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ON A BROWNIAN MOTION PROBLEM OF T. SALISBURY

FRANK B. KNIGHT

ABSTRACT. Let B be a Brownian motion on R, $B(0) = 0$, and let $f(t, x)$ be continuous. T. Salisbury conjectured that if the total variation of $f(t, B(t))$, $0 \le t \le 1$, is finite P-a.s., then f does not depend on x. Here we prove that this is true if the expected total variation is finite.

For real-valued $f(t), t \in I \doteq [0, 1]$, we denote the total variation of $f(\cdot)$ in $[0, t]$ by $V(t; f) = \sup \sum_i |f(t_i) - f(t_{i-1})|$, the supremum being over all finite partitions of $[0, t]$. If f is continuous, it is easy to check that $V(t; f)$ is nondecreasing and continuous before reaching ∞, and $\{t \colon V(t, f) = \infty\}$ has the form either $[t_0, \infty)$ or (t_0, ∞) for some $t_0 \le \infty$. In the IMS Workshop on Brownian motion and analysis held in Chapel Hill, North Carolina, in June 1994, the following problem was raised by T. Salisbury: To show that, if $f(t, x)$ is continuous on $I \otimes R$, and B_t is a (continuous) Brownian motion starting at 0 (with probability $P \doteq P^0$), and if $f(t, B_t)$ is of locally finite variation P-a.s., then f does not depend on x. This problem gains interest in view of the paper [2], in which the assertion is shown to be false if $B(t)$ is replaced by a general continuous martingale M_t such that (t, M_t) is a realization of a Hunt process.

Here we will demonstrate the assertion under the extra

HYPOTHESIS E. $EV\Big(1; f\big(\cdot, B(\cdot, w)\big)\Big) < \infty$.

"Normally" one would expect to remove such hypothesis by reducing the general case to it, either by some localization argument using stopping times, or by some convenient modification of f. But in the present case we have not been able to remove it. So we now state our

MAIN RESULT. If Hypothesis E holds, then f does not depend on x.

Turning to the details, since it suffices to prove for all $\varepsilon > 0$ that f is free of x for $\varepsilon \le t \le 1$, by the Markov property at time ε and the additivity of V it suffices to replace I by $[\varepsilon, 1]$ and assume that $E^\mu V\big(1 - \varepsilon; f(\cdot + \varepsilon, B.)\big) < \infty$ for $\mu = N(0, \varepsilon)$ as initial distribution for B. Denoting $f(\cdot + \varepsilon)$ again by f, and for convenience replacing $1 - \varepsilon$ by 1 (our proof will apply on any finite time interval), we see that it suffices to show that f is free of x under

HYPOTHESIS E′. For some normal $\mu = N(0, \sigma^2)$ (and hence, for all small $\sigma^2 > 0$), $E^\mu V\big(1; f(\cdot, B.)\big) < \infty$.

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We now define $g(a;t,x) = \int_0^a f(t,x+y)\,dy$, $-\infty < a < \infty$, and we wish to show that for some $v = N\big(0,\sigma^2(a)\big)$,

$$E^v V\big(1; g(a;\cdot,B.)\big) < \infty. \tag{1.1}$$

To this end, we need the key

LEMMA 1. *For measurable $f(t,x)$ on $I \otimes R$, and $g(a;t,x) = \int_0^a f(t,x+y)\,dy$, we have $\int_0^a V\big(t; f(\cdot,x+y)\big)\,dy \geq V\big(t; g(a;\cdot,x)\big)$.*

PROOF. For any partition $0 = t_o < t_1 < \cdots < t_{n+1} = t$, we have

$$\begin{aligned}\sum_{j=1}^{n+1} |g(a;t_j,x) - g(a;t_{j-1},x)| &\leq \sum_{j=1}^{n+1} \int_0^a |f(t_j;x+y) - f(t_{j-1};x+y)|\,dy \\ &\leq \int_0^a V\big(t; f(\cdot;x+y)\big)\,dy, \quad \text{as required.}\end{aligned}$$

Now, to complete the proof of (1.1), we replace $f(t,x)$ in Lemma 1 by $f\big(t,x+B(t,w)\big)$ for a fixed point w of the probability space. Setting $t = 1$ and $x = 0$, we obtain $\int_0^a V\Big(1; f\big(\cdot, B(\cdot,w)+y\big)\Big)\,dy \geq V\big(1; g(a;\cdot,B.)\big)$. Apply E^v to both sides, for $v = N\big(o,\sigma^2(a)\big)$ yet to be determined. We need only arrange that

$$E^v \int_0^a V\Big(1; f\big(\cdot, B(\cdot,w)+y\big)\Big)\,dy < \infty.$$

This becomes easily

$$\begin{aligned}&\big(2\pi\sigma^2(a)\big)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp -\frac{z^2}{2\sigma^2(a)} E^0 \int_0^a V\Big(1; f\big(\cdot; y+z+B(\cdot,w)\big)\Big)\,dy\,dz \\ &\qquad = \big(2\pi\sigma^2(a)\big)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \Big(\int_0^a \exp -\frac{(x-y)^2}{2\sigma^2(a)}\,dy\Big) E^0 V\big(1; f(\cdot, x+B.)\big)\,dx.\end{aligned}$$

We separate the last integral into the part over $\{|x| \leq 2a\}$ and that over $\{|x| > 2a\}$. Over the former, for any $\sigma^2(a) > 0$, $\exp -\frac{(x-y)^2}{2\sigma^2(a)}$ is bounded by a constant times $\exp -\frac{x^2}{2\sigma^2}$, where σ^2 is from Hypothesis E$'$. Over the latter we use

$$\exp -\frac{(x-y)^2}{2\sigma^2(a)} = \exp\Big[\frac{-x^2}{2\sigma^2(a)}\Big(\frac{y}{x} - 1\Big)^2\Big] \leq \exp -\frac{x^2}{8\sigma^2(a)}.$$

So if we set $\sigma^2(a) = \frac{\sigma^2}{4}$, it is clear that (1.1) is valid.

For any b, we now introduce

$$g(a,b;t,x) = \int_0^b g(a;t,x+y)\,dy.$$

Under Hypothesis E' for f, we see as above that Hypothesis E$'$ also holds for $g(a,b;t,x)$.

Let us pause to complete the proof in the important special case in which $f(t,x)$ does not depend on t, and the conclusion is that f is a constant. Then we can likewise delete a t in g, and write $g(a;x)$ and $g(a,b;x)$. It is easy to see that $g(a,b;x)$ has two continuous

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derivatives in x, and $\frac{\partial}{\partial x} g(a,b;x) = g(a,b+x) - g(a,x)$. Since $g(a,b;B_t)$ is of finite variation, and Ito's Formula is applicable, we have

$$\int_0^t g(a;B_s+b) - g(a;B_s)\,dB_s = 0, \quad P^\mu\text{-a.s.}, \tag{1.2}$$

where $\mu = N(0,\sigma^2)$ for some $\sigma^2 > 0$. Then $0 = \int_0^t \big(g(a;B_s+b) - g(a;B_s)\big)^2\,ds$, and it follows by continuity that $g(a;B_s+b) = g(a;B_s)$ for all $a,b,s \le 1$, P^μ-a.s. This implies that $g(a;x)$ does not depend on x. Hence, neither does $\frac{d}{da} g(a;x)$, which is $f(z+a)$, and the proof is complete in this case.

REMARK. It is noteworthy that our methods require Hypothesis E even in this special case. According to [2], however, this case has been solved without Hypothesis E by E. Cinlar and J. Jacod (unpublished). A proof is given at the end.

For the general case, take $c > 0$ and set $g(a,b,c;t,x) = \int_{t-c}^t g(a,b;s,x)\,ds = \int_0^c g(a,b;t-s,x)\,ds$, where we set $f(t,x) = 0$ for $t < 0$, so that the same is true of three g-functions. It is now to be shown, under Hypothesis E′, that for small $c > 0$,

$$P^\mu\Big(V(1;g(a,b,c;\cdot,B.)\Big) < \infty) = 1 \tag{1.3}$$

(for normal μ). We again apply Lemma 1, this time with $g\big(a,b;t-x,B_t(w)\big)$ in place of $f(t,x)$, where w is a fixed sample point. We conclude that

$$\int_0^c V\Big(1;g(a,b;\cdot - x - s,B.)\Big)\,ds \ge V\Big(1;g(a,b,c;\cdot - x,B.)\Big). \tag{1.4}$$

Set $x = 0$, and take E^ν on both sides of (1.4), where $\nu = N(0,\sigma^2)$ for a σ^2 to be determined. Then it remains to see that for small $c > 0$

$$\int_0^c V\Big(1;g(a,b;\cdot - s,B.)\Big)\,ds < \infty, P^\nu\text{-a.s.} \tag{1.5}$$

We note that $V\big(t;g(a,b;\cdot - s,B.)\big) = 0$ for $s > t$, and for $s \le t$ we have

$$V\Big(t;g(a,b;\cdot - s,B.)\Big) = \Delta g(a,b;0,B_s) + V\Big(t-s;g(a,b;\cdot,B.\circ\theta_s)\Big), \tag{1.6}$$

where Δ denotes the jump at $t = 0$ and θ_s is the usual translation operator. Since B_s is continuous along with $g(a,b;\cdot,\cdot)$, it is clear that the first term on the right of (1.6) makes only a finite contribution to (1.5). As to the second term, it is bounded by $V\big(t;g(a,b;\cdot,B.\circ\theta_s)\big)$, where

$$\begin{aligned} E^\nu &\int_0^c V\Big(1;g(a,b;\cdot,B.\circ\theta_s)\Big)\,ds \\ &= \int_0^c ds \int_{-\infty}^\infty dz\Big[\big(2\pi(\sigma^2+s)\big)^{-\frac12}\exp -\frac{z^2}{2(\sigma^2+s)}E^z\Big(V\Big(1;g(a,b;\cdot,B.)\Big)\Big)\Big]. \end{aligned}$$

It is routine to check that the normal integrand is increasing in s for $s < c$ and $|z| > (\sigma^2+c)^{\frac14}$, hence setting $\delta^2 = \sigma^2 + c$, we have the bound

$$\begin{aligned} &c(2\pi\delta^2)^{-\frac12}\int_{|z|<\sqrt{\delta}} E^z\Big(V\Big(1;g(a,b;\cdot,B.)\Big)\Big)\,dz \\ &\quad + c\int_{|z|>\sqrt{\delta}} (2\pi\delta^2)^{-\frac12}\exp -\frac12\Big(\frac{z}{\delta}\Big)^2 E^z\Big(V\Big(1;g(a,b;\cdot,B.)\Big)\Big)\,dz \end{aligned}$$

The first term is finite by Hypothesis E′, while the second is also finite if δ^2 is less than the variance assumed in Hypothesis E′ for g. This gives a $\sigma^2 > 0$ if c is small, as needed to prove (1.5). Hence (1.3) is proved.

We now make a (slightly novel) application of Ito's Formula to $g(a,b,c;t,x)$. The sufficient continuous differentiability of g in t holds expect for the jump

$$\Delta \frac{\partial}{\partial t} g(a,b,c;t,x)|_{t=c} = -g(a,b;0,x).$$

However, if we apply Ito's Formula separately in $[0,c)$ and in $[c,\infty)$, using the left-derivative at $t=c$ in the former case, we obtain (by addition for $t>c$) an expression of the form

$$g(a,b,c;t,B_t) = \int_0^t \frac{\partial}{\partial x} g(a,b,c;s,B_s)\,dB_s + \text{(finite variation)} \tag{1.7}$$

It follows, since $g(a,b,c;t,B_t)$ is of finite variation for P^ν, that

$$\int_0^t \frac{\partial}{\partial x} g(a,b,c;s,B_s)\,dB_s = 0, \quad P^\nu\text{-a.s.},$$

or again

$$\int_0^t \Big(\frac{\partial}{\partial x} g(a,b,c;s,B_s)\Big)^2 ds = 0 \quad \text{for } t \ge 0, P^\nu\text{-a.s.}$$

By continuity we see that, P^ν-a.s., $\frac{\partial}{\partial x} g(a,b,c;t,B_t) = 0$ for all a, b, and all $c \ge 0$, $t \ge 0$. Then it follows that $\frac{\partial^2}{\partial c \partial x} g(a,b,c;t,B_t) = 0$, where for $t > c$, $\frac{\partial^2}{\partial c \partial x} g(a,b,c;t,x) = g(a;t-c,x+b) - g(a;t-c,x)$. Now letting $c \longrightarrow 0$ we have $g(a;t,B_t+b) - g(a;t,B_t) = 0$ for $t > 0$, and by varying b it follows readily that $g(a;t,x)$ is free of x. Finally, $\lim_{a\to 0+} a^{-1} g(a;t,x) = f(t,x)$ is also free of x, and the proof is finished.

ADDENDUM. Proof of Main Result when $f = f(x)$, without assuming E. Let $L(x)$ denote the continuous martingale local time of B at $t=1$, and for $k2^{-n} \le x < (k+1)2^{-n}$ let $N_n(x)$ denote the number of successive upcrossings of $[k2^{-n},(k+1)2^{-n}]$ by $t=1$. Then it is known ([1]) that $P\{\lim_{n\to\infty}\big(2^{n+1}N_n(x) - L(x)\big) = 0 \text{ uniformly in } x\} = 1$. Also, since $L(x) > 0$ holds for x inside the range of B, it is clear that for any $a < b$, $P\{L(x) > \epsilon > 0, a < x < b\} > 0$ for some $\epsilon > 0$. Thus $P\{2^{n+1}N_n(x) > \frac{\epsilon}{2}, a < x < b$, for n large$\} > 0$. Now we have for $a = i2^{-n} < b = j2^{-n}$,

$$\begin{aligned} \infty > V(1; f\circ B) &\ge \sum_k \Big| f\big((k+1)2^{-n}\big) - f(k2^{-n})\Big| N_n(k2^{-n}) \\ &\ge |f(b) - f(a)| \min_{i \le k < j} N_n(k2^{-n}). \end{aligned}$$

Keeping a, b fixed, and letting $n \longrightarrow \infty$, the last term tends to ∞ with positive probability unless $f(a) = f(b)$, completing the proof.

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References

1. R. Chacon, Y. Le Jan, E. Perkins, and S. J. Taylor, *Generalized arc length for Brownian motion and Lévy processes*, Z. Wahrsch. Verw. Gebiete **57**(1981), 197–211.

2. T. S. Salisbury, *An increasing diffusion*. In: Seminar on Stochastic Processes, 1984, Birkhauser, Boston, (1986), 173–194.

Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801
U.S.A.