Locus problems concerning centroids of a cyclic quadrilateral and two classic cubic curves

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On his website dedicated to questions and investigations arising out of dynamic geometry technology, Michael de Villiers has a series called Geometry Loci Doodling [1]. These are locus problems connected to the centroids of cyclic quadrilaterals – 'centroids' in the plural, for there are three different kinds of centroid depending whether one understands the quadrilateral in terms of its vertices, perimeter or area. The corresponding centroids are the point-mass centroid, the perimeter-centroid, and the lamina-centroid. In each case, de Villiers keeps three vertices of the quadrilateral fixed on the circumcircle, and then traces the locus of the different centroids as the fourth point moves round the circle. In this paper, I shall take a brief look at the point-mass centroid and then a lingering view of the lamina-centroid.

The case of the point-mass centroid

What the locus is in the case of the point-mass centroid is easy to answer. It is another circle. The proof, though, shows that this is only a special case of more general result. For, keeping in mind that the point-mass centroid is just the intersection of the lines joining the midpoints of the opposite sides, we can easily show the following:

> Suppose $ABCD$ is an arbitrary quadrilateral, D_2DD_1 is an arbitrary curve, and W, X, Y, Z are the midpoints of the sides AB, BC, CD, DA respectively. If A, B, C are fixed (so that W and X are also fixed), then as D moves along the curve D_2DD_1 the intersection, P, of WY and XZ traces a curve similar and parallel to D_2DD_1 and one-fourth its linear dimension (see Figure 1).

Proof: Consider the various positions of the line AD. Since Z is always the midpoint of AD, the map taking D to Z is a contraction with respect to the centre A, so that the figure Z_2ZZ_1 is similar and parallel to D_2DD_1 and half the size.

Recall that the figure formed by joining the midpoints of the sides of a quadrilateral is a parallelogram (the 'Varignon parallelogram'). The lines joining the midpoints of opposite sides of the quadrilateral are the diagonals of the parallelogram and thus bisect one another. Therefore point P is the midpoint of XZ for every position of Z as D moves along the curve. Hence we have a second contraction with respect to centre X, so that the curve traced by P is similar and parallel to that through Z and half the size. Therefore the curve traced by P is similar and parallel to the original figure and $\frac{1}{4}$ the linear dimension.

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In particular, where the original curve is a circle, so that the quadrilateral *ABCD* is cyclic, *P* traces out a circle whose radius is $\frac{1}{4}$ the radius of the circumcircle.

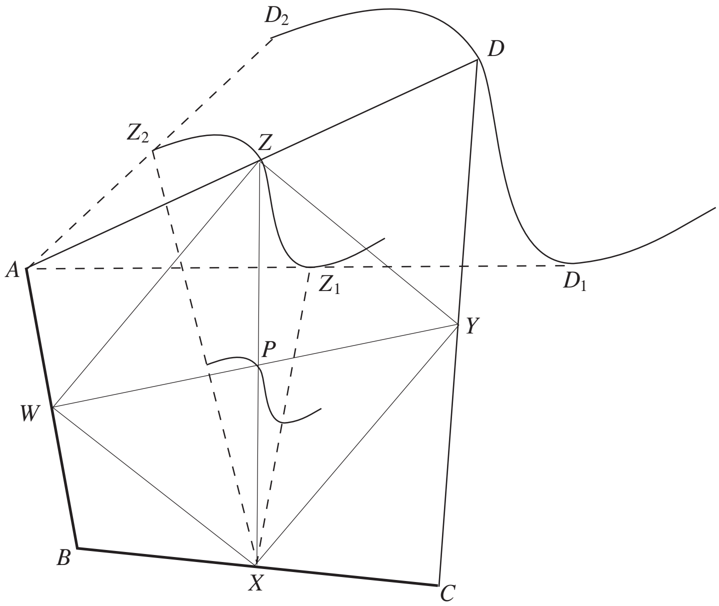

FIGURE 1

The case of the lamina-centroid

In considering the lamina-centroid, by contrast, we will have to stay within the bounds of cyclic quadrilaterals. Even so, this case is in many ways more interesting than the point-mass centroid, not only for the problem itself, but also, as often happens in problem-solving, for the things that the come along the way.

Let us first state the problem: suppose *ABCD* is a cyclic quadrilateral and *U* is its centroid. Let *A*, *B*, *C* be fixed, and let *D* move along the circle circumscribing *ABCD*. What is the curve traced out by *U*?

We shall show that in the special case where *ABC* is a right-isosceles triangle, the locus is the curve known as a *right strophoid*. Moreover, at the end of this paper, we shall show that in a limiting case, namely, where *A* coincides with *C* while *ABC* is isosceles (so that *AB* becomes a diameter in the limit), the locus will be the *cissoid of Diocles*.

Right strophoid

A strophoid curve is defined as follows (for this and the cissoid, more details can be found in [2]). Let a straight line ℓ, a fixed point *H* on ℓ,* and

* A more general definition allows for ℓ to be any curve and *H* any fixed point.

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another point O be given, and consider a straight line m through O intersecting ℓ at point K. The strophoid is the locus of points P and P' on m such that $HK = KP = KP'$. In the case where OH is perpendicular to ℓ the strophoid is said to be right (Figure 2). The point O is called the pole of the strophoid. It is very clear that the right strophoid has an asymptote parallel to l and at distance of HO from it.

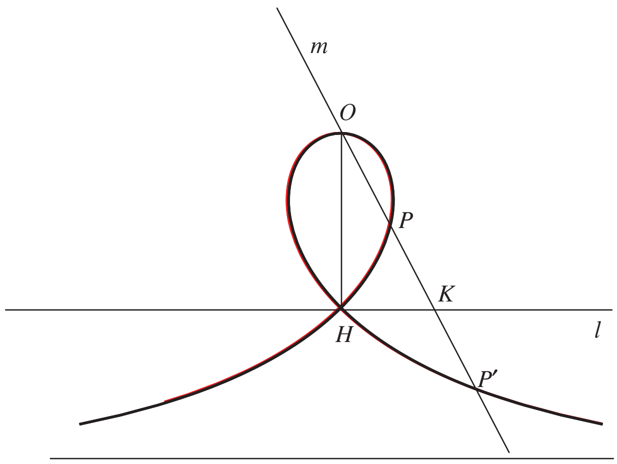

FIGURE 2

Centroid of quadrilaterals and centroid quadrilaterals

Consider now the centroid — more precisely the *lamina centroid* — of a quadrilateral (for more about this, see [3]). Let $ABCD$ be the quadrilateral, and let E, G and H, F denote respectively the centroids of triangles BCD, $DABE$, produced by diagonal BD, and ABC, CDA produced by diagonal AC (see Figure 3). We shall refer to the quadrilateral $EFGH$ as the *centroid quadrilateral* of $ABCD$. The centroid of $ABCD$ must lie along the line joining E and G and similarly along that joining H and F. The lamina centroid of $ABCD$ is, therefore, the point of intersection of the diagonals of its centroid quadrilateral. We shall use this as the formal definition of the lamina centroid of a quadrilateral even in the case of a self-intersecting quadrilateral where one can no longer give the lamina centroid its physical interpretation.*

We observe the following, which will be important afterwards.

Theorem: A quadrilateral is similar to its centroid-quadrilateral.

Proof: Let $ABCD$ be the quadrilateral and let $EFGH$ be its centroid-quadrilateral (Figure 3).

* I thank the anonymous referee for pointing out the necessity of making this clear.

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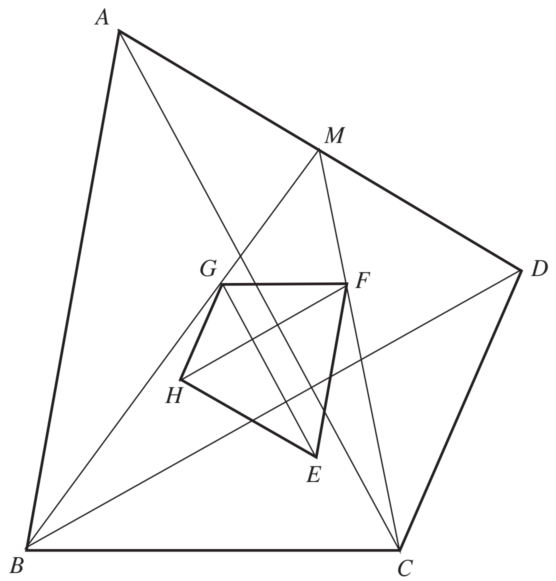

FIGURE 3

Let M be the midpoint of side AD of the quadrilateral, and draw MB and MC. These lines, will pass through the centroid G of ABD and the centroid F of DAC, so that $\frac{MG}{MB} = \frac{MF}{MC} = \frac{1}{3}$. Therefore

$$GF \,//\, BC \text{ and } GF = \frac{1}{3}BC.$$

Similarly

$$GH \,//\, DC \text{ and } GH = \frac{1}{3}DC,$$

$$HE \,//\, AD \text{ and } HE = \frac{1}{3}AD,$$

$$EF \,//\, BA \text{ and } EF = \frac{1}{3}BA.$$

Since the sides of *EFGH* are parallel to the corresponding sides in *ABCD* the corresponding angles are equal. But also the sides are in the same ratio, namely 1 : 3. Therefore *EFGH* is similar to *ABCD*.

The following two diagrams (Figure 4) show that our results hold also for non-convex and self-intersecting planar quadrilaterals.

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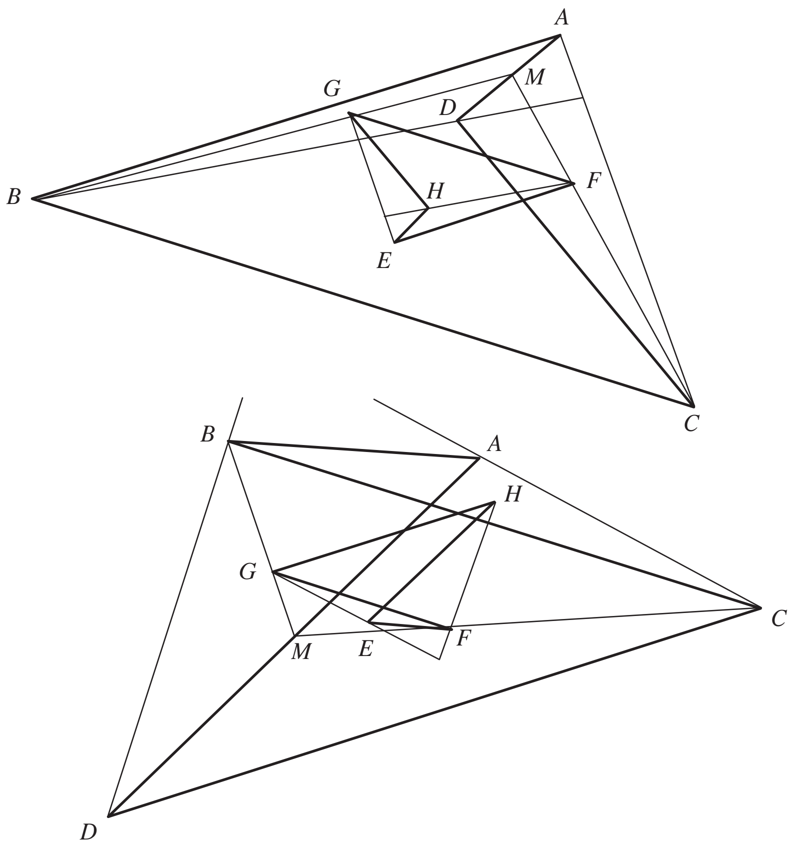

FIGURE 4

Corollary: The centroid-quadrilateral is related to the quadrilateral by a dilatation.

Proof: They are similar and the sides are parallel.

The corollary implies, among other things, that the corresponding diagonals of the quadrilateral and its centroid-quadrilateral are parallel and that if *AE*, *BF*, *CG*, *DH* are drawn they will meet at a point, namely the centre of dilatation.

The locus of the lamina-centroid of a cyclic quadrilateral: a special case

Suppose *ABCD* is a cyclic quadrilateral, *EFGH* its centroid-quadrilateral, and *U* its centroid, which, as we have defined it, is the intersection of the diagonals of *EFGH*.

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Let A, B, C be fixed, and let D move round the circle circumscribing $ABCD$ (Figure 5). We wish to show that if ABC is an isosceles right triangle (right angle at B) the curve traced out by U is a right strophoid. The pole of the right strophoid will be O and fixed line l will be the line through H perpendicular to OB. The fixed point on l will be H, the centroid of ABC.

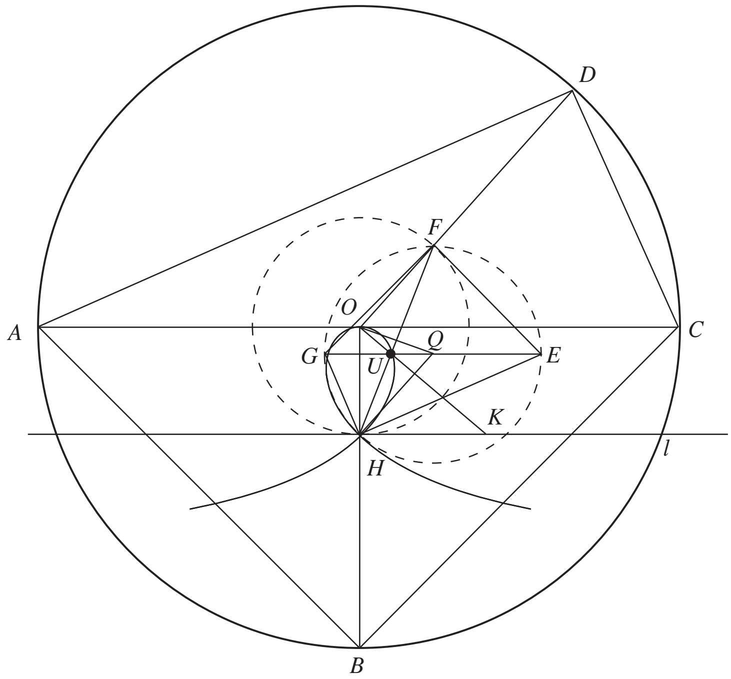

FIGURE 5

The points F and H lie on a circle whose radius is one-third the radius of the circumscribing circle, since $OF = \frac{1}{3}OD = \frac{1}{3}OB = OH$, OB and OD being radii.

Moreover, since the centroid-quadrilateral $EFGH$ is similar to $ABCD$ which is cyclic, it is cyclic as well, and, from what we demonstrated above, its linear dimensions are one-third those of $ABCD$. Therefore the circle circumscribing $EFGH$ with centre Q is congruent to the circle FH with centre O. FH is thus the perpendicular bisector of the segment joining the centres O and Q.

Join OU and extend it to K on l the line perpendicular to OB through H. Since FH is the perpendicular bisector of OQ, it follows that $\angle GUH = \angle KUH$. But $GE \parallel AC \parallel HK$ (HK is ℓ). Therefore

$$\angle KUH = \angle GUH = \angle KHU,$$

so that $KU = HK$. Hence U lies on a right strophoid with pole O, fixed line l, fixed point H on l.

The quadrilateral in Figure 5 is convex since D and B are on opposite sides of the diameter AC. If one adheres strictly to the lettering in the steps above, one observes that the proof is unchanged for the case in which D and B are on the same side of AC, that is, where the quadrilateral is self-intersecting (see Figure 6).

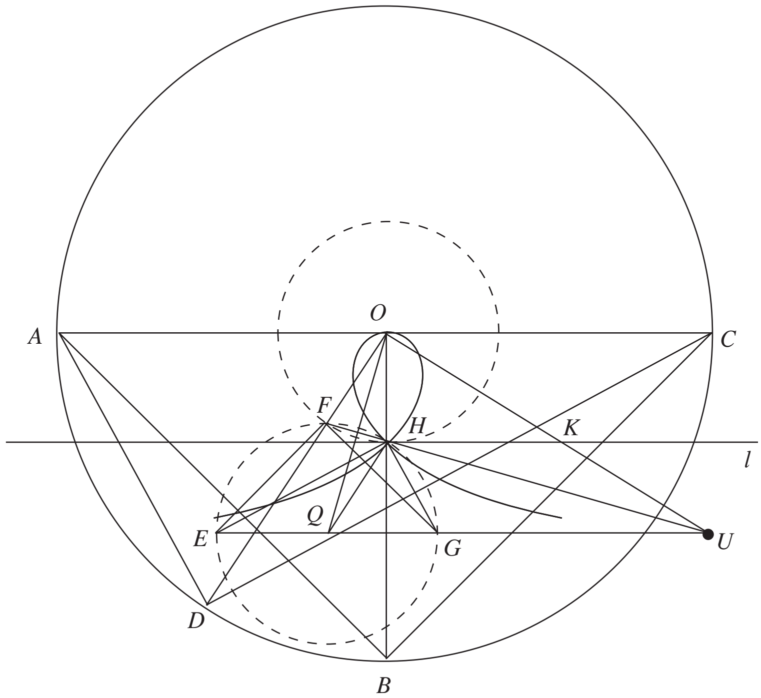

FIGURE 6

One more special case: the cissoid of diocles

The curve known as the cissoid may be defined as follows. Let circle O and point H on its circumference be given (Figure 7). Let ℓ be a line perpendicular to HO. The curve is then generated as follows. Let line m be drawn through H, and suppose m cuts the circle again at A and the line ℓ at B. As m rotates about H, the locus of points P on m such that $HP = AB$ is a cissoid. What kind of cissoid it is depends on the position of ℓ. If ℓ passes through the centre O, it can be shown that the cissoid is in fact the right strophoid discussed above (see the appendix). If the line ℓ is tangent to the circle at the point opposite to H, then the cissoid is the cissoid of Diocles.

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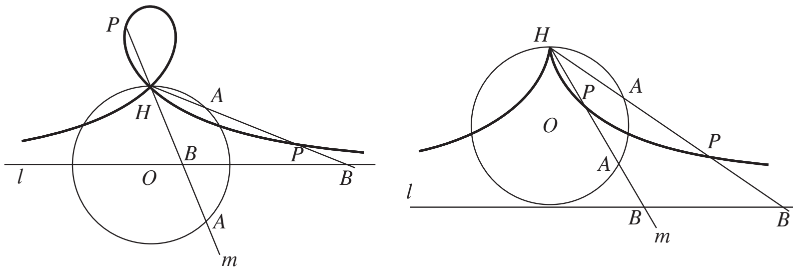

Right strophoid Cissoid of Diocles

FIGURE 7

With that, let us consider one more special case for the locus of the quadrilateral centroid, though our approach will be somewhat heuristic. Using the same notations as sections 2 and 3 above, we consider the case in which $AB = BC$, as in the last case, but where A approaches C so that in the limit $AB = CB$ = diameter of circle O. In this limiting case, the locus of the centroid U will be the cissoid of Diocles.

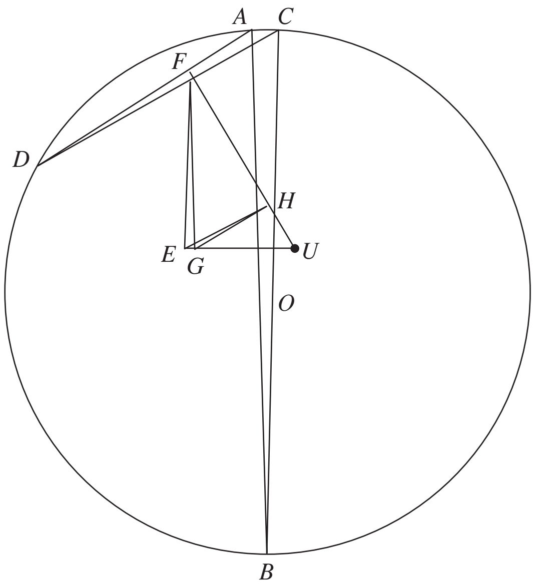

FIGURE 8

As we allow A to approach C, keeping always $AB = CB$, the line joining AC will approach the tangent at A (or C) and the line EGU will remain parallel to AC or to any line perpendicular to the diameter AB (see Figure 8). Of course when A approaches C also E will approach G. The lines EF and EH will remain parallel to AB and DA respectively.

In the limit (Figure 9), the point F will be a point on the line AD such that $AF = \frac{1}{3}DC$, and the point H will be a point on the line AB such that

$AH = \frac{1}{3}BA$, so that FH will be parallel to DB and perpendicular to EH.

In all positions E will remain the centroid of DBC, which becomes the same as DBA. So that in the limit E will be a point along the radius OD such that $OE = \frac{1}{3}OD$ where, as before, O is the centre of the circle. So, as in our first special case, E will trace out a circle concentric with the original circle and having its linear dimensions one-third those of the original circle – a simple central contraction of the original circle.

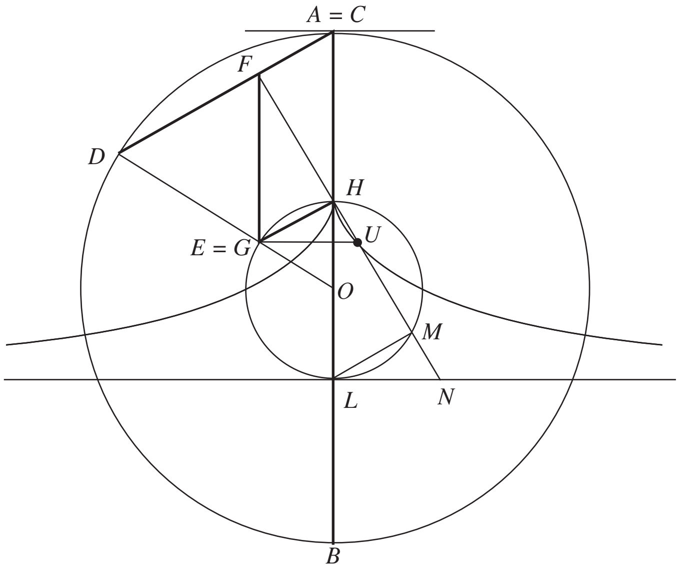

FIGURE 9

Now draw LN tangent to this contracted circle at L along AB and extend FH through H to N. Let this line intersect the contracted circle at M. The lines EH and LM are parallel since they are both perpendicular to HM, and they are equal since they have been drawn from either end of the diameter HL. Also, since EU is parallel to LN, $\angle HEU = \angle MLN$ so that the triangles HEU and MLN are congruent. Therefore $HU = MN$. And this is true for any position of D and, accordingly, any position of the line HN through the fixed point H. Thus, by our definition above, the point U traces out the cissoid of Diocles with its cusp at H and its asymptote LN.

Concluding remark

Both the cissoid of Diocles and the right strophoid are cubic curves: the cissoid can be given algebraically by an equation such as $x(x^2 + y^2) + 2ay^2 = 0$ and the right strophoid by $x^2(x - a) + y^2(x + a) = 0$. The algebraic study of these curves, essentially the study of elliptic curves, is a fascinating area of mathematics, but it is a relatively advanced one. Everything in this paper, on the other hand, is quite elementary and purely geometric. These kinds of locus problems and the dynamic geometry programs that suggest them can, accordingly, serve not only a source of interesting problems for high school students but also as a door to areas of higher mathematics. It is in that double role that it is hoped that the material in this paper will be of value both to students and their teachers.

Appendix

We remarked that the right strophoid was a special case of a cissoid. In effect we have given two definitions of the strophoid.

Definition 1 of the right strophoid:

O is a fixed point; HK is a fixed line perpendicular to OH. A line OK moves about the fixed point O and intersects HK at K. The right strophoid is the locus of points P_1 and P_2 such that $HK = KP_1 = KP_2$ (Figure 10).

Definition 2 of the right strophoid as a kind of cissoid:

H is a fixed point; HA_1A_2 is a circle with radius OH, B_1B_2 passes through the centre of HA_1A_2 and is parallel to HL. A line HB moves about the fixed point H on the circle and intersects the circle at a second point A and the line at a point B. The right strophoid is the locus of points M such that $HP = AB$.

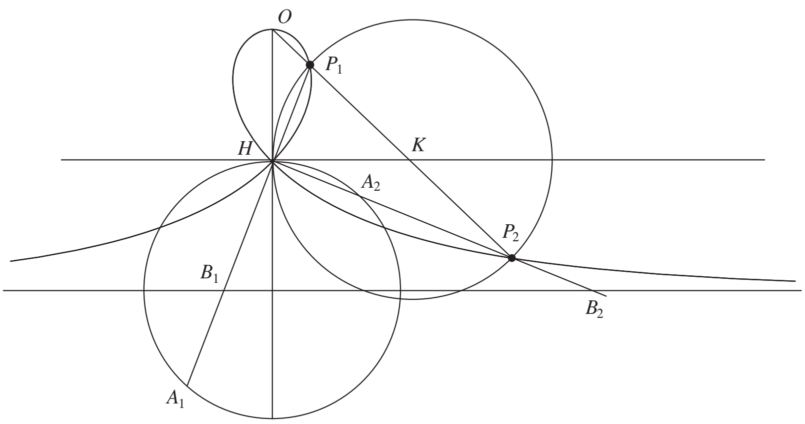

FIGURE 10

We ought to show that these two definitions are truly equivalent.

To start, suppose that $HK = KP_1 = KP_2$. We need to show that $HP_1 = A_1B_1$ and $HP_2 = A_2B_2$. Draw a circle with centre O with radius OH, and draw $B_1'OB_2'$ through O and parallel to B_1B_2 (Figure 11). Extend B_2H to B_2' on $B_1'OB_2'$ and let it intersect the circle again at A_2'. Extend A_1H to A_1' and let it intersect the line $B_1'OB_2'$ at B_1'.

Clearly, by symmetry, $A_1B_1 = A_1'B_1'$ and $A_2B_2 = A_2'B_2'$. Draw OC_2 perpendicular to $A_2'H$ so that it also bisects $A_2'H$. Since triangle HKP_2 is isosceles by assumption, $B_2'OP_2$ is also isosceles. Therefore $B_2'C_2 = C_2P_2$ so

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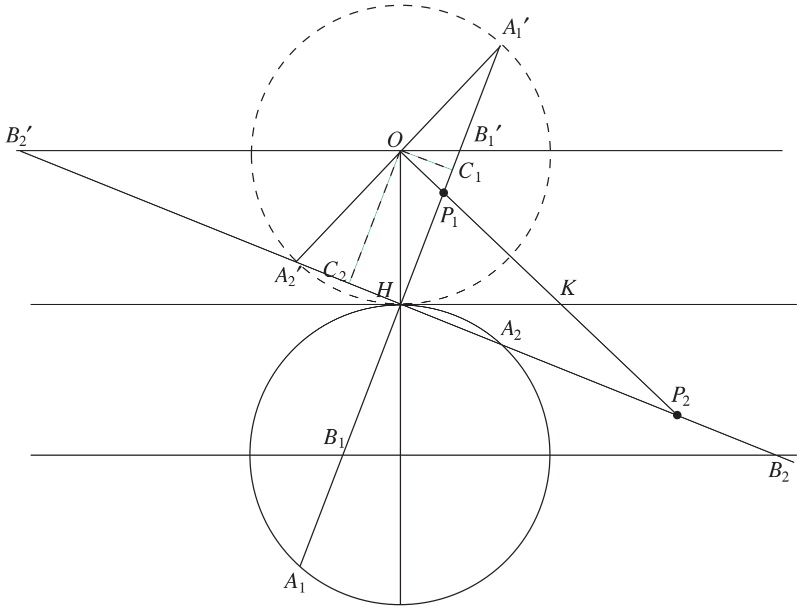

FIGURE 11

that, since $A_2'C_2 = C_2H$, we have $HP_2 = A_2'B_2'$. And since $A_2B_2 = A_2'B_2'$, we have $HP_2 = A_2B_2$. By a similar argument, we have also $HP_1 = A_1B_1$.

Now, assuming the second definition, draw HB_1 and HB_2 perpendicular to one another and intersecting the circle HA_1A_2 at A_1 and A_2. Thus, if P_1 and P_2 are on the locus, $HP_2 = A_2B_2$ and $HP_1 = A_1B_1$.

Using the same construction as above, we have, again, $A_1B_1 = A_1'B_1'$ and $A_2B_2 = A_2'B_2'$, and, as before, if OC_1 and OC_2 are drawn perpendicular to $A_1'H$ and $A_2'H$ respectively, they bisect those segments. Hence $B_2'C_2 = C_2P_2$ and $B_1'C_1 = C_1P_1$, so that the triangles $B_2'OP_2$ and $B_1'OP_1$ are isosceles. Therefore KHP_1 and HKP_2 are isosceles triangles. Therefore $HK = KP_1 = KP_2$.

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