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UNIFORMLY CLOSED ALGEBRAS GENERATED BY BOOLEAN ALGEBRAS OF PROJECTIONS IN LOCALLY CONVEX SPACES

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Introduction and the main result. The theory of operator algebras in Banach spaces generated by Boolean algebras of projections is by now well known. It is systematically exposed in the penetrating studies of W. Bade, [**1**], [**2**] and [**6**, Chapter XVII]. Many of these results, a priori independent on normability of the underlying space, have recently been extended to the setting of locally convex spaces; see [**3**], [**4**], [**5**], [**11**] and [**15**], for example.

However, one of Bade's fundamental results, stating that the closed algebra generated by a complete Boolean algebra in the uniform operator topology is the same as the closed algebra that it generates in the weak operator topology, has remained remarkably resistant in attempts to extend it to locally convex spaces. Recently however, a class of Boolean algebras in non-normable spaces, called boundedly σ-complete Boolean algebras, was exhibited in which the analogue of Bade's result is valid, [**14**; Theorem 5.3]. In this note another class of Boolean algebras is presented, overlapping with but distinct from the boundedly σ-complete Boolean algebras, for which the weakly closed algebra generated by any Boolean algebra in this class coincides with the closed algebra that it generates with respect to the topology of uniform convergence on bounded sets (cf. the theorem below).

The above mentioned result is based on a relatively successful technique which is often used in the study of continuous linear operators on locally convex spaces, namely to realize the given locally convex space as a suitable projective limit of Banach spaces. It is then possible to use the well developed theory for operators in Banach spaces.

Let X be a locally convex Hausdorff space with continuous dual space X' and $L(X)$ denote the space of all continuous linear operators from X into itself. Then $L_s(X)$ and $L_b(X)$ denote $L(X)$ equipped with the topology of pointwise convergence in X and the topology of uniform convergence on bounded sets of X, respectively. It will always be assumed that X is quasicomplete and $L_s(X)$ is sequentially complete.

The concept of a Boolean algebra of projections is not a priori

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connected with normability of the topology of the vector space on which the algebra acts; the definition usually given in Banach spaces can be extended to locally convex spaces in a straight-forward way. If $\mathscr{A} \subseteq L(X)$ is a Boolean algebra, then $\langle \mathscr{A} \rangle_s$ and $\langle \mathscr{A} \rangle_b$ denote the closed algebra generated by $\mathscr{A}$ in $L_s(X)$ and $L_b(X)$, respectively. Since $\langle \mathscr{A} \rangle_s$ is the closure in $L_s(X)$ of the linear hull of $\mathscr{A}$ (which is a convex subset of $L(X)$) it follows that $\langle \mathscr{A} \rangle_s$ is also the closed algebra generated by $\mathscr{A}$ with respect to the weak operator topology in $L_s(X)$. A Boolean algebra $\mathscr{A} \subseteq L(X)$ is said to be *equicontinuous* if it is an equicontinuous subset of $L(X)$. The notions of σ-completeness and completeness of a Boolean algebra used by Bade in [**1**] are topological and algebraic, and consequently extend themselves immediately to the locally convex setting. Just as in the Banach space situation, a σ-complete or complete, equicontinuous Boolean algebra in $L(X)$ may be realized as the range of an $L(X)$-valued spectral measure (for example, on the Baire or Borel sets respectively of its Stone space, [**15**; Proposition 1.3]).

Let $\mathscr{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra and $\mathscr{A}_1$ denote the closure in $L_s(X)$ of the set of all operators of the form $\sum_{i=1}^n \alpha_i A_i$, where n is a positive integer, each α_i, $1 \leqq i \leqq n$, is a complex number satisfying $|\alpha_i| \leqq 1$ and

$$\{A_i\}_{i=1}^n \subseteq \mathscr{A}$$

is a set of mutually disjoint projections. Then it turns out that $\mathscr{A}_1$ is an equicontinuous subset of $L(X)$; see Lemma 3. Accordingly, if $\{V_j; j \in \mathscr{J}\}$ is any neighbourhood basis of zero in X consisting of closed, convex and balanced sets, then for each index $j \in \mathscr{J}$ the set

$$(1) \qquad W_j = \cap\{A^{-1}(V_j); A \in \mathscr{A}_1\}$$

is also a closed, convex and balanced neighbourhood of zero in X. If $q^{(j)}$ denotes the Minkowski gauge functional of W_j, for each $j \in \mathscr{J}$, then the family of seminorms $\{q^{(j)}; j \in \mathscr{J}\}$ determines the topology of X. It turns out that the family of seminorms $\{q^{(j)}; j \in \mathscr{J}\}$, called $\mathscr{A}$-compatible in [**15**], has some rather special properties. Indeed, each subspace

$$(2) \qquad N_j(\mathscr{A}) = \{x \in X; q^{(j)}(x) = 0\}, \quad j \in \mathscr{J},$$

is invariant for each operator $T \in \langle \mathscr{A} \rangle_s$; see Section 2. Accordingly, if X_j, $j \in \mathscr{J}$, denotes the quotient space of X modulo the closed subspace $N_j(\mathscr{A})$ and equipped with the quotient norm, then each operator $T \in \langle \mathscr{A} \rangle_s$ induces a family of operators

$$(3) \qquad T_j: X_j \to X_j, \quad j \in \mathscr{J},$$

in the obvious way. The Boolean algebra $\mathscr{A}$ is said to be *projectively extendable* if, for each $T \in \langle \mathscr{A} \rangle_s$, each of the induced operators specified by (3) is continuous: this depends only on the Boolean algebra $\mathscr{A}$ and not on the particular neighbourhood basis of zero, $\{V_j; j \in \mathcal{J}\}$, that we begin with.

The aim of this note is to establish the following

THEOREM. *Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $\mathcal{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra which is projectively extendable. Then $\langle \mathcal{A} \rangle_b$ and $\langle \mathcal{A} \rangle_s$ are equal as linear subspaces of $L(X)$ and, in particular, $\langle \mathcal{A} \rangle_b$ coincides with the closed algebra generated by $\mathcal{A}$ with respect to the weak operator topology in $L_s(X)$.*

The organization of this note is as follows. In Section 1 we introduce those concepts from the theory of spectral measures which are needed in the sequel. In particular, we show that the set $\mathcal{A}_1$ is equicontinuous and weakly compact in $L_s(X)$. A crucial role is played by the notion of a closed measure, introduced by I. Kluvanek in [7]. In Section 2 we prove the main theorem and give some applications. The final section is devoted to a discussion of some relevant examples.

1. Preliminaries. Let X be a locally convex Hausdorff space. The correspondence

$$\sum_i x_i \otimes x_i' \mapsto \xi \in (L_s(X))'$$

defined by

$$\xi(T) = \sum_i \langle Tx_i, x_i' \rangle, \quad T \in L_s(X), \tag{4}$$

is an (algebraic) isomorphism of the tensor product $X \otimes X'$ onto the dual of $L_s(X)$.

An $L_s(X)$-valued operator measure in a σ-additive map

$$P:\Sigma \to L_s(X),$$

whose domain Σ is a σ-algebra of subsets of a set Ω. It follows from the identification (4) and the Orlicz-Pettis lemma that P is σ-additive if and only if the complex-valued set function

$$\langle Px, x' \rangle : E \mapsto \langle P(E)x, x' \rangle, \quad E \in \Sigma,$$

is σ-additive for each $x \in X$ and $x' \in X'$. The measure P is said to be *equicontinuous* if its range,

$$\mathcal{R}(P) = \{P(E); E \in \Sigma\},$$

is an equicontinuous subset of $L(X)$. If P is multiplicative and $P(\Omega) = I$, the identity operator in X, then P is called a *spectral measure*. Of course, the multiplicativity of P means that

$$P(E \cap F) = P(E)P(F), \quad \text{for every } E \in \Sigma \text{ and } F \in \Sigma.$$

Let $P:\Sigma \to L_s(X)$ be an operator measure. A net $\{E_\alpha\}$ of elements in Σ is said to be *P-convergent* to an element E of Σ (respectively, to be *P-Cauchy*) if, for every neighbourhood V of zero in $L_s(X)$, there is an index α_V such that $P(F) \in V$, for every set $F \subseteq E\Delta E_\alpha$ (respectively, $F \subseteq E_\alpha \Delta E_\beta$), $F \in \Sigma$, whenever $\alpha_V \leqq \alpha$ (respectively, $\alpha_V \leqq \alpha$ and $\alpha_V \leqq \beta$), where $G\Delta H$ denotes the symmetric difference of any two sets G and H. The measure P is said to be *closed* if Σ is P-complete, that is, if every P-Cauchy net in Σ is P-convergent to a member of Σ. This is a special case of the definition for arbitrary vector measures introduced in [**7**]. An equicontinuous spectral measure is a closed measure if and only if its range is a closed subset of $L_s(X)$, [**11**; Proposition 3].

Let $P:\Sigma \to L_s(X)$ be an operator measure. A complex-valued, Σ-measurable function f is said to be *P-integrable* if it is integrable with respect to every measure $\langle Px, x' \rangle$, $x \in X$ and $x' \in X'$, and if, for every $E \in \Sigma$ there exists an element

$$(fP)(E) = \int_E f dP$$

of $L(X)$ such that

$$\langle (fP)(E)x, x' \rangle = \int_E f d\langle Px, x' \rangle,$$

for each $x \in X$ and $x' \in X'$. This agrees with the definition of integrability with respect to arbitrary vector measures, [**9**]. The element $(fP)(\Omega)$ is denoted simply by $P(f)$. Under the assumption of sequential completeness of $L_s(X)$ it follows that every bounded, Σ-measurable function is P-integrable, [**9**; II Lemma 3.1]. If P is a spectral measure, then the multiplicativity of P implies that

$$(5) \qquad (fP)(E) = P(E)P(f) = P(f)P(E), \quad E \in \Sigma,$$

[**4**; Section 1]. In this case a P-integrable function f is *P-null* or equal to zero P-a.e. (cf. [**9**; Chapter II, Section 2] for the definition) if and only if $P(f) = 0$. If, in addition, P is equicontinuous, then the space of (equivalence classes, modulo P-a.e., of) P-integrable functions, denoted by $L^1(P)$, can be equipped with a locally convex Hausdorff topology which turns $L^1(P)$ into a unital, commutative locally convex algebra with respect to pointwise multiplication (of equivalence classes), [**4**; Proposition 1.4]. Furthermore, P is a closed measure if and only if $L^1(P)$ is complete as a locally convex space, in which case the integration mapping

$$(6) \qquad f \mapsto P(f) = \int_\Omega f dP, \quad f \in L^1(P),$$

is a bicontinuous isomorphism of the (complete) locally convex algebra $L^1(P)$ onto the operator algebra $\langle \mathscr{R}(P) \rangle_s$, [**4**; Proposition 1.5]. In particular, $\langle \mathscr{R}(P) \rangle_s$ is a complete subspace of $L_s(X)$.

LEMMA 1. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $P:\Sigma \to L_s(X)$ be an operator measure which is a closed measure. Then the set*

$$(7) \quad P(L_1^\infty(\Sigma)) = \left\{ \int_\Omega f dP;\ f \text{ is } \Sigma\text{-measurable}, |f| \leqq 1, P\text{-a.e.} \right\}$$

is a weakly compact subset of $L_s(X)$. In particular, it is a closed subset of $L_s(X)$.

Proof. By Corollary 13 of [**8**] there is a localizable measure

$$\lambda:\Sigma \to [0, \infty]$$

such that P is absolutely continuous with respect to λ, that is, $P(E) = 0$ whenever $E \in \Sigma$ and $\lambda(E) = 0$. If

$$B = \left\{ \int_\Omega g dP;\ g \in L^\infty(\Sigma, \lambda), \|g\|_\infty \leqq 1 \right\},$$

then it can be shown, using the absolute continuity of P with respect to λ, that B coincides with the set (7) and so it suffices to show that B is weakly compact. But, the localizability of λ guarantees that the Radon-Nikodym theorem is available and that $L^\infty(\Sigma, \lambda)$ is the dual space to $L^1(\lambda)$, from which it follows that the integration mapping

$$g \mapsto \int_\Omega g dP; \quad g \in L^\infty(\Sigma, \lambda),$$

is continuous from the weak-star topology on $L^\infty(\Sigma, \lambda)$ into $L_s(X)$ equipped with its weak topology, that is, the weak operator topology. Since the closed unit ball of $L^\infty(\Sigma, \lambda)$ is weak-star compact (by Alaoglu's theorem) it follows that B, and hence (7), is weakly compact.

LEMMA 2. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete. If $P:\Sigma \to L_s(X)$ and $Q:\Lambda \to L_s(X)$ are any two closed spectral measures whose ranges $\mathscr{R}(P)$ and $\mathscr{R}(Q)$ coincide as subsets of $L(X)$, then also $P(L_1^\infty(\Sigma))$ and $Q(L_1^\infty(\Lambda))$, as defined by* (7), *are equal as subsets of $L(X)$.*

Proof. If $T \in P(L_1^\infty(\Sigma))$, then $T = P(f)$ for some Σ-measurable function f such that $|f| \leqq 1$, P-a.e. By redefining f on a P-null set, if necessary, we may assume that $|f| \leqq 1$ everywhere. Then there exists a sequence $\{f_k\}$ of Σ-simple functions converging pointwise to f (even uniformly) such that $|f_k| \leqq 1$, for each $k = 1, 2, \dots$. By the Dominated Convergence Theorem for vector measures, [**9**; II Theorem 4.2], it follows that

$$T = \lim_{k\to\infty} P(f_k) \quad \text{in } L_s(X).$$

Accordingly, since $P(L_1^\infty(\Sigma))$ is a closed subset of $L_s(X)$ by Lemma 1, to verify the inclusion

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$$P(L_1^\infty(\Sigma)) \subseteq Q(L_1^\infty(\Lambda))$$

it suffices to show that $P(h) \in Q(L_1^\infty(\Lambda))$ whenever h is a Σ-simple function satisfying $|h| \leqq 1$.

So, suppose that

$$h = \sum_{i=1}^{n} \alpha_i \chi_{E(i)}$$

where the α_i, $1 \leqq i \leqq n$, are distinct complex numbers satisfying $|\alpha_i| \leqq 1$ and the sets $E(i)$, $1 \leqq i \leqq n$, are elements of Σ which are pairwise disjoint. Since $\mathscr{R}(P) = \mathscr{R}(Q)$, there are sets $F(i) \in \Lambda$, $1 \leqq i \leqq n$, such that

$$Q(F(i)) = P(E(i)), \quad \text{for each } i = 1, 2, \dots, n.$$

Furthermore, $F(i) \cap F(j)$ is a Q-null set whenever $i \neq j$ since, by multiplicativity

$$Q(F(i) \cap F(j)) = Q(F(i))Q(F(j))$$
$$= P(E(i))P(E(j)) = P(E(i) \cap E(j)) = 0.$$

Hence,

$$\tilde{h} = \sum_{i=1}^{n} \alpha_i \chi_{F(i)}$$

is a Λ-simple function satisfying $|\tilde{h}| \leqq 1$, Q-a.e. and so

$$Q(\tilde{h}) \in Q(L_1^\infty(\Lambda)).$$

Since $P(h) = Q(\tilde{h})$, it follows that $P(h) \in Q(L_1^\infty(\Lambda))$. This shows that

$$P(L_1^\infty(\Sigma)) \subseteq Q(L_1^\infty(\Lambda)).$$

By interchanging the roles of P and Q, a similar argument gives the reverse inclusion.

LEMMA 3. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $\mathscr{A} \subseteq L(X)$ be an equicontinuous, complete Boolean algebra. Then the set $\mathscr{A}_1$ (cf. Introduction) is equal to $P(L_1^\infty(\Sigma))$ for any spectral measure $P:\Sigma \to L_s(X)$ such that $\mathscr{A} = \mathscr{R}(P)$. In particular, $\mathscr{A}_1$ is an equicontinuous subset of $L(X)$.*

Proof. We remark that any such spectral measure P is necessarily equicontinuous and closed; see [**4**; Proposition 4.2] and [**11**; Proposition 3], for example.

If

$$T = \sum_{i=1}^{n} \alpha_i A_i,$$

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where the α_i, $1 \leqq i \leqq n$, are complex numbers satisfying $|\alpha_i| \leqq 1$ and $\{A_i\}_{i=1}^n \subseteq \mathscr{A}$ is a set of pairwise disjoint projections, then $A_i = P(E(i))$ for some sets $E(i)$, $1 \leqq i \leqq n$, belonging to Σ and the identities

$$P(E(i) \cap E(j)) = P(E(i))P(E(j)) = A_iA_j = 0, \quad i \neq j,$$

show that $E(i) \cap E(j)$ is P-null whenever $i \neq j$. Accordingly,

$$h = \sum_{i=1}^{n} \alpha_i \chi_{E(i)}$$

is a Σ-simple function satisfying $|h| \leqq 1$, P-a.e. and so $T = P(h)$ belongs to $P(L_1^\infty(\Sigma))$. Since $P(L_1^\infty(\Sigma))$ is closed in $L_s(X)$ by Lemma 1, it follows from the definition of $\mathscr{A}_1$ that

$$\mathscr{A}_1 \subseteq P(L_1^\infty(\Sigma)).$$

Conversely, if $T = P(f)$ for some Σ-measurable function f satisfying $|f| \leqq 1$, then there exist Σ-simple functions $\{f_k\}$ converging pointwise to f such that $|f_k| \leqq 1$, for each $k = 1, 2, \ldots$. Since each f_k, $k = 1, 2, \ldots$, can be expressed in the form $\sum_{i=1}^n \alpha_i \chi_{E(i)}$ with $|\alpha_i| \leqq 1$, $i = 1, 2, \ldots, n$, and the elements $E(i)$, $1 \leqq i \leqq n$, of Σ, being pairwise disjoint, it is clear that each operator $P(f_k)$, $k = 1, 2, \ldots$, belongs to $\mathscr{A}_1$. Since $\mathscr{A}_1$ is closed in $L_s(X)$ and $P(f_k) \to P(f)$, in $L_s(X)$, as $k \to \infty$ (by the Dominated Convergence Theorem) it follows that $T \in \mathscr{A}_1$. This shows that

$$\mathscr{A}_1 = P(L_1^\infty(\Sigma)).$$

The equicontinuity of $\mathscr{A}_1$ is then immediate from [**15**; Proposition 2.1].

2. Proof of the theorem and some applications. Let $\mathscr{A} \subseteq L(X)$ be an equicontinuous, complete Boolean algebra and $P:\Sigma \to L_s(X)$ be any spectral measure realizing $\mathscr{A}$ as its range, that is, $\mathscr{A} = \mathscr{R}(P)$. As noted before, P is then necessarily closed and equicontinuous and Lemma 3 implies that

$$\mathscr{A}_1 = P(L_1^\infty(\Sigma)).$$

If $\{V_j; j \in \mathscr{J}\}$ is any neighbourhood basis of zero in X as given in the introduction (and assumed fixed from now on), then it follows from the equality $\mathscr{A}_1 = P(L_1^\infty(\Sigma))$ and Proposition 2.3 of [**15**] that the sets W_j, $j \in \mathscr{J}$, defined by (1), form a neighbourhood basis of zero in X, which depends only on the Boolean algebra $\mathscr{A}$ and not on the particular spectral measure P realizing $\mathscr{A}$ as its range (cf. Lemmas 2 and 3). The reason for and advantage of introducing a spectral measure P is that it transforms the proof of the theorem into a consideration of spectral algebras (via the isomorphism (6)), where there is available a powerful functional calculus based on the theory of integration. It is then possible to reduce the problem to a consideration of spectral algebras in Banach spaces by forming a suitable projective limit (cf. [**15**], for example).

Let $q^{(j)}$ denote the Minkowski gauge functional of W_j, $j \in \mathscr{J}$. Then for each $j \in \mathscr{J}$:

(i) if f is Σ-measurable and $|f| \equiv 1$, then

$$q^{(j)}(P(f)x) = q^{(j)}(x), \quad x \in X; \tag{8a}$$

(ii) if f and g are bounded, Σ-measurable functions with $0 \leqq f \leqq g$, then

$$q^{(j)}(P(f)x) \leqq q^{(j)}(P(g)x), \quad x \in X, \tag{8b}$$

[**15**; p. 304]. Furthermore, if f is a bounded, Σ-measurable function, then

$$q^{(j)}(P(f)x) \leqq \|f\|_\infty q^{(j)}(x), \quad x \in X, \tag{9}$$

for each $j \in \mathscr{J}$, [**15**; Proposition 2.4].

Let X_j, $j \in \mathscr{J}$, denote the quotient spaces as defined in the introduction. The image of an element $x \in X$, under the natural inclusion of X onto X_j is denoted by $[x]_j$, $j \in \mathscr{J}$. Then X_j is a normed space with respect to the norm

$$\| [x]_j\|_j = q^{(j)}(x), \quad [x]_j \in X_j,$$

for each $j \in \mathscr{J}$. The completion of X_j with respect to this norm is denoted by $\hat{X}_j$, $j \in \mathscr{J}$. If, for some $i, j \in \mathscr{J}$ there is $\beta > 0$ such that $q^{(j)} \leqq \beta q^{(i)}$, then the natural linear transformation which maps $\hat{X}_i$ into $\hat{X}_j$ is continuous, [**15**; Proposition 2.4], and hence X (in the case when it is complete) can be identified with the projective limit of the Banach spaces $\{\hat{X}_j; j \in \mathscr{J}\}$, [**15**; Theorem 2.7].

If $E \in \Sigma$, then it follows from (8a) and (8b) that each subspace $N_j(\mathscr{A})$, $j \in \mathscr{J}$, is invariant for $P(E)$ and hence, there is induced a family of linear operators $P_j(E):X_j \to X_j$, $j \in \mathscr{J}$, given by

$$P_j(E):[x]_j \mapsto [P(E)x]_j, \quad [x]_j \in X_j. \tag{10}$$

It is clear from (9) that each operator $P_j(E)$, $j \in \mathscr{J}$, is continuous with norm not exceeding one. Hence, each of the induced operators (10) has a unique continuous extension to $\hat{X}_j$, denoted by $\hat{P}_j(E)$, $j \in \mathscr{J}$. It is easily verified that the set function $\hat{P}_j:\Sigma \to L_s(\hat{X}_j)$ given by

$$\hat{P}_j:E \mapsto \hat{P}_j(E), \quad E \in \Sigma,$$

is a spectral measure, for each $j \in \mathscr{J}$, with norms uniformly bounded by one. Since X, when complete, is the projective limit of $\{\hat{X}_j; j \in \mathscr{J}\}$ the measure P can be interpreted as the projective limit of the measures $\{\hat{P}_j; j \in \mathscr{J}\}$, [**15**; Theorem 2.7]. We remark that each induced measure $\hat{P}_j$, $j \in \mathscr{J}$, is necessarily a closed measure (cf. proof of Theorem 2 in **[12]**). It is not difficult to show that a set $E \in \Sigma$ is P-null if and only if it is $\hat{P}_j$-null, for every $j \in \mathscr{J}$.

Let f be a bounded, Σ-measurable function. It follows from (9) that each of the closed subspaces $N_j(\mathscr{A})$, $j \in \mathscr{J}$ (cf. (2)), is invariant for $P(f)$. If f is an arbitrary P-integrable function, not necessarily bounded, then the bounded functions $f\chi_{E(n)}$, $n = 1, 2, \ldots$, where

$$E(n) = \{w;\ |f(w)| \leqq n\} \quad \text{for each } n = 1, 2, \ldots,$$

converge pointwise to f and so the Dominated Convergence Theorem implies that

$$P(f\chi_{E(n)}) \to P(f),$$

in $L_s(X)$, as $n \to \infty$. It follows that each of the closed subspaces $N_j(\mathscr{A})$, $j \in \mathscr{J}$, is invariant for $P(f)$. Accordingly, each element f of $L^1(P)$ induces a family of linear operators

$$P(f)_j : X_j \to X_j, \quad j \in \mathscr{J},$$

defined by

$$(11) \quad P(f)_j : [x]_j \to [P(f)x]_j, \quad [x]_j \in X_j.$$

It is clear from (9) that if f is bounded, then each operator $P(f)_j$, $j \in \mathscr{J}$, is continuous in X_j, with norm not exceeding $\|f\|_\infty$ and hence, has a unique continuous extension to $\hat{X}_j$, denoted by $P(f)_j^\wedge$, satisfying

$$|||P(f)_j^\wedge|||_j \leqq \|f\|_\infty,$$

where $|||\cdot|||_j$ denotes the operator norm of elements in $L(\hat{X}_j)$, $j \in \mathscr{J}$. If f is unbounded, then in the case of an arbitrary Boolean algebra $\mathscr{A}$ it need not follow that the induced operators (11) are continuous (cf. Example 2 in Section 3). However, in the case when $\mathscr{A}$ is projectively extendable, which is what is being assumed, it follows from the definition (cf. (3)) that any operator $T = P(f)$, $f \in L^1(P)$, necessarily an element of $\langle\mathscr{A}\rangle_s$ by (6), induces continuous operators via the formulae (11). The continuous extension to $\hat{X}_j$ of the operator $P(f)_j$ given by (11), for each $j \in \mathscr{J}$, is again denoted by $P(f)_j^\wedge$.

LEMMA 4. *Let $f \in L^1(P)$. Then for each index $j \in \mathscr{J}$, the family of operators $\{P(f\chi_E)_j^\wedge;\ E \in \Sigma\}$ is uniformly bounded in $L(\hat{X}_j)$ by $|||P(f)_j^\wedge|||_j$.*

Proof. Fix $j \in \mathscr{J}$ and $E \in \Sigma$. If $[x]_j \in X_j$, then it follows from the definition of the norm in X_j and (5) that

$$(12) \quad \|P(f\chi_E)_j[x]_j\|_j = q^{(j)}(P(E)P(f)x) = q^{(j)}\left(\int_\Omega \chi_E dPy\right),$$

where $y = P(f)x$. Since $0 \leqq \chi_E \leqq \chi_\Omega$ it follows from (8b) that

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$$q^{(j)}\left(\int_\Omega \chi_E dPy\right) \leqq q^{(j)}(y)$$

and hence (12) implies that

$$\|P(f\chi_E)_j[x]_j\|_j \leqq q^{(j)}(P(f)x)$$
$$= \|P(f)_j[x]_j\|_j \leqq |||P(f)_j^\wedge\|_j\| \, [x]_j\|_j.$$

This shows that $|||P(f)_j^\wedge|||_j$ is a uniform bound for the operator norms of the family $\{P(f\chi_E)_j^\wedge;\ E \in \Sigma\}$.

A Σ-measurable function f on Ω is said to be *P-essentially bounded* if

$$|f|_P = \inf\{\ \|f\chi_E\|_\infty;\ E \in \Sigma,\ P(E) = I\} < \infty.$$

It follows from the σ-additivity of P that there exists a set $E \in \Sigma$ with $P(E) = I$, in which case $\Omega \backslash E$ is P-null, such that

$$|f|_P = \|f\chi_E\|_\infty.$$

Since $\Omega \backslash E$ is then $\hat{P}_j$-null, for each $j \in \mathcal{J}$, it follows that $\hat{P}_j(E) = I, j \in \mathcal{J}$, and hence, that

$$|f|_{\hat{P}_j} \leqq \|f\chi_E\|_\infty = |f|_P, \quad j \in \mathcal{J}.$$

So, we have established the following

LEMMA 5. *For each index $j \in \mathcal{J}$, the natural inclusion of the P-essentially bounded functions equipped with the norm $|\cdot|_P$ into the space of $\hat{P}_j$-essentially bounded functions equipped with the norm $|\cdot|_{\hat{P}_j}$ is norm-decreasing.*

It follows from Lemma 5 that if f is a P-essentially bounded function, then f is also $\hat{P}_j$-essentially bounded, for every $j \in \mathcal{J}$, and hence is necessarily $\hat{P}_j$-integrable, for every $j \in \mathcal{J}$. The natural question then is: what is the relationship, if any, between the operator

$$\hat{P}_j(f) = \int_\Omega f d\hat{P}_j$$

and the operator $P(f)_j^\wedge$ as defined by the continuous extension of (11), for each $j \in \mathcal{J}$?

LEMMA 6. *Let f be a P-essentially bounded function. Then*

$$P(f)_j^\wedge = \int_\Omega f d\hat{P}_j, \quad j \in \mathcal{J}.$$

Proof. Fix an index $j \in \mathcal{J}$. Since both operators $P(f)_j^\wedge$ and $\hat{P}_j(f)$ are continuous, it suffices to show that they agree on the dense subspace of X_j of $\hat{X}_j$.

Let $\{f_n\}$ be a sequence of Σ-simple functions converging to f, P-a.e. Then $\{f_n\}$ also converges to f, $\hat{P}_j$-a.e. It is easily verified, using the fact

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that f_n is Σ-simple, that

$$\int_\Omega f_n d\hat{P}_j[x]_j = \left[\int_\Omega f_n dPx\right]_j, \quad n = 1, 2, \dots,$$

for each $[x]_j \in X_j$, and hence the Dominated Convergence Theorem applied to the vector measure $\hat{P}_j(\cdot)[x]_j$ in $\hat{X}_j$ implies that

$$(13) \quad \left(\int_\Omega f d\hat{P}_j\right)[x]_j = \lim_{n\to\infty} \left[\int_\Omega f_n dPx\right]_j, \quad [x]_j \in X_j.$$

But, the right-hand limit in (13) equals $P(f)_j^\wedge[x]_j$ since, by definition of the norm in X_j it follows that

$$(14) \quad \left|\left|P(f)_j^\wedge[x]_j - \left[\int_\Omega f_n dPx\right]_j\right|\right|_j = q^{(j)}\left(\int_\Omega f dPx - \int_\Omega f_n dPx\right),$$

for each $[x]_j \in X_j$ and $n = 1, 2, \dots$, and the right-hand-side of (14) has limit zero as $n \to \infty$ by the Dominated Convergence Theorem applied to the X-valued measure $P(\cdot)x$.

The main ingredient in the proof of the theorem is the following

LEMMA 7. *Let $f \in L^1(P)$. Then f is integrable with respect to each closed spectral measure*

$$\hat{P}_j:\Sigma \to L_s(\hat{X}_j), \quad j \in \mathscr{J}.$$

Proof. If f is P-null, then there is nothing to prove. So, assume that f is not P-null, in which case $P(f) \neq 0$. Fix an index $j \in \mathscr{J}$. Again we may suppose that $P(f)_j^\wedge \neq 0$, in which case

$$\beta_j = |||P(f)_j^\wedge|||_j$$

is positive. Define sets

$$E(n) = \{w; |f(w)| \leqq n\}$$

and bounded, Σ-measurable functions

$$f_n = f\chi_{E(n)},$$

for each $n = 1, 2, \dots$. Then $\{f_n\}$ converges pointwise to f on Ω. Since $L_s(\hat{X}_j)$ is sequentially complete, to show that f is $\hat{P}_j$-integrable it suffices to show that

$$\left\{\int_E f_n d\hat{P}_j\right\}_{n=1}^\infty$$

is Cauchy in $L_s(\hat{X}_j)$ uniformly with respect to E in Σ, [**10**; Theorem 2.4 (2)].

Let $\mu \in \hat{X}_j$ and $\epsilon > 0$. Choose $\xi = [x]_j \in X_j$ such that

$$(15) \quad ||\mu - \xi||_j < \epsilon/(4\beta_j).$$

If $E \in \Sigma$, then it follows from Lemma 6 that

$$\left|\left|\int_E f_n d\hat{P}_j\xi - \int_E f_m d\hat{P}_j\xi\right|\right|_j = \left|\left|\left(\int_E f_n dP\right)_j\xi - \left(\int_E f_m dP\right)_j\xi\right|\right|_j$$
$$= \left|\left|\left[\int_\Omega \chi_E dPy\right]_j\right|\right|_j,$$

for each m and n, where

$$y = P(f_n - f_m)x.$$

Since $0 \leqq \chi_E \leqq \chi_\Omega$, it follows from the definition of the norm $||\cdot||_j$ and (8b) that

$$\left|\left|\left[\int_\Omega \chi_E dPy\right]_j\right|\right|_j \leqq q^{(j)}(y)$$

and hence

$$\left|\left|\int_E f_n d\hat{P}_j\xi - \int_E f_m d\hat{P}_j\xi\right|\right|_j \leqq q^{(j)}\left(\int_\Omega f_n dPx - \int_\Omega f_m dPx\right),$$

for every $E \in \Sigma$ and each m and n. But, f is $P(\cdot)x$-integrable and hence, there exists a positive integer N, depending on f, ϵ, j, x and ξ, but not on E, such that

$$q^{(j)}\left(\int_\Omega f_n dPx - \int_\Omega f_m dPx\right) < \epsilon/2, \quad m, n \geqq N.$$

Accordingly, for every $m \geqq N$ and $n \geqq N$, it follows that

$$(16) \quad \sup\left\{\left|\left|\int_E f_n d\hat{P}_j\xi - \int_E f_m d\hat{P}_j\xi\right|\right|_j; E \in \Sigma\right\} < \epsilon/2.$$

Then the inequalities

$$\left|\left|\int_E f_n d\hat{P}_j\mu - \int_E f_m d\hat{P}_j\mu\right|\right|_j \leqq \left|\left|\int_E (f_n - f_m)d\hat{P}_j(\mu - \xi)\right|\right|_j$$
$$+ \left|\left|\int_E f_n d\hat{P}_j\xi - \int_E f_m d\hat{P}_j\xi\right|\right|_j,$$

valid for every $E \in \Sigma$ and $m, n \geqq N$, together with (15) and (16) imply that

$$(17) \quad \left|\left|\int_E f_n d\hat{P}_j\mu - \int_E f_m d\hat{P}_j\mu\right|\right|_j$$
$$\leqq (\epsilon/4\beta_j)\left|\left|\left|\int_E f_n d\hat{P}_j - \int_E f_m d\hat{P}_j\right|\right|\right|_j + \epsilon/2.$$

But, an application of the triangle inequality and Lemmas 4 and 6 shows that

$$\left|\left|\left|\int_E f_n d\hat{P}_j - \int_E f_m d\hat{P}_j\right|\right|\right|_j < 2\beta_j,$$

for every $E \in \Sigma$ and all positive integers m and n, and hence it follows from (17) that

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(18) $$\sup\left\{\left\|\int_E f_n d\hat{P}_j\mu - \int_E f_m d\hat{P}_j\mu\right\|_j ; E \in \Sigma\right\} < \epsilon,$$

for every $m \geqq N$ and $n \geqq N$. Since a typical neighbourhood of zero in $L_s(\hat{X}_j)$ is of the form

$$\{T \in L(\hat{X}_j); \|T\mu_k\| < \delta, 1 \leqq k \leqq l\}$$

for some finite set of elements

$$\{\mu_k\}_{k=1}^{l} \subseteq \hat{X}_j$$

and some $\delta > 0$, it follows from (18) that

$$\left\{\int_\Omega f_n d\hat{P}_j\right\}_{n=1}^{\infty}$$

is Cauchy in $L_s(\hat{X}_j)$ uniformly with respect to E in Σ.

Proof of the main theorem. Since the topology of $L_b(X)$ is stronger than that of $L_s(X)$ it is clear that $\langle\mathscr{A}\rangle_b \subseteq \langle\mathscr{A}\rangle_s$.

Conversely, if $T \in \langle\mathscr{A}\rangle_s$, then (6) implies that $T = P(f)$ for some P-integrable function f. Given a neighbourhood U of zero in $L_b(X)$ it is to be shown that there is an operator R of the form

$$\sum_{i=1}^{n} \alpha_i P(E_i),$$

where the α_i, $1 \leqq i \leqq n$, are complex numbers and the E_i, $1 \leqq i \leqq n$, are elements of Σ, such that $(T - R)$ is an element of U. Typically, U is of the form

$$U = \{W \in L(X); \sup\{q^{(j)}(Wx); x \in B\} < \epsilon\},$$

where ϵ is a positive number, j is some index in $\mathscr{J}$ and $B \subseteq X$ is a bounded set. Since the quotient map of X onto X_j is continuous it follows that

$$B(j) = \{[x]_j; x \in B\}$$

is a bounded set in X_j, that is,

$$\delta = \sup\{\|\xi\|_j; \xi \in B(j)\} = \sup\{q^{(j)}(x); x \in B\} < \infty.$$

Accordingly, if $S_1^{(j)}$ denotes the closed unit ball of $\hat{X}_j$, then

$$B(j) \subseteq \delta S_1^{(j)}.$$

Lemma 7 implies that f is $\hat{P}_j$-integrable and hence

$$\int_\Omega f d\hat{P}_j$$

is an element of $\langle\mathscr{R}(\hat{P}_j)\rangle_s$ by (6) applied to the closed, equicontinuous measure $\hat{P}_j$. By the classical Bade theorem applied in the Banach space $\hat{X}_j$ there exists an element H in $L(\hat{X}_j)$ of the form

$$\sum_{i=1}^{n} \alpha_i \hat{P}_j(E_i)$$

with the α_i, $1 \leqq i \leqq n$, complex numbers and the E_i, $1 \leqq i \leqq n$, elements of Σ, such that

$$\left|\left|\left| H - \int_\Omega f d\hat{P}_j \right|\right|\right|_j < \epsilon/\delta.$$

Let

$$R = \sum_{i=1}^{n} \alpha_i P(E_i).$$

Then R belongs to $L(X)$ and

$$\hat{R}_j = \sum_{i=1}^{n} \alpha_i \hat{P}_j(E_i)$$

is precisely the operator H. Furthermore,

$$\begin{aligned} \sup_{x \in B} q^{(j)}(Rx - Tx) &= \sup_{\xi \in B(j)} \left|\left| \hat{R}_j \xi - \int_\Omega f d\hat{P}_j \xi \right|\right|_j \\ &\leqq \delta \sup_{\mu \in S_1^{(j)}} \left|\left| \hat{R}_j \mu - \int_\Omega f d\hat{P}_j \mu \right|\right|_j \\ &= \delta \left|\left|\left| H - \int_\Omega f d\hat{P}_j \right|\right|\right|_j < \epsilon, \end{aligned}$$

which shows that $(T - R) \in U$. This completes the proof of the theorem.

As a simple application of the main theorem we have the following

COROLLARY 1. *Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $\mathscr{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra which is projectively extendable. Then $\langle \mathscr{A} \rangle_b$ is a full subalgebra of $L_b(X)$, that is, if an element of $\langle \mathscr{A} \rangle_b$ is invertible in $L(X)$, then its inverse again belongs to $\langle \mathscr{A} \rangle_b$.*

Proof. Realize $\mathscr{A}$ as the range of a closed, equicontinuous spectral measure, say P. Suppose that $T \in \langle \mathscr{A} \rangle_b$ is invertible in $L(X)$. Since $\langle \mathscr{A} \rangle_b = \langle \mathscr{A} \rangle_s$ by the main theorem, it follows from (6) that $T = P(f)$ for some P-integrable function f. Accordingly, $1/f$ is P-integrable and $T^{-1} = P(1/f)$, [**13**; Lemma 3]. Then (6) again implies that $T^{-1} \in \langle \mathscr{A} \rangle_s$ and hence $T^{-1} \in \langle \mathscr{A} \rangle_b$ by the main theorem.

If $\mathscr{A} \subseteq L(X)$ is a complete, equicontinuous Boolean algebra, then it certainly satisfies the hypotheses for the Bade reflexivity theorem in the setting of locally convex spaces, [**4**; Theorem 3.1]. Combining this observation with the main theorem gives the following

COROLLARY 2. *Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $\mathscr{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra which is projectively extendable. Then an operator $T \in L(X)$ belongs to $\langle \mathscr{A} \rangle_b$ if and only if T leaves invariant every closed subspace of X left invariant by every element of $\mathscr{A}$.*

A Boolean algebra $\mathscr{A} \subseteq L(X)$ is said to be *cyclic* if there exists an element x in X such that the linear span of $\{Ax; A \in \mathscr{A}\}$ is dense in X. The following result is a sharpened version of Corollary 2.

COROLLARY 3. *Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $\mathscr{A} \subseteq L(X)$ be a complete, equicontinuous Boolean algebra which is cyclic and projectively extendable. Then an element of $L(X)$ belongs to $\langle \mathscr{A} \rangle_b$ if and only if it commutes with every element of $\mathscr{A}$.*

Proof. Since $T \in \langle \mathscr{A} \rangle_s$ if and only if T commutes with every element of $\mathscr{A}$, [**14**; Theorem 5.4], the result follows immediately from the equality of $\langle \mathscr{A} \rangle_b$ and $\langle \mathscr{A} \rangle_s$ as subspaces of $L(X)$.

Corollaries 1-3 are natural analogues, for a certain class of Boolean algebras in locally convex spaces, of well known results in the Banach space setting; see Lemma 2.1, Theorem 3.16 and Lemma 3.14 in Chapter XVII of [**6**], respectively.

3. Examples. In [**14**], a Boolean algebra $\mathscr{A} \subseteq L(X)$ is called *boundedly σ-complete* if $A_n \to 0$ in $L_b(X)$ whenever $\{A_n\} \subseteq \mathscr{A}$ is a sequence decreasing to zero in the partial ordering of $\mathscr{A}$. This is equivalent to the σ-additivity in $L_b(X)$ of any spectral measure whose range coincides with $\mathscr{A}$. Such Boolean algebras, for which it is known to be the case that $\langle \mathscr{A} \rangle_b = \langle \mathscr{A} \rangle_s$, [**14**; Theorem 5.3], are of interest mainly in non-normable spaces X since the only spectral measures in Banach spaces which are σ-additive for the uniform operator topology are trivial ones. An examination of Example 3.1 in [**14**] shows that if $\mathscr{A}$ is the range of the spectral measure given there, then $\mathscr{A}$ is a complete, equicontinuous Boolean algebra which is both boundedly σ-complete and projectively extendable. However, the following example shows that there are projectively extendable Boolean algebras which are not boundedly σ-complete. Hence, although these two classes of Boolean algebras do overlap they are, nevertheless, distinct.

Example 1. Let $X = L^1_{\mathrm{loc}}([0, \infty))$ be the space of (equivalence classes of) locally integrable functions on $[0, \infty)$. Then X is a separable Fréchet space when equipped with the topology specified by the seminorms

$$q^{(K)}: f \mapsto \int_K |f(w)|dw, \quad f \in X,$$

where K is any compact subset of $[0, \infty)$. If Σ denotes the Borel subsets of $\Omega = [0, \infty)$, then the set function $P:\Sigma \to L(X)$ defined by

$$P(E)f = f\chi_E, \quad f \in X, \text{ for each } E \in \Sigma,$$

is an equicontinuous spectral measure. Furthermore, since Lebesgue measure, λ, on Ω is localizable and the measure $\langle Px, x'\rangle$ is absolutely continuous with respect to λ, for each $x \in X$ and $x' \in X'$, it follows that P is a closed measure, [**9**; IV Theorem 7.3]. Accordingly, if $\mathscr{A}$ denotes the range $\mathscr{R}(P)$, of P, then $\mathscr{A}$ is a complete, equicontinuous Boolean algebra in $L(X)$.

To see that $\mathscr{A}$ is not boundedly σ-complete, let B denote the closed unit ball of $L^1([0, \infty); \lambda)$, in which case it is clearly a bounded subset of X, and let $E_k = (0, k^{-1})$, for each $k = 1, 2, \ldots$. Then $\{E_k\}$ decreases to the empty set. Since

$$p(T) = \sup_{f \in B} q^{([0,1])}(Tf) = \sup_{f \in B} \int_0^1 |Tf| d\lambda, \quad T \in L(X),$$

is a continuous seminorm in $L_b(X)$ such that

$$p(P(E_k)) \geqq 1, \quad \text{for every } k = 1, 2, \ldots,$$

it is clear that $\{P(E_k)\}$ cannot converge to zero in $L_b(X)$. This shows that P is not σ-additive in $L_b(X)$, that is, $\mathscr{A}$ is not boundedly σ-complete.

However, we claim that $\mathscr{A}$ is projectively extendable. Noting that $\int_\Omega \psi dP$ is the operator

$$\int_\Omega \psi dP: f \mapsto f\psi, \quad f \in X,$$

for each bounded, Σ-measurable function ψ on Ω, it is easily established that the family of seminorms

$$Q = \{q^{(K)}; K \subseteq \Omega, K \text{ compact}\}$$

is $\mathscr{A}$-compatible, that is, satisfies (8a), (8b) and (9).

LEMMA 8. *Let ψ be a measurable function on $[0, \infty)$. Then ψ is P-integrable if and only if ψ is λ-essentially bounded on compact subsets of $[0, \infty)$.*

Proof. If ψ is λ-essentially bounded on compact subsets of $[0, \infty)$, then it is clear that ψ is P-integrable and

$$\int_E \psi dP: f \mapsto \psi f \chi_E, \quad f \in X,$$

for each $E \in \Sigma$.

Conversely, suppose that $\psi \in L^1(P)$. Let K be a compact subset of $[0, \infty)$. If Y_K denotes the Banach space $L^1(K; \lambda_K)$, where λ_K denotes the restriction of λ to K, then the set function

$$\widetilde{P}_K:\Sigma \to L(Y_K)$$

defined by

$$\widetilde{P}_K(E):h \mapsto h\chi_{E\cap K}, \quad h \in Y_K,$$

for each $E \in \Sigma$, is a spectral measure. The claim is that ψ is $\widetilde{P}_K$-integrable, for which it suffices to show that

$$\left\{\int_E \psi_n d\widetilde{P}_K\right\}_{n=1}^{\infty}$$

is Cauchy in $L(Y_K)$ uniformly with respect to E in Σ, where

$$\psi_n = \psi\chi_{E(n)} \quad \text{and} \quad E(n) = \{w; |\psi(w)| \leqq n\},$$

for each $n = 1, 2, \ldots$, [**10**; Theorem 2.4 (2)].

For each bounded, Σ-measurable function g on Ω the operator

$$\int_E gd\widetilde{P}_K, \quad E \in \Sigma,$$

is given by

$$\int_E gd\widetilde{P}_K:h \mapsto hg_K, \quad h \in Y_K,$$

where g_K denotes the restriction of g to K. If $h \in Y_K$, then the function h^* on Ω, defined to be h on K and zero on $\Omega\backslash K$, belongs to X. Now, for each $E \in \Sigma$, we have

$$(19) \quad \left\|\int_E \psi_n d\widetilde{P}_K h - \int_E \psi_m d\widetilde{P}_K h\right\| = \int_K |\psi_n - \psi_m|\chi_E|h|d\lambda_K,$$

for every integer $m \geqq 1$ and $n \geqq 1$, where $\|\cdot\|$ is the norm in Y_K. But, interpreting the integrand in the right-hand-side of (19) as being defined on all of $[0, \infty)$ it follows, for each $E \in \Sigma$, that

$$(20) \quad \left\|\int_E \psi_n d\widetilde{P}_K h - \int_E \psi_m d\widetilde{P}_K h\right\| = q^{(K)}\left(\int_E \psi_n dPh^* - \int_E \psi_m dPh^*\right),$$

for every $m \geqq 1$ and $n \geqq 1$. Since ψ is assumed to be P-integrable in $L(X)$ it is certainly $P(\cdot)h^*$-integrable in X, and hence, if $\epsilon > 0$ is given, then there exists a positive integer N (depending on ψ, h, K and ϵ) such that

$$(21) \quad \sup_{E\in\Sigma} q^{(K)}\left(\int_E \psi_n dPh^* - \int_E \psi_m dPh^*\right) < \epsilon,$$

for every $m \geqq N$ and $n \geqq N$, [**10**; Theorem 2.4]. Since $h \in Y_K$ was arbitrary, it follows from (20) and (21) that

$$\left\{ \int_E \psi_n d\tilde{P}_K \right\}_{n=1}^{\infty}$$

is Cauchy in $L(Y_K)$ uniformly with respect to E in Σ, that is, ψ is $\tilde{P}_K$-integrable. As Y_K is a Banach space, it follows that ψ is $\tilde{P}_K$-essentially bounded, [**6**; Chapter XVIII, Theorem 2.11 (c)]. But, $\tilde{P}_K$ and λ_K are mutually absolutely continuous with respect to each other and so ψ is λ-essentially bounded on K.

Now, the family of $\mathscr{A}$-compatible seminorms Q is indexed by the set $\mathscr{J}$ of all compact subsets of $[0, \infty)$. Fix an index $K \in \mathscr{J}$. If f and g are elements of X, then it follows that $[f]_K = [g]_K$ if and only if

$$\int_K |f - g| d\lambda = 0$$

and the norm $\|\cdot\|_K$ in the quotient space X_K (cf. Section 2) is given by

$$\| [f]_K \|_K = q^{(K)}(f) = \int_K |f| d\lambda, \quad [f]_K \in X_K.$$

Hence, $\hat{X}_K$ can be identified with the Banach space

$$Y_K = L^1(K; \lambda_K)$$

introduced in the proof of Lemma 8 and the induced spectral measure $\hat{P}_K$ (cf. (10)) is just $\tilde{P}_K$. Accordingly, if ψ is any P-integrable function, then Lemma 8 implies that ψ is λ-essentially bounded on K and hence, the restriction, ψ_K, of ψ to K is $\hat{P}_K$-essentially bounded. Since the induced operator $P(\psi)_K$, in X_K, given by (11) is easily shown to be multiplication by ψ_K, it follows that $P(\psi)_K$ is continuous (cf. (9) and Lemma 6). Since $K \in \mathscr{J}$ was arbitrary, this shows that $\mathscr{A}$ is projectively extendable.

Remark. The normed spaces X_K, $K \in \mathscr{J}$, are already complete, that is, $\hat{X}_K = X_K$ for each $K \in \mathscr{J}$. Also, the Boolean algebra $\mathscr{A}$ is cyclic. For example, the constant function 1 on $[0, \infty)$ is a cyclic vector for $\mathscr{A}$.

In the setting of locally convex spaces, the classes of boundedly σ-complete Boolean algebras and projectively extendable Boolean algebras have the property that for any of their members $\mathscr{A}$, the algebra $\langle \mathscr{A} \rangle_b$ coincides with the weakly closed algebra generated by $\mathscr{A}$. We conclude with an example which shows that these two classes do not exhaust the class of all Boolean algebras $\mathscr{A}$ for which $\langle \mathscr{A} \rangle_b = \langle \mathscr{A} \rangle_s$: this remains an open problem.

Example 2. For each $n = 1, 2, \ldots$, let $X^{(n)}$ denote the space

$$\{ f\chi_{[-n,n]};\ f \in L^1(\mathbf{R}) \}$$

equipped with the norm, $\|\cdot\|_1$, induced from $L^1(\mathbf{R})$. Then $X^{(n)} \subseteq X^{(n+1)}$ and the topology of $X^{(n)}$ is that induced by $X^{(n+1)}$, for each $n = 1, 2, \ldots$. Let

$$X = \bigcup_{n=1}^{\infty} X^{(n)}$$

and equip X with the (strict) inductive limit topology induced by the family $\{X^{(n)}; n = 1, 2, \dots\}$. Then X is a complete, separable and barrelled locally convex Hausdorff space. In addition, each $X^{(n)}$, $n = 1, 2, \dots$, is a closed subspace of X and X induces on each $X^{(n)}$, $n = 1, 2, \dots$, its initial (norm) topology. A subset B of X is bounded if and only if there exists a positive integer n such that $B \subseteq X^{(n)}$ and B is bounded in $X^{(n)}$. If

$$u_n: X^{(n)} \to X$$

denotes the natural inclusion map, for each $n = 1, 2, \dots$, then a basis of neighbourhoods of zero in X consists of the balanced, convex hulls of sets of the form

$$\bigcup_{n=1}^{\infty} u_n(V^{(n)})$$

where, for each $n = 1, 2, \dots$, $\{V^{(n)}\}$ runs through a basis of neighbourhoods of zero in $X^{(n)}$, say all positive multiples of the closed unit ball $S_1^{(n)}$, in $X^{(n)}$, for example. The dual space X' can be identified with the measurable functions ξ on $\mathbf{R}$ which are essentially bounded on compact sets with respect to Lebesgue measure λ, on $\mathbf{R}$: the duality is given by

$$\langle f, \xi \rangle = \int_{\mathbf{R}} f \xi d\lambda, \quad f \in X.$$

If Σ denotes the Borel subsets of $\mathbf{R}$, then the set function $P:\Sigma \to L(X)$ defined by

$$P(E)f = f\chi_E, \quad f \in X,$$

for each $E \in \Sigma$, is a spectral measure, necessarily equicontinuous as X is barrelled. Since λ is a localizable measure on Σ and the measure $\langle Pf, \xi \rangle$ is absolutely continuous with respect to λ, for each $f \in X$ and $\xi \in X'$, it follows that P is a closed measure, [**9**; IV Theorem 7.3]. Accordingly, $\mathscr{A} = \mathscr{R}(P)$ is a complete, equicontinuous Boolean algebra in $L(X)$. It is clear that $\mathscr{A}$ is not cyclic. The claim is that $\langle \mathscr{A} \rangle_b = \langle \mathscr{A} \rangle_s$ as linear subspaces of $L(X)$ but, that $\mathscr{A}$ is neither boundedly σ-complete or projectively extendable.

To see that P is not σ-additive in $L_b(X)$, let $B = S_1^{(1)}$, in which case B is certainly a bounded subset of X. If $V^{(n)} = S_1^{(n)}$, for each $n = 1, 2, \dots$, then

$$U = \bigcup_{n=1}^{\infty} u_n(V^{(n)})$$

is already convex and balanced and hence, is a neighbourhood of zero in

X. Accordingly,

$$Q = \{T \in L(X); T(B) \subseteq U\}$$

is a neighbourhood of zero in $L_b(X)$. Let

$$E(k) = (0, k^{-1}), \quad \text{for each } k = 1, 2, \dots .$$

Then $\{E(k)\}$ decreases to the empty set and hence, if P were σ-additive in $L_b(X)$, then it would follow that

$$P(E(k)) \to 0 \quad \text{in } L_b(X) \text{ as } k \to \infty.$$

In particular, there would exist a positive integer K such that $P(E(k)) \in Q$, for every $k \geqq K$. But, this is not possible. Indeed, if

$$f_k = k\chi_{E(k)} \quad \text{for each } k = 1, 2, \dots ,$$

then

$$P(E(k))f_k = f_k \quad \text{for every } k = 1, 2, \dots , \text{ and}$$

$$\int_{-n}^{n} |f_k| d\lambda = 1 \quad \text{for every } k \geqq 1 \text{ and } n \geqq 1,$$

from which it follows (after noting that $\{f_k\} \subseteq B$) that

$$P(E(k))(B) \nsubseteq U \quad \text{for every } k = 1, 2, \dots ,$$

that is,

$$P(E(k)) \notin U \quad \text{for every } k = 1, 2, \dots .$$

So, P is not σ-additive in $L_b(X)$ and hence $\mathscr{A}$ is not boundedly σ-complete.

If $V^{(n)} = S_1^{(n)}$, for each $n = 1, 2, \dots$, then the set

$$W = \bigcup_{n=1}^{\infty} u_n(V^{(n)}),$$

which is convex and balanced, is a neighbourhood of zero in X. If $q^{(W)}$ denotes the Minkowski gauge functional of W, then it is not difficult to see that $q^{(W)}$ is just the restriction of the norm $\|\cdot\|_1$ to X. Observing that for each bounded, measurable function ψ on $\mathbf{R}$, the integrals

$$\int_E \psi dP, \ E \in \Sigma,$$

are simply the operators in X of multiplication by $\psi\chi_E$, it is easily established that $q^{(W)}$ is an $\mathscr{A}$-compatible seminorm. Hence, to show that $\mathscr{A}$ is not projectively extendable it suffices to exhibit a P-integrable function ψ such that the induced operator

$$\left(\int_\Omega \psi dP\right)_W : X_W \to X_W,$$

given by (11), is not continuous. So, let $\psi(w) = w$, for each $w \in \mathbf{R}$. Then ψ is P-integrable and

$$(22) \quad \int_E \psi dP : f \mapsto \chi_E f \psi, \quad f \in X,$$

for every $E \in \Sigma$. If f and g are elements of X, then

$$q^{(W)}(f - g) = \int_{\mathbf{R}} |f - g| d\lambda$$

and hence X_W can be identified with the subspace of $L^1(\mathbf{R})$ consisting of those elements vanishing outside of some compact subset of $\mathbf{R}$. Accordingly, $\hat{X}_W$ is just $L^1(\mathbf{R})$. Since

$$\left(\int_{\mathbf{R}} \psi dP\right)_W$$

is the operator in X_W of multiplication by ψ (cf. (22)), it is clear that

$$\left(\int_{\mathbf{R}} \psi dP\right)_W$$

is not continuous, as required.

To show that $\langle \mathscr{A} \rangle_b = \langle \mathscr{A} \rangle_s$ we require the following

LEMMA 9. *A measurable function ψ on $\mathbf{R}$ is P-integrable if and only if ψ is λ-essentially bounded on each set $[-n, n]$, $n = 1, 2, \ldots$.*

Proof. If ψ is λ-essentially bounded on each set $[-n, n]$, $n = 1, 2, \ldots$, then it is clear that ψ is P-integrable; its integrals are given by the formula (22), for each $E \in \Sigma$.

Suppose then that $\psi \in L^1(P)$ and fix a positive integer n. Since $X^{(n)}$ is a closed subspace of X which is invariant for each operator $P(E)$, $E \in \Sigma$, it follows that the set function

$$P^{(n)} : \Sigma \to L(X^{(n)}),$$

where $P^{(n)}(E)$ is the restriction of $P(E)$ to $X^{(n)}$, for each $E \in \Sigma$, is a spectral measure. It follows that ψ is $P^{(n)}$-integrable in $L(X^{(n)})$ and, for each $E \in \Sigma$, the operator

$$\int_E \psi dP^{(n)}$$

is just the restriction of (22) to $X^{(n)}$, that is, multiplication in $X^{(n)}$ by ψ. Since $X^{(n)}$ is a Banach space (as X induces on $X^{(n)}$ its initial topology) it follows that ψ is $P^{(n)}$-essentially bounded, [**6**; Chapter XVIII, Theorem 2.11 (c)]. But, $P^{(n)}$ is supported in Σ by $[-n, n]$ from which it follows that ψ is $P^{(n)}$-essentially bounded in $[-n, n]$. Since $P^{(n)}$ and λ are mutually absolutely continuous with respect to each other the desired conclusion follows.

The inclusion $\langle \mathscr{A} \rangle_b \subseteq \langle \mathscr{A} \rangle_s$ is always satisfied. So, let $T \in \langle \mathscr{A} \rangle_s$. Then it follows from (6) and Lemma 9 that

$$T = \int_{\mathbf{R}} \psi dP$$

for some measurable function ψ which is λ-essentially bounded on each set $[-n, n]$, $n = 1, 2, \ldots$. A typical neighbourhood of T in $L_b(X)$ is of the form

$$\mathscr{H} = \{S \in L(X); (T - S)(B) \subseteq U\}$$

where $B \subseteq X$ is a bounded set and U is a neighbourhood of zero in X. Then there exists a positive integer n such that $B \subseteq X^{(n)}$ and B is bounded in $X^{(n)}$, that is,

$$\gamma = \sup\{ \|f\|_1; f \in B\} < \infty.$$

As $U \cap X^{(n)}$ is a neighbourhood of zero in $X^{(n)}$ and X induces the L^1-topology on $X^{(n)}$, there is $\alpha > 0$ such that

$$\alpha S_1^{(n)} \subseteq U \cap X^{(n)}.$$

Let $\psi^{(n)}$ denote the restriction of ψ to $[-n, n]$ and

$$\Sigma_n = \Sigma \cap [-n, n].$$

Since $\psi^{(n)}$ is λ-essentially bounded in $[-n, n]$ it follows that $\psi^{(n)}$ is Q-integrable in $L(X^{(n)})$ where

$$Q:\Sigma_n \to L(X^{(n)})$$

is the closed spectral measure (by the same argument as for P) of multiplication by characteristic functions of elements of Σ_n. Hence, (6) implies that the operator

$$\tilde{T}_\psi: X^{(n)} \to X^{(n)}$$

of multiplication in $X^{(n)}$ by $\psi^{(n)}$, being equal to

$$\int_{[-n,n]} \psi^{(n)} dQ,$$

is an element of $\langle \mathscr{R}(Q) \rangle_s$ in $L(X^{(n)})$. Since $X^{(n)}$ is a Banach space and $\mathscr{R}(Q)$ is a complete Boolean algebra in $L(X^{(n)})$, the classical Bade theorem guarantees that

$$\langle \mathscr{R}(Q) \rangle_b = \langle \mathscr{R}(Q) \rangle_s \quad \text{in } L(X^{(n)})$$

and hence, there is an operator

$$\tilde{R} = \sum_{i=1}^{k} \beta_i Q(E_i)$$

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with the $\{E_i\} \subseteq \Sigma_n$ pairwise disjoint, such that

$$\|\tilde{R} - \tilde{T}_\psi\|_n < \alpha/\gamma,$$

where $\|\cdot\|_n$ denotes the operator norm in $L(X^{(n)})$. Then the operator

$$R = \sum_{i=1}^{k} \beta_i P(E_i)$$

belongs to the linear span of $\mathscr{A}$ in $L(X)$ and the restriction of R to $X^{(n)}$ is precisely $\tilde{R}$. Also, the restriction of T to $X^{(n)}$ is the operator $\tilde{T}_\psi$. So, if $f \in B \subseteq X^{(n)}$, in which case $\|f\|_1 \leqq \gamma$, then $(R - T)f$ belongs to $X^{(n)}$ and

$$\|(R - T)f\|_1 = \|(\tilde{R} - \tilde{T}_\psi)f\|_1 \leqq \|\tilde{R} - \tilde{T}_\psi\|_n \|f\|_1 < \alpha,$$

which shows that

$$(R - T)(B) \subseteq \alpha S_1^{(n)} \subseteq U \cap X^{(n)} \subseteq U,$$

that is, $R \in \mathscr{H}$. Accordingly, every neighbourhood of T in $L_b(X)$ contains an element from the linear span of $\mathscr{A}$ in $L(X)$ which shows that $T \in \langle\mathscr{A}\rangle_b$ and hence, that $\langle\mathscr{A}\rangle_b = \langle\mathscr{A}\rangle_s$.

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