

ULTRADISCRETE HAMILTONIAN SYSTEMS

MASATAKA IWAO and DAISUKE TAKAHASHI

School of Science and Engineering, Waseda University, Tokyo 169-8555, Japan
e-mail: masataka.iwao@fuji.waseda.jp

(Received 29 October, 2003; accepted 19 May, 2004)

Abstract. The method of ultradiscrete limit is applied to a series of discrete systems derived from Hamiltonian systems parametrized with corresponding lattice polygons. For every ultradiscrete system, general solution is obtained from the polar set of each lattice polygon.

2000 Mathematics Subject Classification. 37J99.

1. Introduction. The non-analytic limit [1, 3]

$$\lim_{\epsilon \rightarrow +0} \epsilon \log \left(\exp \left[\frac{U}{\epsilon} \right] + \exp \left[\frac{V}{\epsilon} \right] + \cdots + \exp \left[\frac{W}{\epsilon} \right] \right) = \max(U, V, \dots, W) \quad (1)$$

has been introduced into soliton theory and used to translate many fully discrete soliton equations into the corresponding soliton cellular automaton [2, 4]. Nowadays, formula (1) is called the ultradiscrete limit, since the soliton equation of fully discrete independent variables is further transformed through this limit into a system whose dependent variables are also discretized.

We propose applying the ultradiscrete limit to the Hamiltonian system

$$\frac{dQ}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dP}{dt} = -\frac{\partial H}{\partial Q}.$$

We will see that it is successful for some of the Hamiltonian functions in the form

$$H(Q, P) = \epsilon \log \left(\sum_{(j,k) \in \partial \Gamma \cap \mathbf{Z}^2} a_{(j,k)} \exp \left[\frac{jQ + kP}{\epsilon} \right] \right), \quad (2)$$

where $\epsilon > 0$ is a parameter and the weight constants $a_{(j,k)}$ depend on a convex lattice polygon Γ having the origin as a unique internal lattice point. Below we explain the precise correspondence between such polygons and Hamiltonian functions.

1.1. Hamiltonian with polygon Γ . Let Γ be a convex polygon such that

- every vertex $\in \mathbf{Z}^2$,
- the origin $(0, 0)$ is a unique internal lattice point.

Figure 1. shows examples Γ_1 and Γ_2 , where the labels on the lattice points of the edges are the weight constants. Polygons Γ_1 and Γ_2 correspond to the following Hamiltonian

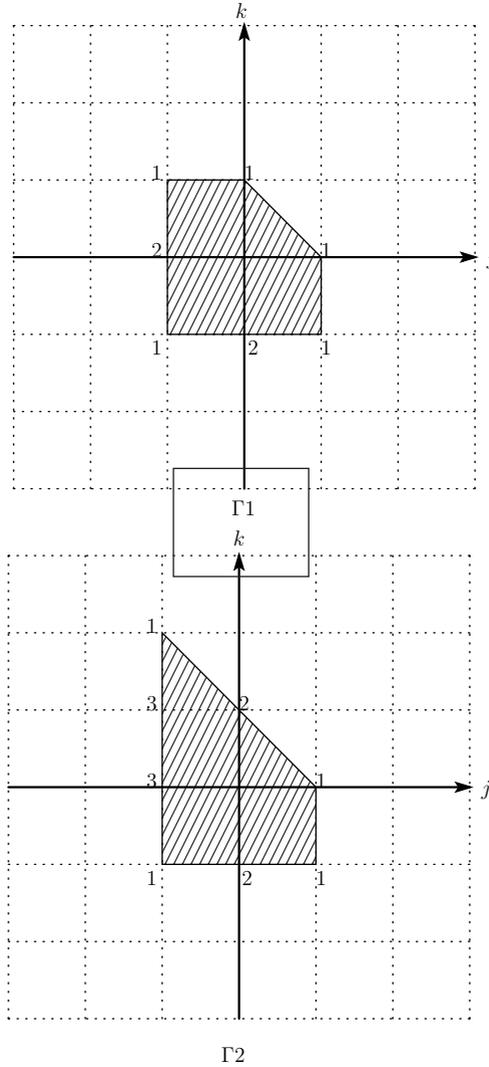


Figure 1. Γ_1 and Γ_2 with weight constants

functions:

$$H_{\Gamma_1}(Q, P) = \epsilon \log \left\{ \begin{array}{l} 1 \cdot e^{(1,0) \cdot (Q,P)/\epsilon} + 1 \cdot e^{(1,-1) \cdot (Q,P)/\epsilon} \\ + 2 \cdot e^{(0,-1) \cdot (Q,P)/\epsilon} + 1 \cdot e^{(-1,-1) \cdot (Q,P)/\epsilon} \\ + 2 \cdot e^{(-1,0) \cdot (Q,P)/\epsilon} + 1 \cdot e^{(-1,1) \cdot (Q,P)/\epsilon} \\ + 1 \cdot e^{(0,1) \cdot (Q,P)/\epsilon} \end{array} \right\}, \tag{3}$$

$$H_{\Gamma_2}(Q, P) = \epsilon \log \left\{ \begin{array}{l} 1 \cdot e^{(1,0) \cdot (Q,P)/\epsilon} + 1 \cdot e^{(1,-1) \cdot (Q,P)/\epsilon} \\ + 2 \cdot e^{(0,-1) \cdot (Q,P)/\epsilon} + 1 \cdot e^{(-1,-1) \cdot (Q,P)/\epsilon} \\ + 3 \cdot e^{(-1,0) \cdot (Q,P)/\epsilon} + 3 \cdot e^{(-1,1) \cdot (Q,P)/\epsilon} \\ + 1 \cdot e^{(-1,2) \cdot (Q,P)/\epsilon} + 2 \cdot e^{(0,1) \cdot (Q,P)/\epsilon} \end{array} \right\}. \tag{4}$$

We remark that the weight constants cannot be chosen arbitrarily, but have a considerable degree of freedom for our purpose of discretization. We use appropriate binomial coefficients as our weight constants for the present, which allow us to discretize the independent variable t . Further research may allow us to determine the suitable general weight constants on general Γ . We give some more examples in the following section.

2. Discretization of independent variable t . We consider the discretization of our Hamiltonian systems.

First we note that our Hamiltonian functions $H = H_{\Gamma_i}$ (for $i = 1, 2$) are strictly convex, that is, $\frac{\partial^2 H}{\partial Q^2} > 0, \frac{\partial^2 H}{\partial Q^2} \frac{\partial^2 H}{\partial P^2} - (\frac{\partial^2 H}{\partial Q \partial P})^2 > 0$ for all $(Q, P) \in \mathbf{R}^2$. Also, they each have a minimum value, at a unique finite point $(Q, P) = (Q_{fix}, P_{fix})$ such that $\frac{\partial H}{\partial Q} = \frac{\partial H}{\partial P} = 0$.

Now, we make general consideration. Let $H(Q, P)$ be a Hamiltonian function given as strictly convex for all $(Q, P) \in \mathbf{R}^2$ and having a minimum value. Then, for arbitrary initial point (Q_0, P_0) , the orbit $\{(Q(t), P(t)) | t \in \mathbf{R}, (Q(0), P(0)) = (Q_0, P_0)\}$ of the time evolution with

$$\frac{dQ}{dt} = \frac{\partial H}{\partial p}, \frac{dP}{dt} = -\frac{\partial H}{\partial Q} \tag{5}$$

is the set $\{(Q, P) \in \mathbf{R}^2 | H(Q, P) = H(Q_0, P_0)\}$, that should be either some simple closed curve or the fixed point $\{(Q_{fix}, P_{fix})\}$. Here, we emphasize that the time evolution $(Q(t), P(t))$ from $(Q(0), P(0)) = (Q_0, P_0)$ always passes through any (Q, P) such that $H(Q, P) = H(Q_0, P_0)$. This situation enables us to obtain a discrete system.

The idea of discretization is simple: Let $(Q(t), P(t))$ be one of the representation of general solution to system (5), and let $(X(t), Y(t))$ be another,

$$\frac{dX}{dt} = \frac{\partial H}{\partial p}(X, Y), \frac{dY}{dt} = -\frac{\partial H}{\partial Q}(X, Y), \tag{6}$$

satisfying

$$H(Q, P) = H(X, Y). \tag{7}$$

Then, for any chosen initial point $(Q(0), P(0))$ of $(Q(t), P(t))$, there exists some point $(X(0), Y(0))$, the initial point of $(X(t), Y(t))$, and some δ such that $(Q(\delta), P(\delta)) = (X(0), P(0))$. Thus we have

$$E_\delta Q = X, E_\delta P = Y, \tag{8}$$

where E_δ is the time shift operator $E_\delta \phi(t) = \phi(t + \delta)$ for a properly chosen fixed parameter δ . We note that this parameter δ is determined respectively for each one-to-one correspondence,

$$(Q, P) \mapsto (X, Y) = (F_\delta(Q, P), G_\delta(Q, P)). \tag{9}$$

Hence, we have found that (8) and (9) form the discrete system

$$E_\delta Q = F_\delta(Q, P), E_\delta P = G_\delta(Q, P), \tag{10}$$

which is satisfied by the general solution $(Q(t), P(t))$ of (5) with a properly chosen fixed parameter δ , if the pair of functions F_δ and G_δ satisfy

$$\frac{dF_\delta}{dt} = \frac{\partial H}{\partial p}(F_\delta, G_\delta), \quad \frac{dG_\delta}{dt} = -\frac{\partial H}{\partial Q}(F_\delta, G_\delta), \quad H(Q, P) = H(F_\delta, G_\delta), \quad (11)$$

obtained from (6), (7) and (9).

REMARK. If a discrete system of the form (10) was given then the condition (11) would be $\frac{d}{dt} \cdot E_\delta(Q, P) = E_\delta \cdot \frac{d}{dt}(Q, P), H(Q, P) = E_\delta H(Q, P)$.

Here, we rewrite the condition (11) into more suitable form for our Hamiltonian functions. Let us introduce the following transformation,

$$(Q(t), P(t), H(Q, P), F_\delta(Q, P), G_\delta(Q, P)) \mapsto (q(t), p(t), h(q, p), f(q, p), g(q, p)) := (e^{Q/\epsilon}, e^{P/\epsilon}, e^{H/\epsilon}, e^{F_\delta/\epsilon}, e^{G_\delta/\epsilon}), \quad (12)$$

which transforms the condition (11) into the following system of equations,

$$\begin{cases} \xi(q, p) \frac{\partial f}{\partial q} + \eta(q, p) \frac{\partial f}{\partial p} = \xi(f, g), \\ \xi(q, p) \frac{\partial g}{\partial q} + \eta(q, p) \frac{\partial g}{\partial p} = \eta(f, g), \\ h(q, p) = h(f, g), \end{cases} \quad (13)$$

where the functions ξ and η are defined by

$$\xi(q, p) = qp \frac{\partial h(q, p)}{\partial p}, \quad \eta(q, p) = -qp \frac{\partial h(q, p)}{\partial q}.$$

The discrete system (10) is transformed into $E_\delta q = f(q, p), E_\delta p = g(q, p)$.

2.1. Discretization for case $\Gamma 1$. The transformation (12) transforms (3) into

$$h_{\Gamma 1}(q, p) = q + \frac{q}{p} + \frac{2}{p} + \frac{1}{qp} + \frac{2}{q} + \frac{p}{q} + p,$$

and we have

$$\xi_{\Gamma 1}(q, p) = qp \frac{\partial h_{\Gamma 1}}{\partial p}, \quad \eta_{\Gamma 1}(q, p) = -qp \frac{\partial h_{\Gamma 1}}{\partial q}.$$

For the above $\xi_{\Gamma 1}$ and $\eta_{\Gamma 1}$ (13) is

$$\begin{cases} \xi_{\Gamma 1}(q, p) \frac{\partial f}{\partial q} + \eta_{\Gamma 1}(q, p) \frac{\partial f}{\partial p} = \xi_{\Gamma 1}(f, g), \\ \xi_{\Gamma 1}(q, p) \frac{\partial g}{\partial q} + \eta_{\Gamma 1}(q, p) \frac{\partial g}{\partial p} = \eta_{\Gamma 1}(f, g), \\ h_{\Gamma 1}(q, p) = h_{\Gamma 1}(f, g). \end{cases}$$

This system has an exact solution

$$f = p, \quad g = \frac{p+1}{q},$$

so, we have the rational map

$$E_\delta q = p, E_\delta p = \frac{p+1}{q},$$

which is transformed using (12) into the discrete system

$$E_\delta Q = P, E_\delta P = \epsilon \log(\exp[P/\epsilon] + 1) - Q,$$

which, for some δ , is satisfied by the general solution of continuous Hamiltonian system.

2.2. Discretization for case Γ_2 . The transformation (12) transforms (4) into

$$h_{\Gamma_2}(q, p) = q + \frac{q}{p} + \frac{2}{p} + \frac{1}{qp} + \frac{3}{q} + \frac{3p}{q} + \frac{p^2}{q} + 2p,$$

and we have

$$\xi_{\Gamma_2}(q, p) = qp \frac{\partial h_{\Gamma_2}}{\partial p}, \eta_{\Gamma_2}(q, p) = -qp \frac{\partial h_{\Gamma_2}}{\partial q}.$$

For this ξ_{Γ_2} and η_{Γ_2} (13) is

$$\begin{cases} \xi_{\Gamma_2}(q, p) \frac{\partial f}{\partial q} + \eta_{\Gamma_2}(q, p) \frac{\partial f}{\partial p} = \xi_{\Gamma_2}(f, g), \\ \xi_{\Gamma_2}(q, p) \frac{\partial g}{\partial q} + \eta_{\Gamma_2}(q, p) \frac{\partial g}{\partial p} = \eta_{\Gamma_2}(f, g), \\ h_{\Gamma_2}(q, p) = h_{\Gamma_2}(f, g). \end{cases}$$

This system has an exact solution

$$f = \frac{p(q+p+1)}{q}, g = \frac{p+1}{q},$$

so we have the rational map

$$E_\delta q = \frac{p(q+p+1)}{q}, E_\delta p = \frac{p+1}{q},$$

which is transformed by (12) into the discrete system

$$\begin{cases} E_\delta Q = P + \epsilon \log(\exp[Q/\epsilon] + \exp[P/\epsilon] + 1) - Q, \\ E_\delta P = \epsilon \log(\exp[P/\epsilon] + 1) - Q, \end{cases}$$

which, for some δ , is satisfied by the general solution of continuous Hamiltonian system.

2.3. Exact solution to system (13): More examples. We can construct other examples. The following are some of them, where we give only h_Γ and the respective exact solution f, g of (13) in each case:

- $h_{\Gamma_3} = q + qp^{-1} + p^{-1} + q^{-1}p + p.$

$$f = \frac{p}{q(q+p+1)}, g = \frac{1}{q+p}.$$

- $h_{\Gamma_4} = qp + 2p^{-1} + q^{-1}p^{-3} + 4q^{-1}p^{-2} + 6q^{-1}p^{-1} + 4q^{-1} + q^{-1}p + 2p.$

$$f = \frac{p(qp^2 + (p+1)^2)^2}{(p+1)^4}, g = \frac{(p+1)^2}{qp^2}.$$

- $h_{\Gamma_5} = q + qp^{-1} + 2p^{-1} + q^{-1}p^{-1} + q^{-1} + p.$

$$f = \frac{qp}{q+1}, g = \frac{qp+q+1}{q(q+1)}.$$

- $h_{\Gamma_6} = q + p^{-1} + q^{-1} + qp.$

$$f = \frac{p}{qp+1}, g = \frac{1}{qp}.$$

- $h_{\Gamma_7} = qp + p^{-1} + q^{-1}p^{-1} + p + qp^2.$

$$f = qp^2(p+1), g = \frac{1}{qp(p+1)}.$$

The following discussion is also applicable to the above examples.

3. Ultradiscrete Hamiltonian systems. We apply the ultradiscrete limit $\epsilon \rightarrow +0$ to our Hamiltonian functions (3) and (4) using (1), to obtain

$$H_{\Gamma_1}(Q, P) = \max(Q, Q - P, -P, -Q - P, -Q, -Q + P, P),$$

$$H_{\Gamma_2}(Q, P) = \max(Q, Q - P, -P, -Q - P, -Q, -Q + P, -Q + 2P, P),$$

and we obtain the following ultradiscrete systems,

$$\begin{aligned} E_\delta Q = P, E_\delta P = \max(P, 0) - Q, & \quad \text{for } H_{\Gamma_1}(Q, P), \\ E_\delta Q = P + \max(Q, P, 0) - Q, E_\delta P = \max(P, 0) - Q, & \quad \text{for } H_{\Gamma_2}(Q, P). \end{aligned}$$

In order to find the general solution to the above ultradiscrete system, we use the polar set of Γ [5].

3.1. Polar set of Γ . Here we describe how to obtain the polar set for a polygon Γ of the form defined in Section 1.1. This procedure is not available for a general polygon.

1. Each edge joins two vertices of Γ . For each edge construct outward normals from each end to the nearest lattice point.
2. Then, for every vertex, construct the triangle spanned by the pair of outward normal vectors at that vertex.
3. Finally, translate each triangle constructed in 2. So that the original vertices coincide at the origin. The resulting convex lattice polygon Γ^Δ is the polar set of Γ .

For our Γ_1 and Γ_2 , we can construct the polar sets as Figure 2.

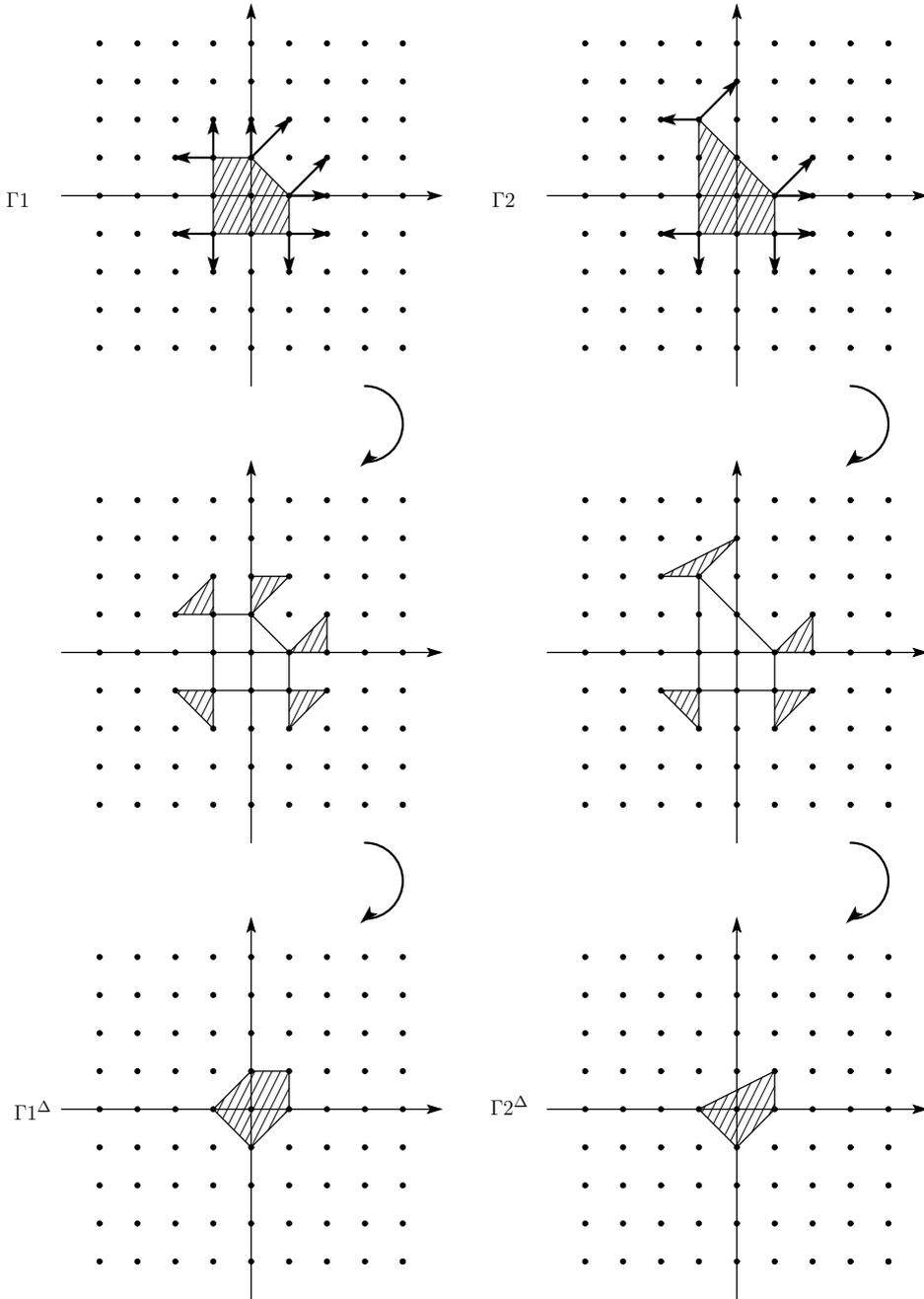


Figure 2. How to make the polar set Γ^Δ

Hereafter, we treat the Hamiltonian function (2) as the ultradiscrete limit,

$$H_\Gamma(Q, P) = \max_{(j,k) \in \partial\Gamma \cap \mathbb{Z}^2} \{jQ + kP\}.$$

Then, we can show that for all Γ ,

$$\{(Q, P) | H_\Gamma(Q, P) = 1\} = \partial\Gamma^\Delta. \tag{14}$$

The formula (14) means that the contour line $\{(Q, P) | H_\Gamma(Q, P) = 1\}$ of the ultradiscrete Hamiltonian function H_Γ corresponds to the boundary of the polar set Γ^Δ . Further, we note that the similarity $H_\Gamma(\lambda Q, \lambda P) = \lambda H_\Gamma(Q, P) = \lambda$ holds for all Γ and $\lambda \geq 0$. Therefore, we have obtained a procedure for constructing arbitrary contour line of the ultradiscrete Hamiltonian function. We remark that any solution orbit of the ultradiscrete system is on a contour line $\{(Q, P) | H_\Gamma(Q, P) = \lambda\}$ for some $\lambda \geq 0$ in the phase space.

We will now verify formula (14): Let Γ be an N -gon with N -vertices $(j_1, k_1), (j_2, k_2), \dots, (j_N, k_N)$, where we choose one of the vertices to be $u_1 := (j_1, k_1)$, and then label the other vertices as $u_i := (j_i, k_i)$ counterclockwise for $i = 2, 3, \dots, N$. We use cyclic notation $u_{N+1} := u_1$ or $u_0 := u_N$. We note that

$$\begin{aligned} H_\Gamma(Q, P) &= \max_{(j,k) \in \partial\Gamma \cap \mathbb{Z}^2} \{(jQ + kP)\} = \max_{(j,k) \in \partial\Gamma \cap \mathbb{Z}^2} \{(j, k) \cdot (Q, P)\} \\ &= \max_{1 \leq i \leq N} \{u_i \cdot (Q, P)\}, \end{aligned}$$

because of the identity,

$$\max\{A \cdot (Q, P), B \cdot (Q, P), C \cdot (Q, P)\} = \max\{A \cdot (Q, P), C \cdot (Q, P)\},$$

which holds when B is on the line segment from A to C . Thus, (14) is equivalent to $\{(Q, P) | \max_{1 \leq i \leq N} \{u_i \cdot (Q, P)\} = 1\} = \partial\Gamma^\Delta$.

To prove this we represent $\partial\Gamma$ using $u_i = (j_i, k_i)$. We recall that Γ includes the origin $(0, 0)$ as an internal lattice point, and that u_i have been arranged counterclockwise, so we have

$$\Gamma = \left\{ (j, k) \mid \det \begin{pmatrix} j_{i-1} - j & k_{i-1} - k \\ j_i - j & k_i - k \end{pmatrix} \geq 0, \text{ for } 1 \leq i \leq N \right\}.$$

If, we write

$$D_i := \det \begin{pmatrix} j_{i-1} & k_{i-1} \\ j_i & k_i \end{pmatrix},$$

for $1 \leq i \leq N$, then, we have that $D_i > 0$ and

$$\Gamma = \{(j, k) | (j, k) \cdot (k_i - k_{i-1}, j_{i-1} - j_i) \leq D_i, \text{ for } 1 \leq i \leq N\}.$$

Thus we have

$$\partial\Gamma = \{(j, k) \mid \max_{1 \leq i \leq N} \{(j, k) \cdot (k_i - k_{i-1}, j_{i-1} - j_i) / D_i\} = 1\}.$$

Now we set $v_i := (k_i - k_{i-1}, j_{i-1} - j_i) / D_i$, for $1 \leq i \leq N$, that have the following properties:

- Since $u_i \cdot v_i = u_{i-1} \cdot v_i = 1$, we have $(u_i - u_{i-1}) \cdot v_i = 0$, hence v_i is the normal for the edge $(u_i - u_{i-1})$ of Γ .

- We have

$$\det \begin{pmatrix} v_i \\ u_i - u_{i-1} \end{pmatrix} = \det \begin{pmatrix} k_i - k_{i-1} & j_{i-1} - j_i \\ j_i - j_{i-1} & k_i - k_{i-1} \end{pmatrix} / D_i > 0,$$

hence, v_i points outward.

- With $\gcd(j_{i-1}, k_{i-1}) = \gcd(j_i, k_i) = 1$, we have

$$D_i = (\text{the number of lattice points on the edge from } u_{i-1} \text{ to } u_i) - 1,$$

hence, $\|v_i\| = \|u_i - u_{i-1}\| / D_i$ is the length of the shortest lattice vector as the normal for $u_i - u_{i-1}$.

Therefore, we have shown that $\{v_i | 1 \leq i \leq N\}$ is the set of all vertices of Γ^Δ constructed by the procedure. Here, we note that

- Γ^Δ has the unique internal lattice point $(0, 0)$.
- v_i are counterclockwise for $i = 1, 2, \dots, N$.
- Γ^Δ is convex. (Because, $u_i \cdot (v_{i+1} - v_i) = 0$ for $i = 1, 2, \dots, N$, and both $\{u_i\}$ and $\{v_i\}$ are counterclockwise.)

Next we represent $\partial\Gamma^\Delta$ using $v_i =: (Q_i, P_i)$. We can do this in the same way as $\partial\Gamma$:

$$\partial\Gamma^\Delta = \{(Q, P) \mid \max_{1 \leq i \leq N} \{(P_{i+1} - P_i, Q_i - Q_{i+1}) \cdot (Q, P) / D'_i\} = 1\},$$

where we have defined

$$D'_i := \det \begin{pmatrix} Q_i & P_i \\ Q_{i+1} & P_{i+1} \end{pmatrix}.$$

Now, we set $w_i := (P_{i+1} - P_i, Q_i - Q_{i+1}) / D'_i$, for $1 \leq i \leq N$, using cyclic notation $w_{n+1} := w_1$. Then, w_i becomes the outward normal for the edge $v_{i+1} - v_i$ of Γ^Δ , as the shortest lattice vector, and $\{w_i | 1 \leq i \leq N\}$ becomes the set of all vertices of a convex N -gon having the unique internal lattice point $(0, 0)$. Since $(w_{i+1} - w_i) \cdot v_{i+1} = 0$, for $1 \leq i \leq N$, and we note that such a convex N -gon should be uniquely determined. On the other hand, we have already found that $(u_{i+1} - u_i) \cdot v_{i+1} = 0$. Hence, we obtain $w_i = u_i$ for $1 \leq i \leq N$ and so

$$\partial\Gamma^\Delta = \{(Q, P) \mid \max_{1 \leq i \leq N} \{u_i \cdot (Q, P)\} = 1\} = \{(Q, P) \mid H_\Gamma(Q, P) = 1\},$$

as required.

3.2. General solution to ultradiscrete system. We restrict the formula (14) to \mathbf{Z}^2 :

$$\{(Q, P) \in \mathbf{Z}^2 \mid H_\Gamma(Q, P) = 1\} = \partial\Gamma^\Delta \cap \mathbf{Z}^2.$$

From this formula, we can find the general solution to our ultradiscrete system as follows. We call it the ‘general solution’ in the sense that the solution includes any time evolution solution from arbitrary chosen initial point $(Q_0, P_0) \in \mathbf{R}^2$.

First, we note that our ultradiscrete system for Γ_1 ,

$$E_\delta Q = P, E_\delta P = \max(P, 0) - Q,$$

acts to permute the set

$$\begin{aligned} \{(Q, P) \in \mathbf{Z}^2 | H_{\Gamma_1}(Q, P) = 1\} &= \partial\Gamma_1^\Delta \cap \mathbf{Z}^2 \\ &= \{(1, 0), (0, -1), (-1, 0), (0, 1), (1, 1)\}. \end{aligned}$$

Actually, we have

$$(1, 0) \mapsto (0, -1) \mapsto (-1, 0) \mapsto (0, 1) \mapsto (1, 1) \mapsto (1, 0).$$

Here, we let $\Phi(Q_0, P_0)$ denote the periodic solution to our system for Γ_1 of an initial value (Q_0, P_0) in $\partial\Gamma_1^\Delta \cap \mathbf{Z}^2$. For instance,

$$\Phi(1, 0) = \begin{cases} (1, 0), & t/\delta \equiv 0 \pmod{5}, \\ (0, -1), & t/\delta \equiv 1 \pmod{5}, \\ (-1, 0), & t/\delta \equiv 2 \pmod{5}, \\ (0, 1), & t/\delta \equiv 3 \pmod{5}, \\ (1, 1), & t/\delta \equiv 4 \pmod{5}. \end{cases}$$

Then, the general solution to our system for Γ_1 is written as

$$\mu\Phi(Q_0, P_0) + \nu E_\delta\Phi(Q_0, P_0), \tag{15}$$

for arbitrary $\mu \geq 0, \nu \geq 0$ and $(Q_0, P_0) \in \partial\Gamma_1^\Delta \cap \mathbf{Z}^2$. This solution is, indeed, the general solution, since,

$$\begin{aligned} \mathbf{R}^2 &= \{\mu(1, 0) + \nu(0, -1)\} \cup \{\mu(0, -1) + \nu(-1, 0)\} \cup \{\mu(-1, 0) + \nu(0, 1)\} \\ &\quad \cup \{\mu(0, 1) + \nu(1, 1)\} \cup \{\mu(1, 1) + \nu(1, 0)\}, \end{aligned}$$

which is to say, an arbitrary point of the phase space \mathbf{R}^2 can be taken as the initial point, and the one-step time evolution of our ultradiscrete system for Γ_1 causes a permutation of the five components of \mathbf{R}^2 decomposed as above.

Second, we note that our ultradiscrete system for Γ_2 ,

$$E_\delta Q = P + \max(Q, P, 0) - Q, \quad E_\delta P = \max(P, 0) - Q.$$

acts to permute the set

$$\begin{aligned} \{(Q, P) \in \mathbf{Z}^2 | H_{\Gamma_2}(Q, P) = 1\} &= \partial\Gamma_2^\Delta \cap \mathbf{Z}^2 \\ &= \{(1, 0), (0, -1), (-1, 0), (1, 1)\}. \end{aligned}$$

Actually, we have

$$(1, 0) \mapsto (0, -1) \mapsto (-1, 0) \mapsto (1, 1) \mapsto (1, 0).$$

Here, we let $\Psi(Q_0, P_0)$ denote the periodic solution to our system for Γ_2 of an initial value (Q_0, P_0) in $\partial\Gamma_2^\Delta \cap \mathbf{Z}^2$. For instance,

$$\Psi(1, 0) = \begin{cases} (1, 0), & t/\delta \equiv 0 \pmod{4}, \\ (0, -1), & t/\delta \equiv 1 \pmod{4}, \\ (-1, 0), & t/\delta \equiv 2 \pmod{4}, \\ (1, 1), & t/\delta \equiv 3 \pmod{4}. \end{cases}$$

Then, the general solution to our system for Γ_2 is written as

$$\mu\Psi(Q_0, P_0) + \nu E_\delta\Psi(Q_0, P_0), \quad (16)$$

for arbitrary $\mu \geq 0$, $\nu \geq 0$ and $(Q_0, P_0) \in \partial\Gamma_2^\Delta \cap \mathbf{Z}^2$.

Both general solutions (15) and (16) are represented in the same form. In the other cases of Γ_3 , Γ_4, \dots , and so on, the general solutions are represented in the similar form without some arrangement of basic periodic solutions for superposition.

4. Summary. The method of ultradiscrete limit is applied to the discrete systems derived from the Hamiltonian systems as parametrized with lattice polygons. For every ultradiscrete system considered, the general solution is obtained from the polar set of each lattice polygon.

Comment: The first author owes his study to *JSPS Research Fellowships for Young Scientists*.

REFERENCES

1. R. Hirota, M. Iwao, A. Ramani, D. Takahashi, B. Grammaticos and Y. Ohta, From Integrability to Chaos in a Lotka-Volterra Cellular Automaton, *Phys. Lett. A* **236** (1997), 39–44.
2. D. Takahashi and J. Satsuma, A Soliton Cellular Automaton, *J. Phys. Soc. Japan* **59** (1990), 3514–3519.
3. T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, From Soliton Equations to Integrable Cellular Automata through Limiting Procedure, *Phys. Rev. Lett.* **76** (1996), 3247–3250.
4. S. Wolfram, *Cellular automata and complexity* (Addison-Wesley, 1994).
5. Günter M. Ziegler, *Lectures on polytopes* (Springer-Verlag, 1995).

