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THE r-MONOTONICITY OF GENERALIZED BERNSTEIN POLYNOMIALS

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Abstract Let $f \in C[0,1]$ and let the $B_n(f,q;x)$ be generalized Bernstein polynomials based on the q-integers that were introduced by Phillips. We prove that if f is r-monotone, then $B_n(f,q;x)$ is r-monotone, generalizing well-known results when $q=1$ and the results when $r=1$ and $r=2$ by Goodman *et al*. We also prove a sufficient condition for a continuous function to be r-monotone.

Keywords: generalized Bernstein polynomial; r-monotonicity; number of sign changes

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1. Introduction

Let $q > 0$. For any $n = 0, 1, 2, \dots$, the integer $[n]_q$ is defined as

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad n = 0, 1, 2, \dots, \qquad [0]_q = 0,$$

the q-factorial $[n]_q!$ is defined as

$$[n]_q! = [1]_q[2]_q \cdots [n]_q, \qquad n = 1, 2, \dots, \qquad [0]_q! = 1,$$

and the q-binomial coefficient $\binom{n}{k}_q$ is defined as

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

for integers $n, k, n \geqslant k \geqslant 0$.

Let $C^r[a,b]$, $r = 1, 2, \dots$, be the class of all functions $f(x)$ which are r-times continuously differentiable on $[a,b]$. $C[a,b]$ is the usual class of continuous functions on $[a,b]$.

For a non-negative integer r and $f \in C[a,b]$, the rth-order divided difference $[x_0, x_1, \dots, x_r]f$ of f at points $x_0, \dots, x_r$ is defined as

$$[x_0, x_1, \dots, x_r]f = \sum_{i=0}^{r} \frac{f(x_i)}{\prod_{j=0, j\neq i}^{r}(x_i - x_j)} = \sum_{i=0}^{r} \frac{f(x_i)}{\omega'_{r+1}(x_i)},$$

where $\omega_{r+1}(x) = \prod_{j=0}^{r}(x - x_j)$. And if the inequality

$$[x_0, x_1, \dots, x_r]f \geqslant 0$$

holds true for all choices of distinct points $x_0, x_1, \dots, x_r \in [a, b]$, then f is said to be r-monotone on $[a, b]$.

In this paper we mainly discuss the r-monotonicity of the generalized Bernstein polynomials defined by

$$B_n(f, q; x) = \sum_{k=0}^{n} f_k \binom{n}{k}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x), \tag{1.1}$$

where an empty product denotes 1, $f \in C[0, 1]$ is r-monotone and

$$f_k = f\left(\frac{[k]_q}{[n]_q}\right)$$

(see [**4**]). In § 2 we prove a sufficient condition for a continuous function to be r-monotone which is different from that in [**1**]. With the proof of the sufficient condition, we discuss the relation between the number of sign changes of an r-monotone function f and the sign-preserving properties of its rth-order divided difference. Finally, it is proved that, for all integers $n, r, n \geqslant r \geqslant 1$ and $q \in (0, 1]$, if f is r-monotone, then $B_n(f, q; x)$ is r-monotone, which is a generalization of the result relating to the classical case $q = 1$ and the result of Goodman *et al*. [**4**]. For more details of q-Bernstein polynomials, see [**7**].

2. Criterion for r-monotonicity

In [**4**], Goodman *et al*. characterized the convexity of a function $f \in C[a, b]$ by its number of sign changes. Motivated by [**4**], we shall characterize the r-monotonicity of a function $f \in C[a, b]$ by its number of sign changes. For this reason, we shall cite some results concerning the number of sign changes, which can be found, for example, in [**3**, **4**].

Definition 2.1. For any real sequence v, finite or infinite, we denote by $S^-(v)$ the number of strict sign changes in v.

Definition 2.2. For a real-valued function f on an interval I, we define $S^-(f)_I$ to be the number of sign changes of f, that is

$$S^-(f)_I = \sup S^-(f(x_0), \dots, f(x_m)), \tag{2.1}$$

where the supremum is taken over all increasing sequences $(x_0, \dots, x_m)$ in I for all m.

In [**4**], Goodman *et al*. obtained the following theorem.

Theorem 2.3. *For any function $f \in C[a, b]$,*

$$S^-(B_n(f, q))_{[0,1]} \leqslant S^-(f)_{[0,1]}. \tag{2.2}$$

The following definitions and results concerning the rth-order divided differences and r-monotonicity can be found, for example, in [**1**, **2**, **8**].

Theorem 2.4. *For a non-negative integer r and any $f \in C[a,b]$, the rth-order divided difference $[x_0, x_1, \dots, x_r]f$ has the following properties.*

(a) *$[x_0, x_1, \dots, x_r]f$ is symmetric in $x_0, x_1, \dots, x_r$.*

(b) *$[x_0, x_1, \dots, x_r]f$ is a constant if f is a polynomial of degree less than or equal to r, and is zero for a polynomial of degree less than r if $r \geqslant 1$.*

(c) *If $f \in C^r[a,b]$, $r \geqslant 1$, $x_i \in [a,b]$, $i = 0, 1, \dots, r$, $x_0 < x_1 < \cdots < x_r$, then, for some $\xi \in [x_0, x_r]$,*

$$[x_0, x_1, \dots, x_r]f = \frac{f^{(r)}(\xi)}{r!}. \tag{2.3}$$

(d) *For $x_i \in [a,b]$, $i = 0, 1, \dots, r$, $r \geqslant 1$, $x_0 < x_1 < \cdots < x_r$, we have the recurrence relation*

$$[x_0, x_1, \dots, x_r]f = \frac{[x_0, x_1, \dots, x_{r-2}, x_r]f - [x_0, x_1, \dots, x_{r-2}, x_{r-1}]f}{x_r - x_{r-1}}. \tag{2.4}$$

(e) *For $x_i \in [a,b]$, $i = 0, 1, \dots, r$, $r \geqslant 1$, $x_0 < x_1 < \cdots < x_r$, $f \in C[a,b]$, let $L_r(f,x)$ be the Lagrange interpolation polynomial of f at $x_0, x_1, \dots, x_r$. Then for any $x \in [a,b]$, $x \neq x_i$, $i = 0, 1, \dots, r$,*

$$f(x) - L_r(f,x) = [x_0, x_1, \dots, x_r, x]f\,\omega_{r+1}(x). \tag{2.5}$$

Theorem 2.5. *For a non-negative integer r and $f \in C[a,b]$, let f be r-monotone on $[a,b]$.*

(a) *When $r \geqslant 2$, $f^{(r-2)}$ exists and is convex and $f^{(r-1)}$ exists almost everywhere in (a,b).*

(b) *If $r \geqslant 1$, and $f \in C^{r-1}[a,b]$, then $f^{(r-1)}$ is increasing and the $(r-1)$th-order divided difference $[t_1, t_2, \dots, t_r]f$ is a increasing function of each of its arguments.*

Using the above results, we can characterize the r-monotonicity of function $f \in C[a,b]$ by its number of sign changes $S^-(f)_{[a,b]}$. Firstly, we have the following theorem.

Theorem 2.6. *Let $f \in C[a,b]$ be r-monotone on $[a,b]$, and integer $r \geqslant 1$. Then the inequality*

$$S^-(f - P_{r-1})_{[a,b]} \leqslant r \tag{2.6}$$

holds true for any polynomial $P_{r-1}(x)$ of degree less than or equal to $r-1$.

Proof. Suppose that there exists a polynomial $P_{r-1}(x)$ of degree less than or equal to $r-1$ such that $S^-(f-P_{r-1})_{[a,b]} \geqslant r+1$. Choose points x_i, $i=0,1,\dots,r+1$ with

$$a \leqslant x_0 < x_1 < \cdots < x_{r+1} \leqslant b$$

and so that

$$\operatorname{sgn}[f(x_i) - P_{r-1}(x_i)] = \varepsilon(-1)^i, \quad i=0,1,\dots,r+1,\ \varepsilon = \pm 1. \tag{2.7}$$

Therefore, there exist $y_i \in (x_i, x_{i+1})$, $i=0,1,\dots,r$, such that

$$f(y_i) = P_{r-1}(y_i), \quad i=0,1,\dots,r. \tag{2.8}$$

However, a unique polynomial $L_{r-1}(f,x)$ of degree less than or equal to $r-1$ exists that interpolates f at y_i, $i=0,1,\dots,r-1$. Thus, we must have

$$L_{r-1}(f,x) \equiv P_{r-1}(x).$$

By Theorem 2.4 (e), we get

$$f(x_r) - P_{r-1}(x_r) = [y_0, y_1, \dots, y_{r-1}, x_r]f \prod_{i=0}^{r-1}(x_r - y_i)$$

and

$$f(x_{r+1}) - P_{r-1}(x_{r+1}) = [y_0, y_1, \dots, y_{r-1}, x_{r+1}]f \prod_{i=0}^{r-1}(x_{r+1} - y_i).$$

Since f is r-monotone,

$$\begin{aligned}\operatorname{sgn}[f(x_r) - P_{r-1}(x_r)]\operatorname{sgn}[f(x_{r+1}) - P_{r-1}(x_{r+1})]&\\ &= \operatorname{sgn}\left[\prod_{i=0}^{r-1}(x_r - y_i)\right]\operatorname{sgn}\left[\prod_{i=0}^{r-1}(x_{r+1} - y_i)\right]\\ &> 0,\end{aligned}$$

which contradicts (2.7). This completes the proof of Theorem 2.6. □

Next, we shall investigate the sign-preserving properties of the rth-order divided difference of the function $f \in C[a,b]$ satisfying (2.6). For this we need the following lemmas.

Lemma 2.7. *Let $f \in C[a,b]$, and let $r \geqslant 1$ be integer. If the inequality*

$$S^-(f - P_{r-1})_{[a,b]} \leqslant r$$

holds true for any polynomial $P_{r-1}(x)$ of degree less than or equal to $r-1$ and there exist points $t_i \in [a,b]$, $i=0,1,\dots,r$, $t_0 < t_1 < \cdots < t_r$, such that

$$[t_0, t_1, \dots, t_r]f > 0, \tag{2.9}$$

then for any $j=0,1,\dots,r$, $x \in [a,b]$, $x \neq t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r$, we have

$$[t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r, x]f \geqslant 0.$$

Proof. For any fixed j, suppose that there exists a point

$$x_j \in [a,b], \quad x_j \neq t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r,$$

such that

$$[t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r, x_j]f < 0.$$

By (2.9) and Theorem 2.4 (b), we know that $x_j \neq t_j$ and f is not a polynomial of degree less than r.

The idea of the proof is as follows. We shall find a polynomial $P_{r-1}(x)$ of degree less than or equal to $r-1$ such that $S^-(f - P_{r-1}) \geqslant r+1$, which leads to a contradiction.

Assume that $x_j \in (t_{k-1}, t_k)$, $k = 0, 1, \dots, r+1$, where $t_{-1} = a$ (if $a < t_0$) and $t_{r+1} = b$ (if $t_r < b$). Let

$$\Omega_j(x) = (x - t_0)(x - t_1) \cdots (x - t_{j-1})(x - t_{j+1}) \cdots (x - t_r),$$

and let c be a positive number depending on j such that

$$c\biggl(\sum_{i=0,\ i\neq j}^{r} \frac{1}{|\Omega_j'(t_i)(t_i - t_j)|}\biggr) < [t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r]f, \tag{2.10}$$

and

$$c\biggl(\sum_{i=0,\ i\neq j}^{r} \frac{1}{|\Omega_j'(t_i)(t_i - x_j)|}\biggr) < |[t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r, x_j]f|. \tag{2.11}$$

We shall construct a different function $\mu(x), x \in [a,b]$ depending on the value of k, so that

$$f(t_i) - P_{r-1}(t_i), \quad i = 0, 1, \dots, r$$

and $f(x_j) - P_{r-1}(x_j)$ have $r+1$ sign alternations, where $P_{r-1}(x)$ is the Lagrange interpolation polynomial of $f(x) - \mu(x)$ at t_i, $i = 0, 1, \dots, j-1, j+1, \dots, r$, that is,

$$f(t_i) - P_{r-1}(t_i) = \mu(t_i), \quad i = 0, 1, \dots, j-1, j+1, \dots, r. \tag{2.12}$$

By the definition of the divided difference and Theorem 2.4 (e), for $x \in [a,b]$, $x \neq t_i$, $i = 0, 1, \dots, j-1, j+1, \dots, r$, we have

$$f(x) - P_{r-1}(x) = \biggl([t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r, x]f - \sum_{i=0, i\neq j}^{r} \frac{\mu(t_i)}{\Omega_j'(t_i)(t_i - x)}\biggr)\Omega_j(x). \tag{2.13}$$

Notice that

$$\operatorname{sgn}[\Omega_j(t_j)] = (-1)^{r-j}, \tag{2.14}$$

$$\operatorname{sgn}[\Omega_j(x_j)] = \begin{cases} (-1)^{r-k}, & k \leqslant j, \\ (-1)^{r-k-1}, & k > j. \end{cases} \tag{2.15}$$

Thus, if $|\mu(t_i)| = c$, $i = 0, 1, \dots, j-1, j+1, \dots, r$, then (2.10)–(2.15) imply

$$\operatorname{sgn}[f(t_j) - P_{r-1}(t_j)] = (-1)^{r-j}, \tag{2.16}$$

$$\operatorname{sgn}[f(x_j) - P_{r-1}(x_j)] = \begin{cases} (-1)^{r-k-1}, & k \leqslant j, \\ (-1)^{r-k}, & k > j. \end{cases} \tag{2.17}$$

Now, we define the function $\mu(x)$, $x \in [a, b]$, only at points t_i, $i = 0, 1, \dots, j-1, j+1, \dots, r$, respectively, in the following cases.

Case 1 ($k = j$). We define

$$\mu(t_i) = \begin{cases} (-1)^{r-i-1}c, & i \leqslant j-1, \\ (-1)^{r-i}c, & i \geqslant j+1. \end{cases}$$

Case 2 ($k = j+1$). We define

$$\mu(t_i) = \begin{cases} (-1)^{r-i}c, & i \leqslant j-1, \\ (-1)^{r-i-1}c, & i \geqslant j+1. \end{cases}$$

Case 3 ($k < j$). We define

$$\mu(t_i) = \begin{cases} (-1)^{r-i-1}c, & i \leqslant k-1, \\ (-1)^{r-i}c, & k \leqslant i \leqslant j-1, \\ (-1)^{r-i}c, & i \geqslant j+1. \end{cases}$$

Case 4 ($k > j+1$). . We define

$$\mu(t_i) = \begin{cases} (-1)^{r-i}c, & i \leqslant j-1, \\ (-1)^{r-i}c, & j+1 \leqslant i \leqslant k-1, \\ (-1)^{r-i-1}c, & i \geqslant k. \end{cases}$$

It is easy to see that in any case the numbers $f(t_0) - P_{r-1}(t_0), \dots, f(t_r) - P_{r-1}(t_r)$ and $f(x_j) - P_{r-1}(x_j)$ have $(r+1)$ sign alternations. This completes the proof of Lemma 2.7. □

Lemma 2.8. *Let $f \in C[a, b]$, and let $r \geqslant 1$ be integer. If the inequality*

$$S^-(f - P_{r-1})_{[a,b]} \leqslant r$$

holds true for any polynomial $P_{r-1}(x)$ of degree less than or equal to $r-1$ and there exist points $t_i \in [a, b]$, $i = 0, 1, \dots, r$, $t_0 < t_1 < \dots < t_r$, such that

$$[t_0, t_1, \dots, t_r]f \geqslant 0, \tag{2.18}$$

then for any $j = 0, 1, \dots, r$, $x \in [t_0, t_r]$, $x \neq t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r$, we have

$$[t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r, x]f \geqslant 0.$$

The proof is omitted as it is similar to that of Lemma 2.7.

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Remark 2.9. In Lemma 2.8, if $[t_0, t_1, \dots, t_r]f = 0$, then $f(x)$, $x \in [t_0, t_r]$, is a polynomial of degree less than or equal to $r-1$.

Indeed, considering f and $-f$, respectively, yields that

$$[t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r, x]f = 0$$

holds true for any $j = 0, 1, \dots, r$, $x \in [t_0, t_r]$, $x \neq t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r$. Let $L_{r-1}(f, x)$ be the Lagrange interpolation polynomial of f at $t_0, t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_r$. By Theorem 2.4 (e), we have

$$f(x) = L_{r-1}(f, x), \quad x \in [t_0, t_r].$$

The next result follows from Lemma 2.8.

Theorem 2.10. *Let $f \in C[a, b]$, and let $r \geqslant 1$ be integer. If the inequality*

$$S^-(f - P_{r-1})_{[a,b]} \leqslant r$$

holds true for any polynomial $P_{r-1}(x)$ of degree less than or equal to $r-1$ and there exist points $t_i \in [a, b]$, $i = 0, 1, \dots, r$, $a = t_0 < t_1 < \dots < t_r = b$, such that

$$[t_0, t_1, \dots, t_r]f \geqslant 0,$$

then f is r-monotone on $[a, b]$.

Proof. Let $x_i \in [a, b]$, $i = 0, 1, \dots, r$, with $x_0 < x_1 < \dots < x_r$. The idea of the proof is as follows. Using Lemma 2.8, we replace $t_r, t_{r-1}, \dots, t_1, t_0$ in $[t_0, t_1, \dots, t_r]f$ by $x_r, x_{r-1}, \dots, x_1, x_0$, successively, where exactly one point is changed at each step. Therefore, without loss of generality, let $x_r \in (t_{k_1-1}, t_{k_1})$, $1 \leqslant k_1 \leqslant r$. By Lemma 2.8, we have

$$[t_0, \dots, t_{k_1-1}, x_r, t_{k_1}, \dots, t_{r-1}]f \geqslant 0. \tag{2.19}$$

In this case, if we define

$$\begin{aligned} t_i^{(1)} &= t_i, & i &\leqslant k_1 - 1, \\ t_{k_1}^{(1)} &= x_r, \\ t_i^{(1)} &= t_{i-1}, & i &\geqslant k_1 + 1, \end{aligned}$$

then (2.19) becomes

$$[t_0^{(1)}, \dots, t_r^{(1)}]f \geqslant 0.$$

Let $x_{r-1} \in (t_{k_2-1}^{(1)}, t_{k_2}^{(1)})$, $1 \leqslant k_2 \leqslant r$. By Lemma 2.8 again, we have

$$[t_0^{(1)}, \dots, t_{k_2-1}^{(1)}, x_{r-1}, t_{k_2}^{(1)}, \dots, t_r^{(1)}]f \geqslant 0,$$

and we continue in this way to derive the inequality

$$[t_0, x_1, \dots, x_r]f \geqslant 0.$$

Finally, by Lemma 2.8, we get

$$[x_0, x_1, \ldots, x_r]f \geqslant 0,$$

which implies that f is r-monotone on $[a, b]$. This completes the proof of Theorem 2.10. □

The following theorem is a consequence of Lemma 2.7 and Theorem 2.10.

Theorem 2.11. *Let $f \in C(I)$, $I = (a, b)$ or $I = R$ and let $r \geqslant 1$ be integer. If the inequality*

$$S^-(f - P_{r-1})_I \leqslant r$$

holds true for any polynomial $P_{r-1}(x)$ of degree less than or equal to $r - 1$ and there exist points $t_i \in I$, $i = 0, 1, \ldots, r$, $t_0 < t_1 < \cdots < t_r$, such that

$$[t_0, t_1, \ldots, t_r]f > 0,$$

then f is r-monotone in I.

Proof. Let $x_i \in I$, $i = 0, 1, \ldots, r$, with $x_0 < x_1 < \cdots < x_r$. If $x_i \in [t_0, t_r]$, $i = 0, 1, \ldots, r$, then it follows from Theorem 2.10 that

$$[x_0, x_1, \ldots, x_r]f \geqslant 0.$$

If $t_i \in [x_0, x_r]$, $i = 0, 1, \ldots, r$, then

$$[x_0, x_1, \ldots, x_r]f \geqslant 0,$$

for otherwise Theorem 2.10 with $-f$ yields

$$[t_0, t_1, \ldots, t_r]f \leqslant 0,$$

which contradicts the assumption $[t_0, t_1, \ldots, t_r]f > 0$. Therefore, without loss of generality, let $x_0 < t_0$ and $x_r < t_r$. In this case, by Lemma 2.7, we have

$$[x_0, t_1, \ldots, t_r]f \geqslant 0.$$

It follows from this and Theorem 2.10 that

$$[x_0, x_1, \ldots, x_r]f \geqslant 0.$$

This completes the proof of Theorem 2.11. □

For $f \in C^r[a, b]$, $r \geqslant 1$, we have the following theorem.

Theorem 2.12. *Let $f \in C^r[a, b]$, and let $r \geqslant 1$ be integer. If the inequality*

$$S^-(f - P_{r-1})_{[a,b]} \leqslant r$$

holds true for any polynomial $P_{r-1}(x)$ of degree less than or equal to $r - 1$, and there exist point $x_0 \in [a, b]$ such that $f^{(r)}(x_0) > 0$, then, for any $x \in [a, b]$, $f^{(r)}(x) \geqslant 0$, and hence f is r-monotone on $[a, b]$.

Proof. Suppose that there exists a point $x_1 \in [a,b]$ such that $f^{(r)}(x_1) < 0$. Then there exists $\delta > 0$ such that $f^{(r)}(x) < 0$ for any $x \in (x_1-\delta, x_1+\delta) \cap [a,b]$. Therefore, taking points $t_i \in (x_1-\delta, x_1+\delta) \cap [a,b]$, $i = 0,1,\dots,r$, $t_0 < t_1 < \cdots < t_r$, from Theorem 2.4 (c) we have

$$[t_0, t_1, \dots, t_r]f < 0.$$

It follows from Theorem 2.11 with $-f$ and Theorem 2.5 (b) that $f^{(r)}(x) \leqslant 0$ for any $x \in [a,b]$, which contradicts the assumption $f^{(r)}(x_0) > 0$. This completes the proof of Theorem 2.12. □

3. The r-monotonicity of generalized Bernstein polynomials

In [**4**], Goodman *et al*. proved the following theorem.

Theorem 3.1. *Let $f \in C[0,1]$, $q \in (0,1]$. If f is increasing on $[0,1]$, then $B_n(f,q;x)$ is increasing on $[0,1]$, and if f is convex on $[0,1]$, then $B_n(f,q;x)$ is convex on $[0,1]$.*

In this section, we shall prove the following theorem, which generalizes Theorem 3.1.

Theorem 3.2. *Let $f \in C[0,1]$, $q \in (0,1]$. For positive integers n, r, with $n \geqslant r$, if f is r-monotone on $[0,1]$, then $B_n(f,q;x)$ is r-monotone on $[0,1]$.*

To prove Theorem 3.2 we need the following lemma.

Lemma 3.3. *For $f \in C[0,1]$, $q \in (0,1]$ and positive integer n, let $x_i = [i]_q/[n]_q$, $i = 0,1,\dots,n$, and let*

$$\Delta^k f = \sum_{i=0}^{k} (-1)^{k-i} q^{(k-i)(k-i-1)/2} \binom{k}{i}_q f_i \tag{3.1}$$

*denote the kth q-difference of f at points $x_0, x_1, \dots, x_k$, $k \leqslant n$, where $f_i = f(x_i)$ [**4**, (2.1)]. Then we have the following formula:*

$$\Delta^k f = \frac{[k]_q!}{[n]_q^k} q^{k(k-1)/2} [x_0, x_1, \dots, x_k] f. \tag{3.2}$$

This is a slight modification of Theorem 1.5.1 in [**6**].

Proof of Theorem 3.2. It is easy to see from [**4**, (2.4)] that $B_n(e_i,q;x)$, $i = 0,1,\dots,r-1$, are linearly independent, where $e_i(x) = x^i$, $i = 0,1,\dots,r-1$. Therefore, for any polynomial $P_{r-1}(x)$ of degree less than or equal to $r-1$, there exists a unique polynomial $\tilde{P}_{r-1}(x)$ of degree less than or equal to $r-1$ such that

$$P_{r-1}(x) = B_n(\tilde{P}_{r-1}, q; x).$$

If f is r-monotone on $[0,1]$, then Theorems 2.3 and 2.6 yield

$$\begin{aligned} S^-(B_n(f,q) - P_{r-1}) &= S^-(B_n(f - \tilde{P}_{r-1}, q)) \\ &\leqslant S^-(f - \tilde{P}_{r-1}) \\ &\leqslant r. \end{aligned} \tag{3.3}$$

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On the other hand, it follows from [**4**, (2.2)] (see also [**5**]) that

$$B_n(f,q;x)=\sum_{i=0}^{n}\binom{n}{i}_q \Delta^i f x^i.$$

By virtue of (3.2), this gives

$$B_n^{(k)}(f,q;0)=k!\binom{n}{i}_q \Delta^k f = k!\binom{n}{i}_q \frac{[k]_q!}{[n]_q^k} q^{k(k-1)/2}[x_0,x_1,\dots,x_k]f, \quad k=0,1,\dots,n. \tag{3.4}$$

Thus, if $f\in C[0,1]$ is r-monotone, then $B_n^{(r)}(f,q;0)\geqslant 0$. Let us write

$$F_k(x)=[x_0,x_1,\dots,x_k,x]f \tag{3.5}$$

for $x\in[0,1]$, $x\neq x_i$, $i=0,1,\dots,k$. Then, from the definition of the divided difference,

$$[x_0,x_1,\dots,x_k]f=[x_r,x_{r+1},\dots,x_k]F_{r-1} \tag{3.6}$$

holds true for any k, $r\leqslant k\leqslant n$.

If $B_n^{(r)}(f,q;0)>0$, then it follows from Theorem 2.11 that $B_n(f,q;x)$ is r-monotone on $[0,1]$.

If $B_n^{(r)}(f,q;0)=0$, then (3.4) gives $[x_0,x_1,\dots,x_r]f=0$. By (3.5) and (3.6) we have

$$\begin{aligned}[x_0,x_1,\dots,x_{r+1}]f&=[x_r,x_{r+1}]F_{r-1}\\ &=\frac{[x_0,x_1,\dots,x_{r-1},x_{r+1}]f}{x_{r+1}-x_r}\\ &\geqslant 0.\end{aligned} \tag{3.7}$$

In this case, if $[x_0,x_1,\dots,x_{r+1}]f>0$, then (3.4) gives $B_n^{(r+1)}(f,q;0)>0$, and there exists $\delta>0$ such that $B_n^{(r+1)}(f,q;x)>0$, $x\in(0,\delta)$, which implies that there exists a point $t\in(0,\delta)$ such that $B_n^{(r)}(f,q;t)>0$. Thus, it follows from Theorem 2.12 that $B_n(f,q;x)$ is r-monotone on $[0,1]$. If $[x_0,x_1,\dots,x_{r+1}]f=0$, then $B_n^{(r+1)}(f,q;0)=0$, and (3.5) and (3.6) give

$$\begin{aligned}[x_0,x_1,\dots,x_{r+2}]f&=[x_r,x_{r+1},x_{r+2}]F_{r-1}\\ &=\frac{[x_0,x_1,\dots,x_{r-1},x_{r+2}]f}{(x_{r+1}-x_r)(x_{r+2}-x_{r+1})}\\ &\geqslant 0.\end{aligned} \tag{3.8}$$

Continuing the process, we have either $B_n^{(k)}(f,q;0)=0$, $k=r,r+1,\dots,m-1$, and $B_n^{(m)}(f,q;0)>0$ for some $n\geqslant m\geqslant r$, or $B_n^{(k)}(f,q;0)=0$ for $k=r$, $r+1,\dots,n$. In the case when $B_n^{(k)}(f,q;0)=0$, $k=r,r+1,\dots,m-1$, and $B_n^{(m)}(f,q;0)>0$ for some $n\geqslant m\geqslant r$, there exists $\delta>0$ such that $B_n^{(m)}(f,q;x)>0$ for $x\in(0,\delta)$. Then Taylor's Formula yields

$$B_n^{(r)}(f,q;x)=\sum_{k=0}^{m-r-1}\frac{B_n^{(k+r)}(f,q;0)}{k!}x^k+\frac{B_n^{(m)}(f,q;\xi)}{(m-r)!}x^{m-r}=\frac{B_n^{(m)}(f,q;\xi)}{(m-r)!}x^{m-r}, \tag{3.9}$$

where $x \in (0,\delta)$, $\xi \in (0,x)$, implies that there exists a point $t \in (0,\delta)$ such that $B_n^{(r)}(f,q;t) > 0$, which shows that $B_n(f,q;x)$ is r-monotone on $[0,1]$. In the case when $B_n^{(k)}(f,q;0) = 0$, $k = r, r+1, \dots, n$, it follows from (3.2) and (3.4) that

$$B_n(f,q;x) = \sum_{i=0}^{r-1} \begin{pmatrix} n \\ i \end{pmatrix}_q \Delta^i f x^i,$$

which implies that $B_n(f,q;x)$ is a polynomial of degree less than or equal to $r-1$, and hence is r-monotone on $[0,1]$. This completes the proof of Theorem 3.2. □

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