperimmune sets, α is hyperimmune iff $D(\alpha) \in \Delta - \omega \cup \{\infty\}$. Let an interval be a set $[A, B] = \{C \mid A \leq C \leq B\}$ where $A < B$ and $A \in \Delta, B \in \Delta - \omega$. By a density theorem of Gonshor and Rice the Boolean algebra of sets of integers modulo finite sets is lattice embeddable in every interval, so that every interval has cardinality $2^{\aleph_0}$ and (using a result of Keisler) assuming the continuum hypothesis every interval is a universal distributive lattice. This result suggests that there may be a model theoretic characterisation of $\langle \Delta, \leq \rangle$. Although it does not appear to be a universal homogeneous structure of any kind it does have many of the properties of such structures. Let $\langle S, P_1, P_2 \rangle$ be a *cut* if $P_1 \cup P_2 \subseteq S \subseteq \Delta - \omega$ and for all $A, B \in S$, (i) $A \cup B \in P_1 \leftrightarrow A, B \in P_1$, (ii) $A \cap B \in P_2 \leftrightarrow A, B \in P_2$, (iii) $A \in P_1$ and $B \in P_2$ implies $A < B$. The recursive density type A satisfies the cut $\langle S, P_1, P_2 \rangle$ if $P_1 = \{B \in S \mid B < A\}$ and $P_2 = \{B \in S \mid A < B\}$.

THEOREM. *For every countable cut there is an interval such that every element of the interval satisfies the cut.* (*Received February 2, 1968.*)

AKIRA NAKAMURA. *On an undecidable Post canonical language.*

The purposes of this paper are [first] to present a rather simple Post canonical language whose decision problem is recursively unsolvable, and by making use of the system [second] to show a nonaxiomatizable propositional calculus.

We consider the following canonical language L:

The alphabet: $\{1, 0, \Delta, *, \#, W, O, \vdash\}$.
The axioms: W_1, O_0.
The productions:
P1. $W\alpha \rightarrow W\alpha\Delta$.
P2. $O\alpha \rightarrow O\alpha\Delta$.
P3. $W\alpha, O\beta\Delta \rightarrow W \# \beta\Delta \# \alpha$.
P4. $W\alpha, W\beta \rightarrow W0\alpha\beta$.
P5. $W\alpha, O\beta \rightarrow W * \beta * \alpha$.
P6. $W\alpha, W\beta, W\gamma, W\delta, W\epsilon \rightarrow \vdash 00000\alpha\beta 0 \# 0\Delta \# \gamma \# 0\Delta \# \delta\gamma\epsilon 00\epsilon\alpha 0\delta\alpha$.
P7. $W\alpha, W\beta, O\gamma \rightarrow \vdash 0 * \gamma * 0\alpha\beta 0 * \gamma * \alpha * \gamma * \beta$.
P8. $W\alpha, O\beta \rightarrow \vdash 0 * \beta * \alpha\alpha$.
P9. $\vdash\alpha, \vdash 0\alpha\beta \rightarrow \vdash\beta$.
P10. $\vdash\alpha, O\beta \rightarrow \vdash * \beta * \alpha$.

[I] Then, it is shown that there is no finite, effective procedure to decide whether or not a given expression is a theorem in L.

The central idea of proof is, roughly speaking, as follows:

(1) We interpret α of $W\alpha$ in the language L as an expression obtained by modifying the definition of wff in the first-order predicate calculus F.

(2) According to the interpretation (1), we construct a propositional logic P such that $W\alpha$, $\vdash\alpha$ mean a wff, a provable wff in P respectively.

(3) Further, we consider a correspondence between wff's in F and wff's in P. Let $\tilde{\omega}$ be a finite set of wff's in P corresponding to a wff ω in F. Then, we show that ω is provable in F iff at least one wff in $\tilde{\omega}$ is provable in P.

(4) Thus, we get the undecidability of language L from that of the first-order predicate calculus F.

[II] Since the axiomatic system P is a kind of propositional logic, it follows from the undecidability of P that the set of refutable wff's in P is not recursively enumerable. Thus, we have a nonaxiomatizable propositional calculus. (*Received January 31, 1968.*)

JURGEN SCHMIDT. *Remark on ordered pairs in the theory of classes.*

The notion of ordered pair is usually restricted to those well-behaving classes called sets. But within a theory of classes admitting proper classes that are not sets (which is convenient, e.g., for categories), one often misses a more extended notion of ordered pairs working for arbitrary classes (be they proper or not). E.g., one might define a category as an ordered pair $(M, \cdot)$, where M is a class (the class of all "morphisms"), $\cdot$ some partial binary operation in class M; thus, a category would become an object of set theory respectively of class theory itself, even in the case when class M is proper. In the same manner, one could define partially ordered classes (which in fact may be considered as special categories as defined above) or other classes—proper or not—with certain types of structures. E.g., (On, $\leq$), where On is the class of all ordinal numbers, $\leq$ its natural ordering, would be a well-ordered class and as such an object of set-theory.

The ordered pair for arbitrary classes should be a term (p, q) in two class variables such that the following *basic properties of pairs hold*:

C1. (p, q) is a set if and only if p and q are sets;

C2. $(p, q) = (r, s)$ if and only if $p = q$ and $r = s$.

Here a set is defined as a class x such that x is a member of some class y, $x \in y$ for some y. For most definitions of the ordered pair given so far (e.g., the well-known definition of Kuratowski), C2 only holds for sets. We are going to observe the following two simple (but not so well-known) facts:

1. One can define a term (p, q) such that C1 and the unrestricted C2 (for arbitrary classes p, q, r and s) hold;

2. class theory, in particular that part concerned with relations and functions, is essentially independent of the actual definition of ordered pairs (even if restricted to sets). (*Received January 30, 1968.*)

HENRY H. CRAPO and DON D. ROBERTS. *Peirce algebras and the distributivity scandal.*

A re-reading of C. S. Peirce's paper ***On the algebra of logic*** (1880) brings to light a number of matters of historical, logical, and mathematical interest.

A *Peirce algebra* (*type* I) is an algebra A with a binary operator $\prec$ and a logic T (or set of "valid" formulas) containing

A1. $x \prec x$

A2. $(x \prec (y \prec z)) \prec (y \prec (x \prec z))$

A3. $(y \prec z) \prec ((x \prec y) \prec (x \prec z))$

and closed with respect to the rules

R1. Detachment: From $x \in T$ and $(x \prec y) \in T$, and to infer $y \in T$.

R2. Uniform substitution.

A Peirce algebra (*type* II) is an algebra A of type I, to which there is added a nullary operator 0, in terms of which negation is defined

DN. $\bar{x} = x \prec 0$

such that the following are "valid"

A4. $\bar{\bar{x}} \prec x$
A5. $0 \prec x$

and in which there exist, for any elements x, y in the algebra, elements $x + y$ and xy satisfying

DA. $(x + y) \prec z$ if and only if $x \prec z$ and $y \prec z$
DM. $z \prec (xy)$ if and only if $z \prec x$ and $z \prec y$.

The scandal, that "Peirce thought that all lattices were distributive," is traced in the correspondence between Peirce, Schröder, and Huntington, and culminates in a footnote to Birkhoff's ***Lattice theory*** (1948). We put the scandal to rest by verifying Peirce's proof of the distributive principle, as recorded in his logic notebook.

In his recognition of the importance of the *unrestricted modular condition*

$$((x + y)z) \prec (x + (yz))$$

and in his intuition concerning the significance of *individual* elements (elements m for which $m \prec (x + y)$ implies $m \prec x$ or $m \prec y$), Peirce anticipates several important lattice-theoretic developments of the nineteen thirties and fifties. For example, Peirce's proof of distributivity also establishes Raney's theorem (1952): If, in a lattice L, every element is expressible as the supremum of some set of individual elements, then the lattice L is distributive.

The Peirce algebra of type II constitutes a multi-valued logic, resembling intuitionist logic in many respects, but agreeing with it only in the trivial case of the classical propositional calculus.

The interval $[0, 1] = \{x : 0 \leq x \leq 1\}$ of the real numbers provides a model of a Peirce algebra of type II. Let

$$x \prec y = \min\{1, 1 - x + y\}.$$

This example contains subalgebras of all finite and countably infinite cardinalities. (*Received January 29, 1968.*)

ROBERT A. DIPAOLA. *Church random sets in subrecursive hierarchies.*

Let C be a subclass of the recursive functions. By a C-random set we mean the set which is defined by replacing the class of recursive functions in the definition of random sequence (and hence random set) given by Church (*On the concept of a random sequence*, ***Bulletin of the American Mathematical Society***, vol. 46 (1940), pp. 130–135) with the class of one-place functions in C.

Let C_1, C_2 be subclasses of the class of recursive functions such that (1) $E \subseteq C_1 \subset C_2$, where E is the class of Kalmar elementary functions; (2) C_1 and C_2 are closed under composition and limited recursion; (3) C_2 contains a binary function f which is universal for the unary functions g in C_1.

THEOREM 1. *C_2 contains a C_1-random set.* COROLLARY 1. *For each $n > 2$, the classes $\mathscr{E}^{n+1}$ of the Grzegorczyk hierarchy of primitive recursive functions contain an $\mathscr{E}^n$-random set.* COROLLARY 2. *For each $x \in \mathcal{O}$, the class C_{2^x} of the Kleene subrecursive hierarchy contains a C_x-random set.* With respect to more restricted subrecursive hierarchies, the results are more special: THEOREM 2. *For each $i > 0$, the classes F_{i+4} of the Ritchie hierarchy of elementary functions contain an F_i-random set.* (*Received January 22, 1968.*)

S. K. THOMASON. *More initial segments of hyperdegrees.*

(I) Every sublattice of the lattice of finite sets of natural numbers is isomorphic to an initial segment of hyperdegrees. (II) The lattice of all finite subsets of the continuum is isomorphic to an initial segment of hyperdegrees, and to an initial segment of degrees. These results extend that of the author's *Some initial segments of hyperdegrees*, presented to the American Mathematical Society January 25, 1968. (*Received January 22, 1968.*)

HUGHES LEBLANC. *Three generalizations of a theorem of Beth's.*

Where A is a wff of F^1 (the first-order functional calculus without identity), and X and Y are not necessarily distinct individual variables of F^1, (i) take $A[Y/X]$ to be the result of substituting Y for every free occurrence of X in A if no free occurrence of X in A is in a component of A of the sort $(\forall Y)B$, (ii) otherwise, take $A[Y/X]$ to be $A'[Y/X]$, where A' is the result of substituting—for every occurrence of Y in every component of A of the sort $(\forall Y)B$ that contains a free occurrence of X in A—an occurrence of the alphabetically earliest individual variable of F^1 that is foreign to that component of A. And, where Asst is an assignment of truth-values to the atomic wffs of F^1, take a wff of F^1 of the sort $(\forall X)A$ to be satisfied by Asst if and only if $A[Y/X]$ is satisfied by Asst for every individual variable Y of F^1. Availing himself of like-minded conventions, Beth showed in ***The foundations of mathematics*** (1959), Section 89, that a set S of *closed wffs* of F^1 is consistent as to provability if and only if there is an assignment of truth-values to the atomic wffs of F^1 that satisfies (each and every) member of S.

Beth's result, which exploits a familiar result of Henkin's, can be generalized in three directions.

(1) Supposing given a suitable characterization of isomorphism between sets of wffs of F^1, it can be shown that *a set S of wffs of F^1 is consistent as to provability if and only if there is an assignment of truth-values to the atomic wffs of F^1 that satisfies some set of wffs of F^1 isomorphic to S.*

(2) With $A[G/F]$ defined along the same lines as $A[Y/X]$, A this time a wff of F^2 (the second-order functional calculus without identity), and F and G not necessarily distinct predicate variables of F^2 of the same degree, with $(\forall F)A$ understood to be satisfied by an assignment Asst of truth-values to the atomic wffs of F^2 if and only if $A[G/F]$ is satisfied by Asst for every predicate variable G of F^2 of the same degree as F, and with a set S of wffs of F^2 taken to be consistent* as to probability if at least one wff of F^2 is not provable from S in Henkin's fragment F^* of F^2 (see this JOURNAL, vol. 18 (1953), pp. 201–208), it can be shown—given a suitable characterization of isomorphism between sets of wffs of F^2—that *a set S of wffs of F^2 is consistent* as to provability if and only if there is an assignment of truth-values to the atomic wffs of F^2 that satisfies some set of wffs of F^2 isomorphic to S.*

(3) With $A[B/F(X_1, X_2, \cdots, X_m)]$ defined essentially like $\check{S}_B^{F(X_1,X_2,\cdots,X_m)}A|$ in Church's ***Introduction to mathematical logic***, Volume I (1956), p. 192, count a function $T \vee$ from the set of the wffs of F^2 to $\{T, F\}$ as a general truth-value function for F^2 if: (i) $T \vee (\sim A) = T$ if and only if $T \vee (A) = F$, (ii) $T \vee (A \supset B) = T$ if and only if $T \vee (A) = F$ or $T \vee (B) = T$, (iii) $T \vee ((\forall X)A) = T$ if and only if $T \vee (A[Y/X]) = T$ for every individual variable Y of F^2, and (iv) $T \vee ((\forall F)A) = T$ if and only if $T \vee (A[B/F(X_1, X_2, \cdots, X_m)]) = T$ for every wff B of F^2 and every m (m the degree of F) distinct individual variables $X_1, X_2, \cdots$, and X_m of F^2. And take a set S of wffs of F^2 to be consistent** as to provability if at least one wff of F^2 is not provable from S in Henkin's version F^{**} of F^2. It can be shown—given a suitable characterization of isomorphism between sets of wffs of F^2—that *a set S of wffs of F^2 is consistent** as to provability if and if there is a general truth-value function for F^2 that satisfies some set of wffs of F^2 isomorphic to S.* *(Received January 19, 1968; corrected June 19, 1968.)*

YIANNIS N. MOSCHOVAKIS. *The lack of hierarchies on the second projective class (preliminary report).*

A subset of ${}^{\omega}\omega$ is $\Sigma_k^1(\Pi_k^1)$ if and only if it is explicitly definable by a formula with $k + 1$ function quantifiers applied to a recursive matrix with outer quantifier existential (universal); $\Delta_k^1 = \Sigma_k^1 \cap \Pi_k^1$ and boldface $\boldsymbol{\Sigma}, \boldsymbol{\Pi}, \boldsymbol{\Delta}$ indicate that the matrix can be recursive in an arbitrary element of ${}^{\omega}\omega$.

It is known that $\boldsymbol{\Delta}_1^1$ = Borel subsets of ${}^{\omega}\omega$ and this puts a *hierarchy* on $\boldsymbol{\Delta}_1^1$. Similar hierarchies have been sought unsuccessfully for the classes $\boldsymbol{\Delta}_k^1$ $(k > 1)$; we give here a precise definition of a hierarchy and we show that there are none on $\boldsymbol{\Delta}_2^1$.

Sets in $\boldsymbol{\Delta}_k^1$ are naturally *coded* by elements of ${}^{\omega}\omega$. There are also natural *reducibilities* on $\boldsymbol{\Delta}_k^1$, e.g.

(1) $A \leq (\alpha)B \Leftrightarrow A = f^{-1}[B]$, *where f is the continuous function with code* α,

(2) $A \leq (\alpha)B \Leftrightarrow$ *A is recursive in* α, **R**, *B with index* $\alpha(0)$ *in the sense of recursive functionals of type-2, where* **R** *is some fixed set in* Δ_k^1.

DEFINITION. A *trail* through Δ_k^1 (relative to a reducibility $\leq$ defined by (1) or (2)) is a sequence $\langle G, o, \chi_1, \cdots, \chi_4\rangle$ where $G \subseteq {}^\omega\omega$, o maps G onto some ordinal κ, $\chi_1, \cdots, \chi_4$ are functions in Δ_k^1 and:

H1. For each $\alpha \in G$, $\chi_1(\alpha)$ is a Δ_k^1-code for some set, call it G_α.

H2. If α is a Δ_k^1-code for some set A, then $\chi_2(\alpha) \in G$ and $A \leq (\chi_3(\alpha))G_{\chi_2(\alpha)}$.

H3. For each $\alpha \in G$, $\{\beta : \beta \in G \;\&\; o(\beta) \leq o(\alpha)\}$ is Δ_k^1 with code $\chi_4(\alpha)$.

A trail is *ascending* and defines a *canonical hierarchy* if for some function χ_5 in Δ_k^1,

H4. If $\alpha, \beta \in G$ and $o(\alpha) \leq o(\beta)$, then $G_\alpha \leq (\chi_5(\alpha, \beta))G_\beta$.

H5. If $\alpha, \beta \in G$ and $o(\alpha) < o(\beta)$, then for each γ, $\neg[G_\beta \leq (\gamma)G_\alpha]$.

THEOREM 1. *There is no trail through* Δ_2^1.

THEOREM 2. *If every set is constructible in the sense of Gödel, then there is no trail through* Δ_k^1, *for every* $k \geq 3$.

THEOREM 3. *If every projective subset of* ${}^\omega\omega$ *is determinate, then there are ascending trails through* Δ_k^1 *for all odd k and there are no trails through* Δ_k^1 *for all even k.*

The last result uses methods developed recently by J. W. Addison and the author and independently by D. A. Martin. The same results hold for the classes Δ_k^1 (*of subsets of* ${}^\omega\omega$).
(*Received February 15, 1968.*)

CARL E. GORDON. *A comparison of abstract computability theories.*

In recent years a number of theories of computability on sets other than the integers have been developed. In particular Moschovakis defines in *Abstract first order computability*, ***Transactions of the American Mathematical Society*** (to appear) the class $SC(A, \varphi)$ of partial functions (and relations) on an arbitrary set B, search computable in a given finite sequence φ of partial functions (and relations) on B, from a given subset A of B. The definition is a generalization of the one in Kleene, *Recursive functionals and quantifiers of finite type.* I, ***Transactions of the American Mathematical Society***, vol. 91 (1959), pp. 1–52. Moschovakis has shown that search computability is equivalent to some known abstract notions of computability, e.g., Fraisé-recursiveness.

An entirely different approach to computability has been presented by Richard Montague. Given a set B, a set P of relations on B, and a subset A (of distinguished elements) of B, the relations *Montague-*$\aleph_0$*-recursive in P from A* are those relations "Δ_1-definable" in a certain higher-type language from the given relations and distinguished elements. The $\aleph_0$, above, refers to a restriction on the interpretation of the higher-type variables, cf. R. Montague, *A generalization of recursion theory*, this JOURNAL, vol. 32 (1967), pp. 443–444, (abstract).

THEOREM 1. *If B is an infinite set, P a finite sequence of relations on B, one of which is "equality", A a subset of B and R a relation on B, then R is Montague-*$\aleph_0$*-recursive in P from A if and only if R is search computable in P from A.*

The theory of functions metarecursive on an admissible ordinal has been studied by Kripke and others, cf. S. Kripke, *Transfinite recursions on admissible ordinals.* I, this JOURNAL, vol. 29 (1964), p. 161, (abstract). In order to compare this theory with search computability, we extend the definition of $SC(A, \varphi)$, in a natural way, to the case when φ contains type-2 objects.

THEOREM 2. *If B is a set,* $\preceq$ *an admissible well-ordering of B, and f a partial function on B, then f is metarecursive on B if and only if* $f \in SC(B, \preceq, bE)$, *where bE is the type-2 object that represents bounded quantifications,*

$$bE(x, f) \simeq \begin{cases} 0 \text{ if } (\exists y \prec x)(f(y) \simeq 0), \\ 1 \text{ if } (\forall y \prec x)(f(y) \simeq 1), \\ \text{undefined otherwise.} \end{cases}$$

(*Received January 17, 1968.*)

RICHARD D. MOSIER. *Recursive functions and the inference calculus.*

Where x is any proposition and Ex is a proposition in calculus E of implication and Ax is a proposition in calculus A of entailment, we can form the sequences:

$$E(Ex_1 \cdots Ex_2 \cdots Ex_3 \cdots Ex_n) \qquad A(Ax_1 \cdots Ax_2 \cdots Ax_3 \cdots Ax_n)$$

If we assign β as the recursive function of the propositional implications in calculus E and γ as the recursive function of the propositional entailments in calculus A, then (E_β) can replace (Ex_n) in calculus E and (A_γ) can replace (Ax_n) in calculus A. Henceforth, calculus E of propositional implications and calculus A of propositional entailments can be represented by:

$$E(E_\beta) \qquad A(A_\gamma)$$

Thus, (E_β) is an "abstract" of the whole calculus E taken at a specified cumulative point (Ex_n); and (A_γ) is an "abstract" of the whole calculus A taken at a specified cumulative point (Ax_n). We have in effect abstracted "implies" as a recursive function (β) from the matrix of propositional implications in calculus E; and, similarly, we have in effect abstracted "entails" as a recursive function (γ) from the matrix of propositional entailments in calculus A.

Consequently, we can now form new sequences consisting only of the "abstracts" from our original sequences:

$$E(E_{\beta_1} \cdots E_{\beta_2} \cdots E_{\beta_3} \cdots E_{\beta_n}) \qquad A(A_{\gamma_1} \cdots A_{\gamma_2} \cdots A_{\gamma_3} \cdots A_{\gamma_n}).$$

Just as we replaced (Ex_n) by (E_β) in our original sequence for calculus E, and just as we replaced (Ax_n) for (A_γ) in our original sequence for calculus A, we now substitute (β_E) for (E_β) in our calculus of "abstracts" of recursive functions in calculus E, and we substitute (γ_A) for (A_γ) in our calculus of recursive functions of calculus A, so that the "abstracts" of the sequences are now represented simply by (β_E) for calculus E and by (γ_A) for calculus A.

In our first abstraction from the original sequences, we abstracted the recursive function (β) for the E calculus and the recursive function (γ) for the A calculus. But the "abstracts" (E_β) and (A_γ) have the defect of being determined by the immediately preceding values $(Ex_1 \cdots Ex_2 \cdots Ex_3$ and $Ax_1 \cdots Ax_2 \cdots Ax_3)$. But if we substitute (β_E) for (E_β) and (γ_A) for (A_γ), we shall discover *recursive functions which are themselves recursive*—genuine "power" functions (β_A and γ_A) rather than "order" functions (E_β and A_γ).

What we here call "power" functions Quine calls "course-of-values" recursion, which he describes as defining an infinite sequence, or function over all natural numbers, such that generation means specification of successive values not in terms of the respective single values just preceding (E_β and A_γ), but in terms of the whole sequence of values which have been generated (β_E and γ_A). Consequently, we conclude that it is possible to discover not merely "order" recursion but also "power" recursion in the inference calculus. (*Received before January 1, 1968.*)

R. L. STANLEY. *Local sharpness in cut-free systems.*

"Cut-free" natural deduction proofs (well-known term) contain no steps taken under authority of any general cut-rule. A "sharp" formula (new term) is one which, occurring suitably at a point in a proof, allows a full-strength cut-step to be taken, *in effect*, at that point, although strictly the proof remains cut-free, since no cut-rule is used. A proof-line may be sharp altogether, only in some parts, or nowhere; correspondingly, cut-steps are available throughout that proof, only locally therein, or not at all.

This paper examines sharpness in a system based on Quine's "Set theory and its logic", (1) displaying varieties, distribution, and general effects of sharp formulas, (2) establishing, from sharpness of certain numerical formulas, the closure of elementary number theory under *Modus Ponens* (that is, establishing Takeuti's conjecture, as restricted to elementary number theory), (3) constructing a sharp model to prove that Quine's system is consistent if the natural deduction sub-system without cut-rule is consistent, and (4) building sharpness into atomic formulas to form a system isomorphic to Quine's, for which Takeuti's conjecture is true without restriction.

Sharpness is not peculiar to this system, indeed occurs more extensively in some others. The methods and results of (2), (3) and (4) extend to some other strong natural deduction systems—conjecturally, to most.

Local sharpness' extent in any system enlarges simply and artificially, but importantly. To build systems, however, whose structure seriously diminishes sharpness appears difficult. Conjectured possibilities deserve exploration. (*Received December 7, 1967.*)

NICHOLAS RESCHER. *Autodescriptive systems of many-valued logic.*

Consider an entry in an n-valued truth-table of some two-place connective $\otimes$:

$p \setminus q$	$p \otimes q$: j
i	$\langle i \otimes j \rangle$

where i and j and $\langle i \otimes j \rangle$ are all elements of $\{1, 2, \cdots, n\}$

Such an entry says: "If the truth-value of p is i and that of q is j, then that of $p \otimes q$ is to be $\langle i \otimes j \rangle$." We can translate this quoted statement into the vocabulary of the many-valued system itself whenever this system affords us three pieces of machinery: (i) an implication-connective $\rightarrow$ for "if then" (which is assumed to have the *modus ponens* feature that if p and $p \rightarrow q$ take designated truth-values, then so does q), (ii) a conjunction-connective $\wedge$ for "both-and" (which is assumed to have the feature that if p and q both take designated truth-values, then $p \wedge q$ does so), and (iii) a truth-value assignment operator Vip for "the truth-value of p is i" (which is assumed to have the feature that Vip takes a designated truth value iff i is the truth-value of p). The above-quoted statement can then be rendered intrasystematically as:

$$(1) \qquad (Vip \wedge Vjq) \rightarrow V\langle i \otimes j \rangle (p \otimes q).$$

(The translation of the truth-table for a one-place connective is to be handled analogously.)

A many-valued system that affords this machinery will be termed *autodescriptive* with respect to its truth-value assignment operator Vip if the translation into the system the information enshrined in any and every one of its truth-table entries in the manner of (1) will be a tautology of the system.

The truth-value assignment operator of an n-valued system will be said to be *diversified* if statements of the form Vip can assume *all* of the truth-values (not just the two corresponding to *truth* and *falsity*). The defining matrix for a diversified truth-value assignment operator has the crucial and interesting feature that it contains entries other than 1 (true) and 2^n (false), so that truth-value assigning statements of the form "The truth-value of p is i" are not viewed as inherently two-valued (in a way alien to the spirit of many-valued logic).

The three-valued system $Ł_3$ of Łukasiewicz is an example of a system that is autodescriptive in this sense with respect to the now-to-be-specified diversified truth-value assignment operator:

$i \setminus p$	Vip 1	2	3
+1	1	2	3
2	2	1	2
3	3	2	1

$p \setminus q$	$\neg p$	$p \wedge q$ 1	2	3	$p \rightarrow q$ 1	2	3
+1	3	1	2	3	1	2	3
2	2	2	2	3	1	1	2
3	1	3	3	3	1	1	1

In the case of an autodescriptive system of many-valued logic with a diversified truth-value assignment operator, one can take the view that the system can be presented by means of a many-valued metalanguage, and does not require the usual two-valued one. The existence of such many-valued systems which could themselves serve as their own metalanguage refutes the position of various writers who view two-valued logic as fundamental *vis à vis* many-valued logic on the grounds that many-valued systems must invariably be developed by means of two-valued logical machinery used at the metalinguistic level. (*Received January 19, 1968.*)

ANDRZEJ MOSTOWSKI. *A theorem on β-models.*

Let A be the set of axioms of the second order arithmetic as described, e.g., in this JOURNAL, vol. 23 (1958), p. 189 and containing, in addition, the axiom-scheme of choice $(x)(E\alpha^1)F(x, \alpha^1) \equiv (E\alpha^2)(x)F(x, \lambda y\alpha^2(x, y))$. The following theorem is proved: If a set $X \supseteq A$ has a β-model, then it also has an ω-model which is not a β-model.

In the proof we use an extended language L containing a symbol c for a function with 2 arguments. Let M be a β-model of X and W the set of all functions $f \in M$ such that the relation $R_f = \{\langle x, y \rangle : f(x, y) = 0\}$ well orders ω. We call a set $C \subseteq W$ unbounded if for every f in W there is a g in C such that $\langle \omega, R_f \rangle$ is isomorphic to a submodel of $\langle \omega, R_g \rangle$.

LEMMA. *If $C = \bigcup C_n$ is unbounded and the relation $\{\langle n, x \rangle : x \in C_n\}$ is definable in M, then at least one C_n is unbounded.*

Using this lemma repeatedly we show that there is a consistent and complete set X_0 consisting of sentences of L and an infinite sequence of integers i_n such that (i) if $F \in X \cup A$, then the closure of F is in X_0; (ii) the sentences Bord(c) (i.e. "c is a well ordering") and $c(i_{n+1}, i_n) = 0$ belong to X_0 for $n = 0, 1, 2, \cdots$; (iii) if F has one free variable and $(Ex)F(x) \in X_0$, then there is an integer such that $F(n) \in X_0$.

The set X_0 has an ω-model but no ω-model of X_0 is a β-model. *(Received January 30, 1968.)*

A. A. MULLIN. *On an application of recursive arithmetic to philosophy.*

The Pythagorean doctrine pervades the works of Plato (e.g., ***Republic***, Book VII), Aristotle (e.g., ***Metaphysics***, Book I, Chapter 5), Philo Judeus (e.g., ***The creation***, vol. I), Nicomachus of Gerasa, Virgil (e.g., ***Eclogue***, VIII), and Aurelius Augustinus (e.g., ***City of God***, Chapter XXX) to name only a few pre-medieval inquirers. In these works the simple odd-even dichotomy is utilized, among other properties of numbers. The present study develops a motley of metaphysical theories, simple ethical theories, and elementary aesthetical theories, based, in part, upon computable number-theory, with special emphasis on the prime-composite dichotomy. The ethical theories are admittedly naive since they hinge exclusively on computable properties of amicable numbers and so-called agapistic numbers [***Notices of the American Mathematical Society***, vol. 12 (1965), pp. 217–218]. The author did *not* have access to the little-known results by A. Comte, who apparently constructed parts of his metaphysics from properties of prime numbers. In the aesthetical domain, new syllable-count forms of poetry are constructed based upon use of recursive sets, so as to supplement recent results using prime numbers [***Notices of the American Mathematical Society***, vol. 14 (1967), pp. 941–942]. Recursion theory is chosen as a frame of reference partly because of its logical relevance and partly because of the central role its philosophical analogue plays in various parts of the modern works of Vico, Nietzsche, Whitehead, and Joyce, among others. *(Received February 6, 1968.)*

PETER ACZEL. *The universal properties of recursive density types.*

Notation as in the previous abstract. Call a function on ω a *density function* if it is bounded by a recursive function and it is nondecreasing in each argument. If $R \subseteq^n \omega$ and $\alpha_1, \cdots, \alpha_n \subseteq \omega$, then $\langle \alpha_1, \cdots, \alpha_n \rangle$ is *R-attainable* if there are density functions $f_1, \cdots, f_n$ such that $(\forall x)x \leq f_i(x)$ for $i = 1, \cdots, n$ and $(\forall x)\langle \alpha_1[f_1(x)], \cdots, \alpha_n[f_n(x)]\rangle \in R$. The extension R_Δ of R to Δ is defined by $R_\Delta = \{\langle D(\alpha_1), \cdots, D(\alpha_n)\rangle \mid \langle \alpha_1, \cdots, \alpha_n\rangle$ is R-attainable$\}$.

The recursive density types behave in many respects like the recursive equivalence types. In fact using the above extension procedure there are analogues to most of Nerode's results in ***Extensions to isols, Annals of mathematics***, vol. 73 (1961). The role of the isols is here played by the set $\Gamma = \{A \in \Delta \mid (\forall B)B < \infty \rightarrow A \cup B < \infty\}$. The density functions play the role of Myhill's recursive combinatorial functions. The graph of every density function f extends to the graph of a function f_Δ. The extension procedure commutes with composition of functions and Γ is closed under every f_Δ. Let f_Γ denote the restriction of f_Δ to Γ and let $\mathscr{C}$ be the class of recursive combinatorial functions.

THEOREM. *The same universal sentences are true in* $\langle \Delta, f_\Delta\rangle_{f\in\mathscr{C}}$ *and* $\langle \Gamma, f_\Gamma\rangle_{f\in\mathscr{C}}$.

NOTE 1. The existence of an extension procedure for Δ analogous to the Myhill-Nerode procedure for R.E.T.'s was first conjectured by H. Gonshor.

NOTE 2. Almost all of the results of this and the previous abstract depend only on the fact that the set of recursive functions form a countable primitive recursively closed set. *(Received February 2, 1968.)*

LAWRENCE WOS AND GEORGE ROBINSON. *Maximal model theorem.*

A *clause* is a set of atomic formulae (of the first order predicate calculus) and negations of atomic formulae—and is thought of as representing the disjunction of its members (literals). Each existential variable has been replaced by a Skolem function and each other variable is universally quantified over the clause in which it occurs. For any set $\mathscr{S}$ of clauses, let P be the set of atoms over its Herbrand universe. For undefined terms refer to J. A. Robinson, "A machine-oriented logic based on the resolution principle," ***Journal of the Association for Computing Machinery***, vol. 12 (1965), pp. 23–41. Where x' is the negation of a literal x, let $N =$

$\{b' \mid b \in P\}$, and for $W \subseteq P \cup N$ let $W' = \{b' \mid b \in W\}$. If $W \subseteq P$, $W \cup (P - W)'$ is an *interpretation of* $\mathscr{S}$. An interpretation of $\mathscr{S}$ is called a *model of* $\mathscr{S}$ if it has a nonempty intersection with each variable-free instance of each clause in $\mathscr{S}$. Relative to each $Q \subseteq P$, there is a partial ordering of the set of interpretations and hence of the models of $\mathscr{S}$:

$$M_1 \leq {}_Q M_2 \quad \text{iff} \quad M_1 \cap [Q' \cup (P - Q)] \subseteq M_2 \cap [Q' \cup (P - Q)].$$

MAXIMAL MODEL THEOREM. *If $\mathscr{S}$ has a model, then given any $Q \subseteq P$, $\mathscr{S}$ has a maximal model M relative to Q. Furthermore, for each literal b in $M \cap [Q \cup (P - Q)']$ there exists a clause in $\mathscr{S}$ having an instance D with $D \cap M = \{b\}$.*

PROOF. Let $T = Q \cup (P - Q)'$; then $M_1 \leq {}_Q M_2$ iff $M_1 \cap T' \subseteq M_2 \cap T'$. Let $\{H_\alpha\}_{\alpha \in \Gamma}$ be a nonempty, simply ordered (relative to Q) set of models of $\mathscr{S}$. Then

$$\Big(\bigcup_{\alpha \in \Gamma} H_\alpha \cap T'\Big) \cup \Big(\bigcap_{\alpha \in \Gamma} H_\alpha \cap T\Big)$$

is a model of $\mathscr{S}$ and an upper bound for $\{H_\alpha\}_{\alpha \in \Gamma}$. Hence by Zorn's lemma there is a maximal model M of $\mathscr{S}$. Furthermore, if there were an element b in $M \cap T$ that falsified the second part of the theorem, $(M - \{b\}) \cup \{b'\}$ would be a model of $\mathscr{S}$ strictly greater than M. (The proof makes no appeal to the properties that the members of P are atomic formulae and the clauses in $\mathscr{S}$ are disjunctions, but does use $x'' = x$.)

It also follows that given, not a subset Q of P, but an interpretation T of $\mathscr{S}$, there is a model M of $\mathscr{S}$ maximal with respect to T, i.e., for which no model M^* of $\mathscr{S}$ exists with $M \cap T \subset M^* \cap T$ (properly).

With $Q = P$, the maximal model theorem plays a central role in establishing that certain properly paramodulation-based systems are semidecision procedures for first-order predicate calculus with equality (see abstract following). (*Received February 1, 1968.*)

GEORGE ROBINSON AND LAWRENCE WOS. *Completeness of paramodulation.*

For terminology and for the maximal model theorem see preceding abstract. A finite set $\mathscr{S}$ of clauses has a model M in the sense of the preceding abstract iff it has a model in the usual sense. If M satisfies $\mathscr{S}$ under the usual definition for first-order predicate calculus with equality, M will be called an *R-model* of $\mathscr{S}$ and $\mathscr{S}$ termed *R-satisfiable.*

Paramodulation. Given clauses A and $\alpha' = \beta' \vee B$ (or $\beta' = \alpha' \vee B$) such that A contains a term δ with δ and α' having a most general common instance α identical to $\alpha'[s_i/u_i]$ identical to $\delta[t_j/w_j]$, where A' is obtained by replacing in $A[t_j/w_j]$ some single occurrence of α (resulting from an occurrence of δ) by $\beta'[s_i/u_i]$, infer $A' \vee B[s_i/u_i]$.

THEOREM 1. *If an R-unsatisfiable set $\mathscr{S}$ of clauses is closed under paramodulation and contains all instances of $x = x$ over the Herbrand universe H of $\mathscr{S}$, then $\mathscr{S}$ is unsatisfiable.*

PROOF. Suppose $\mathscr{S}$ is satisfiable. Let $\mathscr{S}^*$ be the set of all (variable-free) instances of members of $\mathscr{S}$ over H and let P be the set of atoms of $\mathscr{S}^*$. By the maximal model theorem there is a model M of $\mathscr{S}$ such that for each b in $M \cap P$ there is a clause D in $\mathscr{S}^*$ with $D \cap M = \{b\}$. From this it can be shown that the relation $\{(\alpha, \beta) | (\alpha = \beta) \in M\}$ is in fact reflexive, symmetric, transitive, and (with respect to all functions and predicates occurring in $\mathscr{S}$) substitutive. Hence M is an R-model of $\mathscr{S}$, contradicting its R-unsatisfiability.

THEOREM 2. *Based upon paramodulation, taken together with resolution and the axiom schema $\alpha = \alpha$, one can construct a semidecision procedure for first-order predicate calculus with equality, i.e., a procedure which yields a refutation for any finite set $\mathscr{S}$ iff $\mathscr{S}$ is R-unsatisfiable.*

PROOF. Adjoin to $\mathscr{S}$ all instances of $x = x$ over the Herbrand universe of $\mathscr{S}$ and close under paramodulation, obtaining a set $\mathscr{S}'$. $\mathscr{T}$, the closure of $\mathscr{S}'$ under resolution, can be effectively enumerated. If $\mathscr{S}$ is R-satisfiable, then $\mathscr{T}$ is satisfiable since paramodulation and resolution are both sound. If $\mathscr{S}$ is R-unsatisfiable, then by Theorem 1, $\mathscr{T}$ must be unsatisfiable. Then since $\mathscr{T}$ is closed under resolution it must, by a theorem of J. A. Robinson in the paper cited in the preceding abstract, contain a pair of contradictory one-member clauses. (*Received February 1, 1968.*)