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Exponents of Diophantine Approximation in Dimension Two

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Abstract. Let $\Theta = (\alpha, \beta)$ be a point in $\mathbf{R}^2$, with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$. We attach to Θ a quadruple $\Omega(\Theta)$ of exponents that measure the quality of approximation to Θ both by rational points and by rational lines. The two "uniform" components of $\Omega(\Theta)$ are related by an equation due to Jarník, and the four exponents satisfy two inequalities that refine Khintchine's transference principle. Conversely, we show that for any quadruple Ω fulfilling these necessary conditions, there exists a point $\Theta \in \mathbf{R}^2$ for which $\Omega(\Theta) = \Omega$.

1 Introduction and Results

Let α and β be real numbers. We first introduce four exponents which quantify various notions of rational approximation to the point (α, β) in the plane $\mathbf{R}^2$.

Define $\omega(\alpha, \beta)$ as the supremum (possibly infinite) of all real numbers ω such that there exist infinitely many integers H for which the inequalities

$$|x\alpha + y\beta + z| \leq H^{-\omega} \quad \text{and} \quad \max\{|x|, |y|, |z|\} \leq H$$

admit a non-zero integer solution (x, y, z). Following the general notations of [4], we define moreover $\hat{\omega}(\alpha, \beta)$ as the supremum of all real numbers ω such that for any sufficiently large integer H, the above system of inequations has a non-zero integer solution. Considering as well the simultaneous rational approximation to α and β, we define similarly two further exponents $\omega\binom{\alpha}{\beta}$ and $\hat{\omega}\binom{\alpha}{\beta}$ by repeating word for word the previous sentences and replacing the above inequalities by

$$\max\{|z\alpha - x|, |z\beta - y|\} \leq H^{-\omega} \quad \text{and} \quad \max\{|x|, |y|, |z|\} \leq H.$$

The exponents $\omega(\alpha, \beta)$ and $\omega\binom{\alpha}{\beta}$ are those which occur most frequently in Diophantine approximation. Substituting $\max\{|x|, |y|, |z|\}$ for H in the preceding inequations, we observe that these two exponents measure the sharpness of the approximation to the point (α, β) by rational lines and by rational points respectively, in terms of their height. The corresponding uniform exponents $\hat{\omega}(\alpha, \beta) \geq 2$ and $\hat{\omega}\binom{\alpha}{\beta} \geq 1/2$ were first introduced by Jarník. They quantify the possible improvements to Dirichlet box principle when applied to the two systems of linear inequalities.

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Set $\Theta = (\alpha, \beta)$ and denote by ${}^t\Theta = \binom{\alpha}{\beta}$ the transposed vector. For brevity, we shall often write

$$\omega(\Theta) = \omega(\alpha, \beta), \quad \omega({}^t\Theta) = \omega\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad \hat{\omega}(\Theta) = \hat{\omega}(\alpha, \beta), \quad \hat{\omega}({}^t\Theta) = \hat{\omega}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

The goal of our article is to describe the *spectrum* of these four exponents, that is, the set of values taken by the quadruples

$$\Omega(\Theta) = (\omega(\Theta), \omega({}^t\Theta), \hat{\omega}(\Theta), \hat{\omega}({}^t\Theta)),$$

when $\Theta = (\alpha, \beta)$ ranges over $\mathbf{R}^2$, with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$. We have conventionally excluded from the spectrum the points with $1, \alpha, \beta$ linearly dependent over $\mathbf{Q}$, for which the four exponents behave as for real numbers. In this latter case, observe that the exponent $\hat{\omega}\binom{\alpha}{0}$ of uniform rational approximation to α is equal to 1, whenever α is irrational (Satz 1 of Khintchine's seminal paper [14]). Thus, if the numbers $1, \alpha, \beta$ are linearly dependent over $\mathbf{Q}$ and at least one of the numbers α or β is irrational, the quadruple $\Omega(\Theta)$ has the form

$$\Omega(\Theta) = (+\infty, v, +\infty, 1)$$

with $v \geq 1$, and any value v in the interval $[1, +\infty]$ may be reached for some point Θ. When both α and β are rational, we obviously have

$$\Omega(\Theta) = (+\infty, +\infty, +\infty, +\infty).$$

From now on, we shall assume that the numbers $1, \alpha, \beta$ are linearly independent over $\mathbf{Q}$.

Jarník has studied the relations between the exponents ω and $\hat{\omega}$ in a series of papers [11–13] dealing with any system of real linear forms. We refer to [4, 5] for a detailled survey of his results on this topic. In dimension two [11], he proved the formula

$$\hat{\omega}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\hat{\omega}(\alpha, \beta) - 1}{\hat{\omega}(\alpha, \beta)}.$$

The exponents $\omega(\alpha, \beta)$ and $\omega\binom{\alpha}{\beta}$ are related by Khintchine's transference inequalities

$$\frac{\omega(\alpha, \beta)}{\omega(\alpha, \beta) + 2} \leq \omega\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \leq \frac{\omega(\alpha, \beta) - 1}{2}.$$

See for instance [14, Satz VI]. Our theorem refines this latter estimate.

Theorem *For any row vector $\Theta = (\alpha, \beta)$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$, the four exponents*

$$v = \omega(\Theta), \quad v' = \omega({}^t\Theta), \quad w = \hat{\omega}(\Theta), \quad w' = \hat{\omega}({}^t\Theta),$$

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satisfy the relations

$$2 \le w \le +\infty, \quad w' = \frac{w-1}{w}, \quad \frac{v(w-1)}{v+w} \le v' \le \frac{v-w+1}{w}.$$

When $w < v = +\infty$, we have to understand these relations as $w - 1 \le v' \le +\infty$, and when $w = +\infty$, the set of constraints should be interpreted as $v = v' = +\infty$ and $w' = 1$. Conversely, for each quadruple (v, v', w, w') in $(\mathbf{R}_{>0} \cup \{+\infty\})^4$ satisfying the previous conditions there exists a row vector $\Theta = (\alpha, \beta)$ of real numbers with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$ such that $\Omega(\Theta) = (v, v', w, w')$.

Notice that the estimate

$$\frac{v(w-1)}{v+w} \le v' \le \frac{v-w+1}{w}$$

refines Khintchine's inequalities since $w \ge 2$.

Few explicit computations of quadruples $\Omega(\Theta)$ have actually been achieved. It follows from Roy's works [16, 17] that

$$\hat{\omega}(\alpha, \alpha^2) = \frac{3+\sqrt{5}}{2}, \quad \hat{\omega}\begin{pmatrix}\alpha \\ \alpha^2\end{pmatrix} = \frac{\sqrt{5}-1}{2},$$

when α is a so-called Fibonacci continued fraction. Next, Bugeaud and Laurent [3] explicitly determined the quadruple $\Omega((\alpha, \alpha^2))$ for any sturmian continued fraction α. Further (very partial) information on quadruples of the form $\Omega((\alpha, \alpha^2))$, where α is a real transcendental number, may also be derived from [5, 7, 18].

Jarník [12, 13] has improved the obvious lower bounds $\omega(\Theta) \ge \hat{\omega}(\Theta)$ and $\omega({}^t\Theta) \ge \hat{\omega}({}^t\Theta)$. We deduce his results from our theorem and we show that they are optimal.

Corollary 1 *For any row vector $\Theta = (\alpha, \beta)$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$, the lower bounds*

$$\hat{\omega}(\Theta) \ge 2 \quad \textit{and} \quad \omega(\Theta) \ge \hat{\omega}(\Theta)(\hat{\omega}(\Theta) - 1)$$

hold. Conversely, for any $v \in \mathbf{R}_{>0} \cup \{+\infty\}$ and any $w \in \mathbf{R}_{>0} \cup \{+\infty\}$ satisfying $2 \le w \le +\infty$ and $w(w-1) \le v \le +\infty$, there exists a row vector $\Theta = (\alpha, \beta)$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$, such that

$$\omega(\Theta) = v \quad \textit{and} \quad \hat{\omega}(\Theta) = w.$$

Corollary 2 *For any column vector $\Theta = \binom{\alpha}{\beta}$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$ we have*

$$\frac{1}{2} \le \hat{\omega}(\Theta) \le 1 \quad \textit{and} \quad \omega(\Theta) \ge \frac{\hat{\omega}(\Theta)^2}{1 - \hat{\omega}(\Theta)}.$$

Conversely, for any $w' \in \mathbf{R}_{>0}$ and any $v' \in \mathbf{R}_{>0} \cup \{+\infty\}$ satisfying

$$\frac{1}{2} \le w' \le 1 \quad \textit{and} \quad \frac{w'^2}{1-w'} \le v' \le +\infty$$

there exists a column vector $\Theta = \binom{\alpha}{\beta}$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$ such that $\omega(\Theta) = v'$ and $\hat{\omega}(\Theta) = w'$.

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The existence of a column (resp. row) vector Θ for which $\hat{\omega}(\Theta)$ takes an arbitrary value in the interval $[1/2, 1]$ (resp. $[2, +\infty]$) follows from [13]. Jarník's approach, which is based on some explicit construction of continued fractions, differs from ours.

In order to derive both corollaries from the theorem, observe that for given positive real numbers v and w, the interval

$$\frac{v(w-1)}{v+w} \leq v' \leq \frac{v-w+1}{w}$$

occurring in our theorem is non-empty exactly when $v \geq w(w-1)$. For the minimal value $v = w(w-1)$, it reduces to the point

$$\frac{(w-1)^2}{w} = \frac{w'^2}{1-w'}.$$

Corollaries 1 and 2 immediately follow, noting that the extremal values

$$\frac{v(w-1)}{(v+w)} \quad \text{and} \quad \frac{(v-w+1)}{w}$$

are increasing functions of v, when $v \geq w(w-1)$.

The proof of our theorem splits into two parts. We first establish the two transference inequalities by means of simple geometrical constructions involving the best rational approximations ("minimal points" in the terminology of Davenport and Schmidt [6]) to the point Θ. The determination of a point Θ with prescribed $\Omega(\Theta)$, needs more elaborate arguments. We simultaneously construct a sequence of rational lines $\Delta_{n,k}$ and a Cauchy sequence of rational points $P_{n,k}$, which approximate the limit $\Theta = \lim P_{n,k}$ in a controlled way. The geometrical configuration of these two sequences of lines and of points (colinear points and concurrent lines) reflects duality relations between two sequences of best approximations by lines and by points to a given point $\Theta \in \mathbf{R}^2$.

To conclude this introduction, let us address the problem of extending the theorem to higher dimensions. Then Θ should stand for any real linear proper subvariety of a projective space $\mathbf{P}^m(\mathbf{R})$, to which we can attach various (usual and uniform) exponents of approximation by rational linear subvarieties of fixed dimension μ, $0 \leq \mu \leq m-1$, as in [5, 20]. We refer to [5, Section 4] for precise definitions and ask for a description of the spectrum determined by the vector $\Omega(\Theta)$ of these exponents when Θ ranges over the set of all real linear subvarieties of $\mathbf{P}^m(\mathbf{R})$ with given dimension. As a next step after the present situation dealing with a point Θ in $\mathbf{P}^2(\mathbf{R})$, it should be interesting to investigate the case of a point in $\mathbf{P}^3(\mathbf{R})$ which gives rise to six exponents.

2 Transference Inequalities

We prove in this section the transference inequalities

$$\frac{v(w-1)}{v+w} \leq v' \leq \frac{v-w+1}{w}$$

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for any point $\Theta = (\alpha, \beta)$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$. Two specific sequences of best approximations will serve our purpose. We define in our setting the notion of best approximation as follows.

For any triple $\underline{X} = (x, y, z)$ of real numbers, set

$$L(\underline{X}) = |x\alpha + y\beta + z|,\ M(\underline{X}) = \max(|z\alpha - x|, |z\beta - y|), \text{ and } \|\underline{X}\| = \max\{|x|, |y|, |z|\}.$$

We say that the sequences of integer triples $(\underline{\Delta}_n)_{n>1}$ and $(\underline{P}_n)_{n\geq 1}$ are *best approximations* to L and M respectively, if they satisfy the following properties. Put

$$h_n = \|\underline{\Delta}_n\|, \quad q_n = \|\underline{P}_n\|, \quad L_n = L(\underline{\Delta}_n), \quad M_n = M(\underline{P}_n).$$

Then

$$1 < h_1 < h_2 < \cdots \quad \text{and} \quad 1 < q_1 < q_2 < \cdots,$$
$$1 > L_1 > L_2 > \cdots \quad \text{and} \quad 1 > M_1 > M_2 > \cdots.$$

Moreover, for any $n \geq 1$ and any non-zero integer triple $\underline{\Delta}$ (resp. $\underline{P}$) with norm $\|\underline{\Delta}\| < h_{n+1}$ (resp. $\|\underline{P}\| < q_{n+1}$), we have the lower bounds

$$L(\underline{\Delta}) \geq L_n \quad \text{and} \quad M(\underline{P}) \geq M_n.$$

We refer to [4, 15] for further information on the notion of best approximation.

Now define the positive exponents v_n, v'_n, w_n, w'_n by the equations

$$L_n = h_n^{-v_n} = h_{n+1}^{-w_n} \quad \text{and} \quad M_n = q_n^{-v'_n} = q_{n+1}^{-w'_n}, \quad (n \geq 1).$$

Our interest in these two sequences of best approximations rests on the formulas

$$\omega(\Theta) = \limsup_{n\to+\infty} v_n, \quad \omega({}^t\Theta) = \limsup_{n\to+\infty} v'_n,$$
$$\hat{\omega}(\Theta) = \liminf_{n\to+\infty} w_n, \quad \hat{\omega}({}^t\Theta) = \liminf_{n\to+\infty} w'_n,$$

which, thanks to the sequences of test points $\underline{\Delta}_n$ and $\underline{P}_n$, enable us to compute $\Omega(\Theta)$ as is easily seen from the above properties.

A geometrical point of view may be enlightening. Write

$$\underline{\Delta}_n = (r_n, s_n, t_n), \quad \underline{P}_n = (a_n, b_n, c_n), \quad n \geq 1.$$

We denote by Δ_n the line in $\mathbf{R}^2$ with equation $r_n x + s_n y + t_n = 0$, and by P_n the rational point with coordinates $P_n = (a_n/c_n, b_n/c_n)$. In the sequel we shall follow these conventions of notations. An underlined symbol will always stand for some non-zero real triple. The same symbol without underlining will indicate either the associated line (as for Δ_n), or the point obtained by dehomogenization with respect to the third coordinate (as for P_n). The alternative will be clear from the context.

Observe now that two consecutive best approximations $\underline{\Delta}_n$ and $\underline{\Delta}_{n+1}$ are not proportional. Therefore the vector product

$$\underline{Q}_n = \underline{\Delta}_n \wedge \underline{\Delta}_{n+1} = \left(\begin{vmatrix} s_n & s_{n+1} \\ t_n & t_{n+1} \end{vmatrix}, - \begin{vmatrix} r_n & r_{n+1} \\ t_n & t_{n+1} \end{vmatrix}, \begin{vmatrix} r_n & r_{n+1} \\ s_n & s_{n+1} \end{vmatrix} \right),$$

is a non-zero triple, so that Δ_n cuts Δ_{n+1} at the point Q_n. Since both lines Δ_n and Δ_{n+1} are close to Θ, their intersection Q_n should also be close to Θ. More precisely, write

$$\begin{vmatrix} r_n & r_{n+1} \\ s_n & s_{n+1} \end{vmatrix} \alpha - \begin{vmatrix} s_n & s_{n+1} \\ t_n & t_{n+1} \end{vmatrix} = s_{n+1}(r_n\alpha + s_n\beta + t_n) - s_n(r_{n+1}\alpha + s_{n+1}\beta + t_{n+1}),$$

$$\begin{vmatrix} r_n & r_{n+1} \\ s_n & s_{n+1} \end{vmatrix} \beta + \begin{vmatrix} r_n & r_{n+1} \\ t_n & t_{n+1} \end{vmatrix} = -r_{n+1}(r_n\alpha + s_n\beta + t_n) + r_n(r_{n+1}\alpha + s_{n+1}\beta + t_{n+1}).$$

It follows that

$$M(\underline{Q}_n) \le h_{n+1}L(\underline{\Delta}_n) + h_n L(\underline{\Delta}_{n+1}) \le 2h_{n+1}L_n = 2h_n^{-v_n+v_n/w_n}.$$

Bounding from above the norm

$$\|\underline{Q}_n\| \le 2\|\underline{\Delta}_n\|\|\underline{\Delta}_{n+1}\| \le 2h_n h_{n+1} = 2h_n^{1+v_n/w_n},$$

we find that

$$M(\underline{Q}_n) \le 2(\|\underline{Q}_n\|/2)^{-v_n(w_n-1)/(v_n+w_n)}.$$

For any $\epsilon > 0$, we know that $w_n \ge w - \epsilon$, provided n is large enough. Selecting an arbitrarily large index n such that v_n is arbitrarily close to the upper limit v, we obtain the lower bound $v' \ge v(w-1)/(v+w)$.

The proof of the inequality $v' \le (v - w + 1)/w$ is quite similar, making use now of the sequence $(\underline{P}_n)_{n\ge 1}$. Define the non-zero integer triple $\underline{D}_n$ by

$$\underline{D}_n = \underline{P}_n \wedge \underline{P}_{n+1} = \left(\begin{vmatrix} b_n & b_{n+1} \\ c_n & c_{n+1} \end{vmatrix}, - \begin{vmatrix} a_n & a_{n+1} \\ c_n & c_{n+1} \end{vmatrix}, \begin{vmatrix} a_n & a_{n+1} \\ b_n & b_{n+1} \end{vmatrix} \right),$$

so that D_n is the line joining P_n and P_{n+1}. Writing

$$\begin{vmatrix} b_n & b_{n+1} \\ c_n & c_{n+1} \end{vmatrix} = \begin{vmatrix} b_n - c_n\beta & b_{n+1} - c_{n+1}\beta \\ c_n & c_{n+1} \end{vmatrix},$$

$$\begin{vmatrix} a_n & a_{n+1} \\ c_n & c_{n+1} \end{vmatrix} = \begin{vmatrix} a_n - c_n\alpha & a_{n+1} - c_{n+1}\alpha \\ c_n & c_{n+1} \end{vmatrix},$$

$$\begin{vmatrix} a_n & a_{n+1} \\ b_n & b_{n+1} \end{vmatrix} = \begin{vmatrix} a_n - c_n\alpha & a_{n+1} - c_{n+1}\alpha \\ b_n & b_{n+1} \end{vmatrix} + \alpha \begin{vmatrix} c_n & c_{n+1} \\ b_n - c_n\beta & b_{n+1} - c_{n+1}\beta \end{vmatrix},$$

and

$$\begin{vmatrix} b_n & b_{n+1} \\ c_n & c_{n+1} \end{vmatrix} \alpha - \begin{vmatrix} a_n & a_{n+1} \\ c_n & c_{n+1} \end{vmatrix} \beta + \begin{vmatrix} a_n & a_{n+1} \\ b_n & b_{n+1} \end{vmatrix} = \begin{vmatrix} c_n\alpha - a_n & c_{n+1}\alpha - a_{n+1} \\ c_n\beta - b_n & c_{n+1}\beta - b_{n+1} \end{vmatrix},$$

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we obtain the upper bounds

$$\|\underline{D}_n\| \le (1+|\alpha|)\big(q_{n+1}M(\underline{P}_n) + q_n M(\underline{P}_{n+1})\big) \le 2(1+|\alpha|)q_{n+1}^{1-w'_n},$$
$$L(\underline{D}_n) \le 2M(\underline{P}_n)M(\underline{P}_{n+1}) \le 2q_{n+1}^{-(v'_{n+1}+w'_n)},$$

from which we deduce the expected inequality

$$v \ge \frac{v'+w'}{1-w'} = v'w + w - 1,$$

taking into account Jarník's relation $w' = (w-1)/w$.

3 The Inverse Problem

We have to construct a point $\Theta \in \mathbf{R}^2$ for which the quadruple $\Omega(\Theta)$ takes a prescribed value (v, v', w, w') as in the theorem. We restrict the discussion in this part to real numbers w, w', v, v'. The case of possibly infinite exponents is postponed to Section 7. To this end, we shall establish the following proposition in Sections 4–6.

Proposition *Let $w, \tau_0, \tau_1, \sigma$ be positive real numbers satisfying the inequalities*

$$w \ge 2, \quad \tau_1 \le 1, \quad w\tau_0 \le \sigma \le \tau_0 + \tau_1.$$

Then there exists $\Theta \in \mathbf{R}^2$ such that

$$\Omega(\Theta) = \Big(\frac{w-1+\tau_1}{\tau_0}, \frac{w-1}{\sigma}, w, \frac{w-1}{w}\Big).$$

Let us show that the quadruples

$$(v, v', w, w') = \Big(\frac{w-1+\tau_1}{\tau_0}, \frac{w-1}{\sigma}, w, \frac{w-1}{w}\Big)$$

given by the proposition are exactly those for which the conditions

$$w \ge 2, \quad w' = \frac{w-1}{w} \quad \text{and} \quad \frac{v(w-1)}{v+w} \le v' \le \frac{v-w+1}{w}$$

of our theorem hold.

Let us fix $w \ge 2$. Observe first that for given real numbers $\tau_0 > 0$ and $0 < \tau_1 \le 1$, the interval $w\tau_0 \le \sigma \le \tau_0 + \tau_1$ occurring in the proposition is non-empty exactly when (τ_0, τ_1) belongs to the triangle $\mathcal{T} \subset \mathbf{R}^2$ defined by the inequalities $1 \ge \tau_1 \ge (w-1)\tau_0 > 0$. Now fix $v \ge w(w-1)$. The necessity of this last assumption follows from Corollary 1. Then the intersection of $\mathcal{T}$ with the line of equation $v\tau_0 = w-1+\tau_1$ in the plane $\mathbf{R}^2$, is the segment whose extremities are the points

$$\Big(\frac{w-1}{v-w+1}, \frac{(w-1)^2}{v-w+1}\Big) \quad \text{and} \quad \Big(\frac{w}{v}, 1\Big).$$

The set of all admissible values σ, when the point (τ_0, τ_1) ranges along this segment, coincides with the interval

$$\frac{w(w-1)}{v-w+1} \le \sigma \le \frac{w}{v} + 1.$$

Therefore $v' = (w-1)/\sigma$ takes any assigned value in the interval

$$\frac{v(w-1)}{v+w} \le v' \le \frac{v-w+1}{w}.$$

In order to construct a point Θ as in the proposition, it will be relevant to assume the stronger conditions

$$(3.1) \qquad w \ge 2, \quad 0 < \tau_0 < \tau_1 \le 1, \quad w\tau_0 \le \sigma \le \tau_0 + \tau_1, \quad \sigma < w - 1 + \tau_0.$$

Notice that the assumptions of the proposition imply the slightly weaker inequalities

$$0 < \tau_0 \le \tau_1 \le 1 \quad \text{and} \quad \sigma \le w - 1 + \tau_0.$$

Hence the additional constraints in (3.1) exclude only the choices of parameters

$$w = 2, \quad \tau_0 = \tau_1, \quad \sigma = 2\tau_0 \quad \text{and} \quad w = 2, \quad \tau_1 = 1, \quad \sigma = 1 + \tau_0,$$

which lead to extremal quadruples of the form

$$\Big(v, \frac{v-1}{2}, 2, \frac{1}{2}\Big) \quad \text{and} \quad \Big(v, \frac{v}{v+2}, 2, \frac{1}{2}\Big)$$

for some $v \ge 2$. It turns out that Jarník [8–10] has established for any $v \ge 2$ the existence of points Θ for which

$$(\omega(\Theta), \omega({}^t\Theta)) = \Big(v, \frac{v-1}{2}\Big) \quad \text{and} \quad (\omega(\Theta), \omega({}^t\Theta)) = \Big(v, \frac{v}{v+2}\Big).$$

Then we deduce from our refined transference inequalities that

$$\hat{\omega}(\Theta) = 2 \quad \text{and} \quad \hat{\omega}({}^t\Theta) = 1/2.$$

We shall therefore assume that (3.1) holds without any loss of generality.

4 Constructing Points and Lines in the Plane

We shall construct in the next section a sequence of points and a sequence of lines which may be viewed as analogues of the sequences $(P_n)_{n\ge1}$ and $(\Delta_n)_{n\ge1}$ considered in Section 2. To that aim, we establish here some preliminary results. Lemma 1 provides us with families of rational points which are close together and lie on a given rational line. Next, we rephrase our result dually to obtain families of close

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rational lines passing through a given rational point. As a main tool, we take again standard arguments arising from the theory of continued fractions.

Let us first introduce various notions of distance between points and lines in the projective plane $\mathbf{P}^2(\mathbf{R})$, and state some of their (easily proved) properties. It is convenient to view $\mathbf{R}^2$ as a subset of $\mathbf{P}^2(\mathbf{R})$ via the usual embedding $(x, y) \mapsto (x : y : 1)$. With some abuse of notation, we shall identify a point in $\mathbf{R}^2$ with its image in $\mathbf{P}^2(\mathbf{R})$.

For any pair of points P and P' in $\mathbf{P}^2(\mathbf{R})$, with homogeneous coordinates $\underline{P}$ and $\underline{P}'$, denote by

$$d(P, P') = \frac{\|\underline{P} \wedge \underline{P}'\|}{\|\underline{P}\| \|\underline{P}'\|}$$

the so-called *projective distance* between P and P', which is obviously independent on the choice of $\underline{P}$ and $\underline{P}'$. Inside the square $[-1/2, +1/2]^2$ the projective distance coincides with the distance associated to the norm of supremum. In other words, the formula

$$d((x, y), (x', y')) = \max(|x - x'|, |y - y'|)$$

holds whenever $\max(|x|, |y|) \le 1/2$ and $\max(|x'|, |y'|) \le 1/2$. Moreover, for any $0 \le R < 1$, the projective ball $\{ P \in \mathbf{P}^2(\mathbf{R})\,;\ d(P, (0, 0)) \le R \}$, centered at the origin of $\mathbf{R}^2$ with radius R, is equal to the square $[-R, +R]^2$. Note also that the triangle inequality

$$d(P, P') - 2d(P', P'') \le d(P, P'') \le d(P, P') + 2d(P', P'')$$

holds for any points P, P', P'' in $\mathbf{P}^2(\mathbf{R})$ (see formula (5) of [18]). Now let Δ be a line in $\mathbf{P}^2(\mathbf{R})$ with equation $rx + sy + tz = 0$. We set $\underline{\Delta} = (r, s, t)$ and define the (projective) distance $d(\Delta, \Delta')$ between two lines Δ and Δ' by the formula

$$d(\Delta, \Delta') = \frac{\|\underline{\Delta} \wedge \underline{\Delta}'\|}{\|\underline{\Delta}\| \|\underline{\Delta}'\|}.$$

The distance $d(\Delta, \Delta')$ is again independent of the choice of the triples $\underline{\Delta}$ and $\underline{\Delta}'$ respectively associated with Δ and Δ'. Suppose that Δ intersects Δ' inside the square $[-1, +1]^2$. Then, denoting by $\langle \Delta, \Delta' \rangle$ the acute angle determined by the two lines in $\mathbf{R}^2$, we have the estimate

$$\frac{1}{2} \sin\langle \Delta, \Delta' \rangle \le d(\Delta, \Delta') \le 2 \sin\langle \Delta, \Delta' \rangle.$$

Finally, we define the distance $d(P, \Delta)$, between a point P with homogeneous coordinates $\underline{P} = (x, y, z)$ and a line Δ with leading coefficients $\underline{\Delta} = (r, s, t)$, to be the quantity

$$d(P, \Delta) = \frac{|rx + sy + tz|}{\|\underline{P}\| \|\underline{\Delta}\|}.$$

In the next sections, we shall make use of the formula

$$d(P, \Delta) = d(P, P')\, d(\Delta, \Delta'), \tag{4.1}$$

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which is valid for any point P' of Δ, distinct from P, and where Δ' stands for the line joining P and P'. This equality, which follows from the formula for the double vector product in $\mathbf{R}^3$, shows moreover that $d(P,\Delta)$ compares with the minimal projective distance between P and the points of Δ.

We call *normalized* homogeneous coordinates of a rational point P in $\mathbf{P}^2(\mathbf{R})$, any triple $\underline{P} = (a,b,c)$ of homogeneous coordinates of P, such that a,b,c are coprime integers. The triple $\underline{P}$ is clearly defined up to a multiplicative factor ± 1, and we denote by

$$H(P) = \|\underline{P}\| = \max(|a|,|b|,|c|)$$

the usual height of the rational point P. Note that $H(P) = |c|$ when P is located in the unit square $[-1,+1]^2$. Similarly, we normalize the equation $rx + sy + tz = 0$ of any rational projective line Δ by requiring that r,s,t be coprime integers, and we define its height $H(\Delta)$ as being the norm $\max(|r|,|s|,|t|)$.

Lemma 4.1 *Let Δ be a rational line in $\mathbf{P}^2(\mathbf{R})$ with height h, and let P_0 be a rational point belonging to Δ with height q_0. Let ℓ be a positive integer and let $q_1,\dots,q_\ell$ be a sequence of positive real numbers satisfying*

$$q_1 \geq 14q_0, \quad q_0q_1 \geq 4h \quad \textit{and} \quad q_{k+1} \geq 3q_k \quad (0 \leq k \leq \ell - 1). \tag{4.2}$$

There exist rational points $P_1,\dots,P_\ell$ located on Δ, such that the estimates

$$\frac{1}{2}q_k \leq H(P_k) \leq 2q_k \quad (0 \leq k \leq \ell)$$

and

$$\frac{1}{32}\frac{h}{q_kq_{k+1}} \leq d(P_k,P_{k'}) \leq 16\frac{h}{q_kq_{k+1}} \quad (0 \leq k < k' \leq \ell)$$

are verified. On the other hand, for any pair of distinct rational points P and P' lying on Δ, we have the lower bound

$$d(P,P') \geq \frac{h}{H(P)H(P')}.$$

Proof Fix an equation $rx + sy + tz = 0$ of Δ whose coefficients r,s,t are coprime integers, so that $h = \max(|r|,|s|,|t|)$. We denote by $\Delta(\mathbf{Z})$ the additive group of integer triples (a,b,c) for which $ra + sb + tc = 0$. Then a rational point P lies on Δ, if and only if its normalized homogeneous coordinates $\underline{P}$ belong to $\Delta(\mathbf{Z})$. Thanks to [2, Theorem 2], we can find a basis $\underline{A},\underline{B}$ of the $\mathbf{Z}$-module $\Delta(\mathbf{Z})$ such that

$$\|\underline{A}\| \leq \|\underline{B}\| \quad \text{and} \quad \|\underline{A}\|\|\underline{B}\| \leq \sqrt{3}h.$$

An integer triple $m\underline{A} + n\underline{B}$ is primitive if and only if the coefficients m and n are relatively prime integers. Note that $\underline{A} \wedge \underline{B} = \pm(r,s,t)$, so that $\|\underline{A} \wedge \underline{B}\| = h$.

We first prove *Liouville's inequality*, which is the last assertion of Lemma 4.1. Suppose that P and P' are distinct rational points located on the line Δ. Let $\underline{P}$ and $\underline{P}'$ be

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normalized homogeneous coordinates of P and P'. Then the vector product $\underline{P} \wedge \underline{P}'$ is a non-zero integer multiple of $\underline{A} \wedge \underline{B}$. Therefore, we obtain the lower bound

$$d(P, P') = \frac{\|\underline{P} \wedge \underline{P}'\|}{\|\underline{P}\| \|\underline{P}'\|} \geq \frac{h}{H(P)H(P')}.$$

Let $\underline{P}_0$ be normalized homogeneous coordinates of the point P_0. Since $\underline{P}_0$ belongs to $\Delta(\mathbf{Z})$, we may write $\underline{P}_0 = m\underline{A} + n\underline{B}$ for some coprime integer coefficients m and n. Using Cramer's formula, we easily obtain the upper bounds

$$|m| \leq \frac{2q_0\|\underline{B}\|}{h} \leq \frac{2\sqrt{3}q_0}{\|\underline{A}\|} \quad \text{and} \quad |n| \leq \frac{2q_0\|\underline{A}\|}{h} \leq \frac{2\sqrt{3}q_0}{\|\underline{B}\|}.$$

Let e and f be integers satisfying the equation $mf - ne = 1$, chosen so that f has minimal absolute value. Suppose first that n is non-zero. Noting that f is an element of smallest absolute value in some coset modulo n, we bound

$$|f| \leq \frac{|n|}{2} \leq \frac{\sqrt{3}q_0}{\|\underline{B}\|} \quad \text{and} \quad |e| \leq \frac{|m||f| + 1}{|n|} \leq \frac{|m_|}{2} + 1 \leq \frac{\sqrt{3}q_0}{\|\underline{A}\|} + 1.$$

Thus

$$\|e\underline{A} + f\underline{B}\| \leq 2\sqrt{3}q_0 + \|\underline{A}\| \leq 4\sqrt{3}q_0 \leq q_1/2,$$

since $\|\underline{A}\| \leq \|\underline{B}\| \leq |n|\|\underline{B}\| \leq 2\sqrt{3}q_0$. When $n = 0$, we have $\underline{A} = \pm\underline{P}_0$. Then $e = 0$, $f = \pm 1$, and we bound again

$$\|e\underline{A} + f\underline{B}\| = \|\underline{B}\| \leq \frac{\sqrt{3}h}{\|\underline{A}\|} = \frac{\sqrt{3}h}{q_0} \leq q_1/2.$$

We are now able to construct the sequence of points $P_1, \dots, P_\ell$. Define

$$g_1 = \left\lceil \frac{q_1}{q_0} \right\rceil \quad \text{and} \quad \underline{P}_1 = g_1\underline{P}_0 + e\underline{A} + f\underline{B}.$$

The integer triple $\underline{P}_1$ is primitive. Let P_1 be the rational point in $\mathbf{P}^2(\mathbf{R})$ with homogeneous coordinates $\underline{P}_1$. Its height $H(P_1)$ is therefore equal to $\|\underline{P}_1\|$, and satisfies the required estimate

$$q_1/2 \leq q_1 - \|e\underline{A} + f\underline{B}\| \leq \|\underline{P}_1\| = H(P_1) \leq q_1 + q_0 + \|e\underline{A} + f\underline{B}\| \leq 2q_1.$$

Next, when $\ell \geq 2$, we define recursively a sequence of primitive integer triples $\underline{P}_2, \dots, \underline{P}_\ell$ by the relations

$$\underline{P}_k = g_k\underline{P}_{k-1} + \underline{P}_{k-2}, \quad (2 \leq k \leq \ell),$$

where we have set

$$g_k = \left\lceil \frac{q_k}{\|\underline{P}_{k-1}\|} \right\rceil \geq 1.$$

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Let P_k be the rational point with homogeneous coordinates $\underline{P}_k$. Arguing by induction on k, we obtain similarly the estimate of height

$$(4.3)\qquad \frac{1}{2}q_k \le q_k - \|P_{k-2}\| \le H(P_k) = \|\underline{P}_k\| \le q_k + \|P_{k-1}\| + \|P_{k-2}\| \le 2q_k.$$

It remains to estimate the distances between the points P_k. Let us write

$$\underline{P}_k = u_k\underline{P}_0 + v_k\underline{P}_1$$

for integer coefficients u_k, v_k satisfying the usual recurrence relations

$$u_k = g_k u_{k-1} + u_{k-2} \quad (u_0 = 1, u_1 = 0),$$
$$v_k = g_k v_{k-1} + v_{k-2} \quad (v_0 = 0, v_1 = 1),$$

occurring in the theory of continued fractions. We therefore have the formula

$$\frac{u_k}{v_k} = [0; g_2, \dots, g_k], \quad (1 \le k \le \ell).$$

Observe next that the estimates of norms

$$(4.4)\ \frac{1}{2}v_k\|\underline{P}_1\| \le v_k(\|\underline{P}_1\| - [0; g_2, \dots, g_k]\|\underline{P}_0\|) = v_k\|\underline{P}_1\| - u_k\|\underline{P}_0\| \le \|\underline{P}_k\|$$
$$\le u_k\|\underline{P}_0\| + v_k\|\underline{P}_1\| = v_k([0; g_2, \dots, g_k]\|\underline{P}_0\| + \|\underline{P}_1\|) \le 2v_k\|\underline{P}_1\|$$

hold as well for $1 \le k \le \ell$. Now for any $0 \le k < k' \le \ell$, we have

$$d(P_k, P_{k'}) = \frac{\|\underline{P}_k \wedge \underline{P}_{k'}\|}{\|\underline{P}_k\|\|\underline{P}_{k'}\|} = \frac{|u_k v_{k'} - u_{k'} v_k| h}{\|\underline{P}_k\|\|\underline{P}_{k'}\|},$$

since

$$\begin{aligned}\underline{P}_k \wedge \underline{P}_{k'} &= (u_k v_{k'} - u_{k'} v_k)\underline{P}_0 \wedge \underline{P}_1 \\ &= (u_k v_{k'} - u_{k'} v_k)(mf - ne)\underline{A} \wedge \underline{B} = \pm(u_k v_{k'} - u_{k'} v_k)(r, s, l).\end{aligned}$$

By a standard result on continued fractions, we know that

$$\frac{1}{2v_{k+1}} \le \left| v_k \frac{u_{k'}}{v_{k'}} - u_k \right| \le \frac{1}{v_{k+1}}.$$

It follows that

$$\frac{1}{2}\frac{v_{k'} h}{v_{k+1}\|\underline{P}_k\|\|\underline{P}_{k'}\|} \le d(P_k, P_{k'}) \le \frac{v_{k'} h}{v_{k+1}\|\underline{P}_k\|\|\underline{P}_{k'}\|}.$$

The required estimate for $d(P_k, P_{k'})$ follows from (4.3) and (4.4). ■

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We state now a dual version of Lemma 4.1, in which the roles of lines and points are exchanged.

Lemma 4.2 *Let Δ_0 be a rational line with height h_0, and let P be a rational point lying on Δ_0, with height q. Let ℓ be a positive integer and let $h_1, \dots, h_\ell$ be a sequence of positive real numbers satisfying*

$$h_1 \geq 14h_0, \quad h_0h_1 \geq 4q \quad \text{and} \quad h_{k+1} \geq 3h_k \quad (0 \leq k \leq \ell - 1). \tag{4.5}$$

There exist rational lines $\Delta_1, \dots, \Delta_\ell$ passing through the point P, such that the estimates of height

$$\frac{1}{2}h_k \leq H(\Delta_k) \leq 2h_k \quad (0 \leq k \leq \ell)$$

and of distance

$$\frac{1}{32}\frac{q}{h_kh_{k+1}} \leq d(\Delta_k, \Delta_{k'}) \leq 16\frac{q}{h_kh_{k+1}} \quad (0 \leq k < k' \leq \ell)$$

are verified. On the other hand, for any pair of distinct rational lines Δ and Δ' containing P, we have the lower bound

$$d(\Delta, \Delta') \geq \frac{q}{H(\Delta)H(\Delta')}.$$

Proof The proof is completely parallel to that of Lemma 4.1. The formalism remains exactly the same, and we omit the details. ■

5 The Basic Construction

Recall the stronger assumptions (3.1) relating the data $w, \tau_0, \tau_1, \sigma$. Observing that $0 < \tau_0 < \tau_1 \leq 1$, we put $\ell = 1$ if $\tau_1 = 1$, and otherwise pick an integer $\ell \geq 2$ and an increasing sequence of real numbers $\tau_2, \dots, \tau_\ell$, such that

$$0 < \tau_0 < \tau_1 < \cdots < \tau_\ell = 1 \quad \text{and} \quad \frac{w - 1 + \tau_{k+1}}{\tau_k} \leq \frac{w - 1 + \tau_1}{\tau_0}, \quad (0 \leq k \leq \ell - 1). \tag{5.1}$$

As an example, we may choose $\tau_k = \min(1, \tau_0 + k(\tau_1 - \tau_0))$ for $k = 1, \dots, \ell$, where ℓ is the smallest integer such that $\tau_0 + \ell(\tau_1 - \tau_0) \geq 1$. Note that, in any case, the sequence $(\tau_k)_{0\leq k\leq \ell}$ increases and ends at $\tau_\ell = 1$. Now set $\sigma_0 = \sigma$ and $\sigma_1 = w$. It follows from (3.1) that $0 < \sigma_0 < \sigma_1 \leq \sigma/\tau_0$. Similarly, we extend this second sequence into an (eventually longer) increasing sequence

$$0 < \sigma_0 < \cdots < \sigma_{\ell'} = \sigma/\tau_0, \tag{5.2}$$

for some integer $\ell' \geq 1$ selected so that the growth conditions

$$\frac{\sigma_{k+1} - 1}{\sigma_k} \leq \frac{w - 1}{\sigma}, \quad (0 \leq k \leq \ell' - 1) \tag{5.3}$$

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hold. The constraints (5.2) and (5.3) can be simultaneously fulfilled by taking bounded ratios

$$1 < \frac{\sigma_{k+1}}{\sigma_k} \le \frac{w-1+\tau_0}{\sigma} \quad \text{for} \quad 1 \le k \le \ell' - 1.$$

Next, we introduce two sequences of positive real numbers $(h_{n,k})_{n\ge 1, 0\le k\le \ell}$ and $(q_{n,k})_{n\ge 1, 0\le k\le \ell'}$ in the following way. For simplicity, set $h_n = h_{n,0}$. We start with any large initial value h_1, and define inductively $h_{n,k}$ and $q_{n,k}$ thanks to the recurrence relations

$$\text{(5.4)} \qquad h_{n+1} = h_n^{1/\tau_0}, \ h_{n,k} = h_{n+1}^{\tau_k}, (0 \le k \le \ell), \ q_{n,k} = h_{n+1}^{\sigma_k}/16, \quad (0 \le k \le \ell').$$

Taking into account the equalities $\tau_\ell = 1$ and $\sigma_{\ell'} = \sigma/\tau_0$, observe that the branching equations

$$h_{n,\ell} = h_{n+1,0} \quad \text{and} \quad q_{n,\ell'} = q_{n+1,0}$$

hold for any $n \ge 1$. With the exception of these equalities, the sequences $(h_{n,k})$ and $(q_{n,k})$, where the indices (n,k) have been lexicographically ordered, are strictly increasing, since so are the sequences of exponents (τ_k) and (σ_k). Notice as well that both sequences $(h_{n,k})$ and $(q_{n,k})$ increase at least as a double exponential.

For further use, let us quote the estimates

$$\text{(5.5)} \qquad \frac{h_{n+1}}{h_n} < \frac{q_{n,1}}{q_{n,0}} \quad \text{and} \quad \frac{h_{n,k}h_{n+2}}{q_{n+1,0}q_{n+1,1}} = o\Big(\frac{h_{n+1}}{h_{n,k+1}q_{n,1}}\Big), \quad (0 \le k \le \ell - 1),$$

which follow from (3.1) and (5.4). In order to check the second part of (5.5), write both ratios in terms of h_{n+1}, and observe that

$$\tau_k + \frac{1-\sigma-w}{\tau_0} < 1 + \frac{1-w\tau_0-w}{\tau_0} = -(w-1)(1+\frac{1}{\tau_0}) < -2(w-1) \le 1 - \tau_{k+1} - w.$$

Lemma 5.1 *There exists a sequence of rational lines $(\Delta_{n,k})_{n\ge 1, 0\le k\le \ell}$ and a sequence of rational points $(P_{n,k})_{n\ge 1, 0\le k\le \ell'}$ satisfying for any $n \ge 1$ the compatibility relations*

$$\Delta_{n,\ell} = \Delta_{n+1,0}, \quad P_{n,\ell'} = P_{n+1,0},$$

and the following properties. The points $P_{n,0}, \dots, P_{n,\ell'}$ are pairwise distinct and lie on the line $\Delta_{n+1,0}$. The lines $\Delta_{n,0}, \dots, \Delta_{n,\ell}$ are pairwise distinct and pass through the point $P_{n,0}$. Moreover the estimates of distances[1]

$$\text{(5.6)} \qquad d((0,0), P_{1,0}) \gg\ll \frac{h_1}{q_{1,0}},$$

$$d(P_{n,k}, P_{n,k'}) \gg\ll \frac{h_{n+1}}{q_{n,k}q_{n,k+1}}, \quad (0 \le k < k' \le \ell'),$$

$$d(\Delta_{n,k}, \Delta_{n,k'}) \gg\ll \frac{q_{n,0}}{h_{n,k}h_{n,k+1}}, \quad (0 \le k < k' \le \ell),$$

[1]The implicit constants involved in the forthcoming symbols $\gg$ and $\ll$ are absolute. Their computation is however useless for our purpose. We frequently write $F(n) \gg\ll G(n)$ to signify that $F(n) \gg G(n)$ and $F(n) \ll G(n)$ for all sufficently large n.

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are satisfied, as well as the estimates of heights

$$\frac{1}{2}q_{n,k} \le H(P_{n,k}) \le 2q_{n,k} \quad \textit{and} \quad \frac{1}{2}h_{n,k} \le H(\Delta_{n,k}) \le 2h_{n,k}. \tag{5.7}$$

Proof We carry out simultaneously the construction of both sequences $\Delta_{n,k}$ and $P_{n,k}$ by successive steps.

Start (for instance) by defining $\Delta_{1,0}$ as the line with equation $\lceil h_1 \rceil x - y = 0$, and by choosing the point $P_{1,0} = (1/\lceil q_{1,0} \rceil, \lceil h_1 \rceil / \lceil q_{1,0} \rceil)$ on $\Delta_{1,0} \cap \mathbf{R}^2$. Observe that

$$q_{1,0} = h_2^{\sigma}/16 \ge h_1^{w}/16$$

is much bigger than h_1 when h_1 is large. Then the required estimates

$$d((0,0), P_{1,0}) \gg\ll \frac{h_1}{q_{1,0}},$$

$$\frac{1}{2}h_1 \le H(\Delta_{1,0}) = \lceil h_1 \rceil \le 2h_1 \quad \text{and} \quad \frac{1}{2}q_{1,0} \le H(P_{1,0}) = \lceil q_{1,0} \rceil \le 2q_{1,0},$$

clearly hold for sufficiently large initial values h_1. Note also that $P_{1,0}$ may be taken arbitrarily close to the origin $(0,0)$ of $\mathbf{R}^2$, provided h_1 is large enough.

Suppose now that $P_{1,0}, P_{1,1}, \dots, P_{n,0}$ and $\Delta_{1,0}, \Delta_{1,1}, \dots, \Delta_{n,0}$ have already been selected for some $n \ge 1$. We first use Lemma 4.2, applied to the point $P_{n,0}$ lying on the line $\Delta_{n,0}$ and to the sequence $h_{n,1}, \dots, h_{n,\ell}$. The main assumption $H(\Delta_{n,0})h_{n,1} \ge 4H(P_{n,0})$ occurring in (4.5), follows from (5.4), (5.7) and from the inequality $\sigma \le \tau_0 + \tau_1$ in (3.1). Therefore we may find rational lines $\Delta_{n,1}, \dots, \Delta_{n,\ell} = \Delta_{n+1,0}$ passing through $P_{n,0}$, for which the third estimate of (5.6) and the second one of (5.7) are verified. Next, starting with the point $P_{n,0} \in \Delta_{n+1,0}$, we apply Lemma 4.1 to the sequence $q_{n,1}, \dots, q_{n,\ell'}$. We obtain rational points $P_{n,1}, \dots, P_{n,\ell'} = P_{n+1,0}$ belonging to $\Delta_{n+1,0}$ and satisfying (5.6) and (5.7). Notice that the condition $H(P_{n,0})q_{n,1} \ge 4H(\Delta_{n+1,0})$ occurring in the assumptions (4.2) of Lemma 4.1 is easily fulfilled, since

$$q_{n,1} = h_{n+1}^{\sigma_1}/16 \ge h_{n+1}^2/16.$$

The two sequences have thus been extended up to the rank $(n+1,0)$. ■

Let us show that the sequence of points $(P_{n,k})_{n\ge 1, 0\le k\le \ell'-1}$ furnished by Lemma 5.1 is a Cauchy sequence in $\mathbf{P}^2(\mathbf{R})$. Observe first that the sequence

$$\Big(\frac{h_{n+1}}{q_{n,k}q_{n,k+1}}\Big)_{n\ge 1, 0\le k\le \ell'-1}$$

occurring in (5.6) is decreasing when the indices (n,k) are lexicographically ordered. The only non-obvious inequality,

$$\frac{h_n}{q_{n-1,\ell'-1}q_{n-1,\ell'}} > \frac{h_{n+1}}{q_{n,0}q_{n,1}},$$

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follows from the first part of (5.5). Moreover this sequence tends to 0 much more quickly than any geometric sequence with ratio < 1. Take any index (n,k) smaller than (n',k') for that lexicographic order. Combining the triangle inequality with the upper bounds (5.6), we find

$$d(P_{n,k}, P_{n',k'}) \le \sum_{(n,k)\le(\nu,\kappa)<(n',k')} 2^{\mathrm{rk}(\nu,\kappa)} d(P_{\nu,\kappa}, P_{\nu,\kappa+1})$$
$$\ll \sum_{(n,k)\le(\nu,\kappa)<(n',k')} 2^{\mathrm{rk}(\nu,\kappa)} \frac{h_{\nu+1}}{q_{\nu,\kappa}q_{\nu,\kappa+1}} \ll \frac{h_{n+1}}{q_{n,k}q_{n,k+1}},$$

where $\mathrm{rk}(\nu,\kappa)$ denotes the rank of (ν,κ) in the ordered sequence $(n,k) < \cdots < (n',k')$ of all indices between (n,k) and (n',k'), starting with the initial value $\mathrm{rk}(n,k) = 0$.

Let Θ be the limit of the Cauchy sequence $(P_{n,k})$. The same argument as above yields the estimates

$$d(P_{n,k}, \Theta) \gg\ll \frac{h_{n+1}}{q_{n,k}q_{n,k+1}}, \quad (0 \le k \le \ell' - 1), \tag{5.8}$$

and

$$d(P_{n,0}, P_{n+1,0}) \gg\ll \frac{h_{n+1}}{q_{n,0}q_{n,1}}. \tag{5.9}$$

Moreover, taking h_1 large enough, we may assume that $d((0,0), P_{n,k}) \le 1/4$ for any index $n \ge 1, 0 \le k \le \ell' - 1$, so that all points $P_{n,k}$ lie in the square $[-1/4, +1/4]^2$. Then Θ obviously belongs to $[-1/4, +1/4]^2$.

Put now $\Theta = (\alpha, \beta)$ and recall the notations

$$L(\underline{X}) = |x\alpha + y\beta + z|, \quad M(\underline{X}) = \max(|z\alpha - x|, |z\beta - y|), \quad \underline{X} = (x, y, z),$$

introduced in Section 2. Let

$$\underline{P}_{n,k} = (a_{n,k}, b_{n,k}, c_{n,k}), \text{ with } q_{n,k}/2 \le |c_{n,k}| \le 2q_{n,k} \text{ and } \gcd(a_{n,k}, b_{n,k}, c_{n,k}) = 1,$$
$$\underline{\Delta}_{n,k} = (r_{n,k}, s_{n,k}, t_{n,k}), \text{ with } h_{n,k}/2 \le \|\underline{\Delta}_{n,k}\| \le 2h_{n,k} \text{ and } \gcd(r_{n,k}, s_{n,k}, t_{n,k}) = 1,$$

be normalized integer triples respectively associated to the rational point $P_{n,k}$ and to the rational line $\Delta_{n,k}$. Recall also that the projective distance d coincides inside the square $[-1/2, +1/2]^2$ with the distance associated to the norm of supremum. The estimate of distance (5.8) is therefore equivalent to

$$M(\underline{P}_{n,k}) = |c_{n,k}| d(\Theta, P_{n,k}) \gg\ll \frac{h_{n+1}}{q_{n,k+1}} \gg\ll h_{n+1}^{1-\sigma_{k+1}}, \quad (0 \le k \le \ell' - 1). \tag{5.10}$$

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Observe next that the point $P_{n+1,0} = (a_{n+1,0}/c_{n+1,0}, b_{n+1,0}/c_{n+1,0})$ belongs to the line $\Delta_{n+1,0}$ which intersects $\Delta_{n,k}$ at the point $P_{n,0}$. Employing the formula (4.1) to estimate the distance $d(P_{n+1,0}, \Delta_{n,k})$, we deduce from (5.6), (5.7), (5.9) that

$$|r_{n,k}\frac{a_{n+1,0}}{c_{n+1,0}} + s_{n,k}\frac{b_{n+1,0}}{c_{n+1,0}} + t_{n,k}| \gg\ll \|\underline{\Delta}_{n,k}\| d(\Delta_{n,k}, \Delta_{n+1,0}) d(P_{n,0}, P_{n+1,0})$$

$$\gg\ll \frac{h_{n+1}}{h_{n,k+1}q_{n,1}} \gg\ll h_{n+1}^{1-w-\tau_{k+1}}.$$

Using (5.8) and the second part of (5.5), we may replace $a_{n+1,0}/c_{n+1,0}$ and $b_{n+1,0}/c_{n+1,0}$ in the above inequalities by their limits α and β. We therefore obtain the estimate

$$L(\underline{\Delta}_{n,k}) \gg\ll \frac{h_{n+1}}{h_{n,k+1}q_{n,1}} \gg\ll h_{n+1}^{1-w-\tau_{k+1}}, \quad (0 \le k \le \ell - 1). \tag{5.11}$$

At the present stage, we have constructed two sequences of integer triples $\underline{P}_{n,k}$ and $\underline{\Delta}_{n,k}$ that provide good approximations to Θ with respect to the functions M and L. Since the norm of $\underline{P}_{n,k}$ (resp. $\underline{\Delta}_{n,k}$) compares to $h_{n+1}^{\sigma_k}$ (resp. $h_{n+1}^{\tau_k}$), the upper bounds given by (5.10)–(5.11) yield the lower bounds

$$\begin{aligned}
(5.12)\qquad \omega(\Theta) &\ge \max_{0\le k\le \ell-1}\Big(\frac{w-1+\tau_{k+1}}{\tau_k}\Big) = \frac{w-1+\tau_1}{\tau_0},\\
\omega({}^t\Theta) &\ge \max_{0\le k\le \ell'-1}\Big(\frac{\sigma_{k+1}-1}{\sigma_k}\Big) = \frac{w-1}{\sigma}.\\
\hat\omega(\Theta) &\ge \min_{0\le k\le \ell-1}\Big(\frac{w-1+\tau_{k+1}}{\tau_{k+1}}\Big) = w,\\
\hat\omega({}^t\Theta) &\ge \min_{0\le k\le \ell'-1}\Big(\frac{\sigma_{k+1}-1}{\sigma_{k+1}}\Big) = \frac{w-1}{w}.
\end{aligned}$$

Notice that the two first equalities on the right-hand side of (5.12) follow from (5.1)–(5.3).

It turns out that the lower bounds (5.12) are actually equalities, as we shall prove in the next section.

6 Upper Bounds

In order to bound from above the exponents $\omega({}^t\Theta)$ and $\hat\omega({}^t\Theta)$, (resp. $\omega(\Theta)$ and $\hat\omega(\Theta)$), we establish that the rational points, (resp. the rational lines), which well approximate the point Θ belong necessarily to the set of points $P_{n,k}$, (resp. the set of lines $\Delta_{n,k}$), previously considered. That is the underlying principle for the proof of the next two lemmas.

Lemma 6.1 *For any non-zero integer triple $\underline{P}$ which is not proportional to some triple $\underline{P}_{n,k}$, and having sufficiently large norm $\|\underline{P}\|$, we have the lower bound* [2]

$$M(\underline{P}) \gg \|\underline{P}\|^{-\lambda} \quad \text{with} \quad \lambda = \max\Big(\frac{1}{w-1+\tau_0}, \frac{\sigma-\tau_0}{\sigma}\Big).$$

[2] In this section, the constants involved in the symbols $\gg$ and $\ll$ may possibly depend upon the data $w, \tau_0, \tau_1, \sigma$.

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There exists a positive real number ϵ such that for any sufficiently large integer n and for any non-zero integer triple $\underline{P}$ with norm $\|\underline{P}\| \leq \epsilon q_{n,1}$, we have the uniform lower bound

$$M(\underline{P}) \gg \epsilon q_{n,1}^{-(w-1)/w}.$$

Proof Let us first observe that the sequence

$$\gamma_{(n,k)} = q_{n,1}\frac{h_{n,k}}{h_{n+1}}, \quad (n \geq 1, 0 \leq k \leq \ell),$$

indexed by the couples of integers (n,k) lexicographically ordered, increases (the inequality $\gamma_{(n+1,0)} > \gamma_{(n,\ell)}$ follows from the first part of (5.5)), and tends to infinity. We denote by $(n,k)+1$ the successor of (n,k) for the lexicographic order. Then the estimate (5.11) may be written as

$$L(\underline{\Delta}_{n,k}) \gg\ll \frac{1}{\gamma_{(n,k+1)}} = \frac{1}{\gamma_{(n,k)+1}}, \quad (0 \leq k \leq \ell-1).$$

Write $\underline{P} = (a,b,c)$ and put $q = \|\underline{P}\|$. Suppose first that the point P with homogeneous coordinates $\underline{P}$ lies ouside the square $[-1/2,+1/2]^2$. Since Θ belongs to $[-1/4,+1/4]^2$, we bound from below

$$M(\underline{P}) = |c|\max\bigl(|\alpha - \frac{a}{c}|, |\beta - \frac{b}{c}|\bigr) \geq \frac{1}{4},$$

when c is non-zero, and $M(P) \geq 1$ if $c = 0$. We shall therefore assume that P belongs to the square $[-1/2,+1/2]^2$, so that $q = |c|$. The identity

$$r_{n,k}a + s_{n,k}b + t_{n,k}c = c(r_{n,k}\alpha + s_{n,k}\beta + t_{n,k}) - r_{n,k}(c\alpha - a) - s_{n,k}(c\beta - b)$$

yields for any index (n,k) the upper bound

$$|r_{n,k}a + s_{n,k}b + t_{n,k}c| \leq qL(\underline{\Delta}_{n,k}) + 2h_{n,k}M(\underline{P}) \ll \frac{q}{\gamma_{(n,k)+1}} + h_{n,k}M(\underline{P}). \tag{6.1}$$

Let ϵ be a positive real number. Assuming q is large enough, we define (n,k) as the unique index for which

$$\epsilon\gamma_{(n,k)} < q \leq \epsilon\gamma_{(n,k)+1}.$$

We use a different argumentation depending on whether $0 \leq k \leq \ell - 1$ or $k = \ell$.

Suppose first that $k \leq \ell - 1$. Replacing the index k by $k+1$ in (6.1), we obtain the two upper bounds

$$\begin{aligned} |r_{n,k}a + s_{n,k}b + t_{n,k}c| &\ll \epsilon + h_{n,k}M(\underline{P}) \\ |r_{n,k+1}a + s_{n,k+1}b + t_{n,k+1}c| &\ll \epsilon + h_{n,k+1}M(\underline{P}). \end{aligned} \tag{6.2}$$

If we suppose that $M(\underline{P}) \leq \epsilon h_{n,k+1}^{-1}$ and ϵ is small enough, the left-hand sides of both inequalities (6.2) must vanish, since these are integers. It follows that

$$P = \Delta_{n,k} \cap \Delta_{n,k+1} = P_{n,0},$$

which is impossible since we have assumed that P differs from all points $P_{n,k}$. Therefore

$$M(\underline{P}) > \epsilon h_{n,k+1}^{-1} = \epsilon h_{n+1}^{-\tau_{k+1}} \quad \text{and} \quad q > \epsilon \gamma_{(n,k)} = \frac{\epsilon}{16} h_{n+1}^{w-1+\tau_k},$$

so that

$$M(\underline{P}) \gg \epsilon (q/\epsilon)^{-\tau_{k+1}/(w-1+\tau_k)} \gg \epsilon (q/\epsilon)^{-\lambda}.$$

We now consider the case $k = \ell$. Then q belongs to the interval

$$\epsilon \gamma_{(n,\ell)} = \epsilon q_{n,1} < q \leq \epsilon q_{n+1,1} \frac{h_{n+1}}{h_{n+2}} = \epsilon \gamma_{(n+1,0)}.$$

In this situation, (6.1) yields the upper bound

$$|r_{n+1,0}a + s_{n+1,0}b + t_{n+1,0}c| \ll \frac{q}{\gamma_{(n+1,1)}} + h_{n+1} M(\underline{P}) \ll \epsilon + h_{n+1} M(\underline{P}).$$

When the point P does not lie on the line $\Delta_{n+1,0}$, the sum $r_{n+1,0}a + s_{n+1,0}b + t_{n+1,0}c$ is a non-zero integer. Thus, if ϵ is small enough, we obtain in this case the stronger lower bound

$$M(\underline{P}) \gg h_{n+1}^{-1} = (16 q_{n,1})^{-1/w} \gg (q/\epsilon)^{-\lambda}.$$

It remains to deal with points P located on $\Delta_{n+1,0}$. Since $\epsilon q_{n,1} < q$, define k' as the largest positive integer $k \leq \ell'$ such that $\epsilon q_{n,k} < q$. Therefore $\epsilon q_{n,k'} < q \leq \epsilon q_{n,k'+1}$ when $1 \leq k' \leq \ell' - 1$, and $\epsilon q_{n+1,0} < q \leq \epsilon q_{n+1,1} h_{n+1}/h_{n+2}$ when $k' = \ell'$. Now the Liouville inequality (on the line $\Delta_{n+1,0}$) provides us with the lower bound

$$d(P, P_{n,k'}) \gg \frac{h_{n+1}}{q\, q_{n,k'}} \gg \epsilon^{-1} \frac{h_{n+1}}{q_{n,k'} q_{n,k'+1}}$$

when $1 \leq k' \leq \ell' - 1$, or

$$d(P, P_{n,\ell'}) \gg \frac{h_{n+1}}{q\, q_{n+1,0}} \gg \epsilon^{-1} \frac{h_{n+2}}{q_{n+1,0}\, q_{n+1,1}}$$

when $k' = \ell'$. On the other hand, (5.8) gives the upper bounds

$$d(P_{n,k'}, \Theta) \ll \frac{h_{n+1}}{q_{n,k'} q_{n,k'+1}}, \quad (1 \leq k' \leq \ell' - 1) \quad \text{and} \quad d(P_{n,\ell'}, \Theta) \ll \frac{h_{n+2}}{q_{n+1,0}\, q_{n+1,1}}.$$

In both cases, the triangle inequality shows that

$$d(P, \Theta) \gg \frac{h_{n+1}}{q\, q_{n,k'}},$$

provided ϵ is small enough. It follows that

$$M(\underline{P}) = q\, d(P, \Theta) \gg \frac{h_{n+1}}{q_{n,k'}} = 16 h_{n+1}^{1-\sigma_{k'}} \gg (q/\epsilon)^{-(\sigma_{k'}-1)/\sigma_{k'}} \gg (q/\epsilon)^{-\lambda},$$

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noting that the exponent $1 - \sigma_{k'}$ is negative, since $k' \geq 1$ and $\sigma_{k'} \geq \sigma_1 = w \geq 2$. Finally, fixing ϵ small enough so that the previous estimates are valid, we have thus proved the first assertion of Lemma 6.1.

As for the second part of Lemma 6.1, we take again the same argumentation in a simpler way. Observe that

$$\epsilon q_{n,1} = \epsilon \gamma_{(n,\ell)} = \frac{\epsilon}{16} h_{n+1}^{w}.$$

Let $\underline{P} = (a, b, c)$ be any non-zero integer triple with norm $\|\underline{P}\| \leq \epsilon q_{n,1}$. Now (6.1) gives

$$\max_{k \in \{\ell-1,\ell\}} |r_{n,k}a + s_{n,k}b + t_{n,k}c| \ll \epsilon + h_{n+1}M(\underline{P}).$$

If $M(\underline{P}) \leq \epsilon h_{n+1}^{-1}$ and ϵ is small enough, the left-hand side of the above inequality vanishes, and we find that $P = \Delta_{n,\ell-1} \cap \Delta_{n,\ell} = P_{n,0}$. Then by (5.10) $M(\underline{P}) \geq M(\underline{P}_{n,0}) \gg h_{n+1}^{1-w} \gg q_{n,1}^{-(w-1)/w}$. Otherwise $M(\underline{P}) > \epsilon h_{n+1}^{-1} \gg \epsilon q_{n,1}^{-1/w}$. Therefore the lower bound $M(\underline{P}) \gg \epsilon q_{n,1}^{-(w-1)/w}$ holds for any non-zero integer triple $\underline{P}$ with norm $\|\underline{P}\| \leq \epsilon q_{n,1}$. ■

The next result may be viewed as a dual version of Lemma 6.1.

Lemma 6.2 *For any non-zero integer triple $\underline{\Delta}$ whose norm $\|\underline{\Delta}\|$ is large enough, and which is not proportional to some triple $\underline{\Delta}_{n,k}$, we have the lower bound*

$$L(\underline{\Delta}) \gg \|\underline{\Delta}\|^{-\mu} \quad \text{with} \quad \mu = \max\left(\frac{\sigma}{(w-1)\tau_0}, \frac{w-1+\tau_0}{\tau_0}\right).$$

There exists a positive real number ϵ such that for any sufficiently large integer n and for any non-zero integer triple $\underline{\Delta}$ with norm $\leq \epsilon h_n$, we have the uniform lower bound

$$L(\underline{\Delta}) \gg h_n^{-w}.$$

Proof We take again the same arguments as in Lemma 6.1, exchanging the roles of lines and points. Set now (note that $k \geq 1$ here)

$$\delta_{(n,k)} = \frac{q_{n-1,k}}{h_n}, \quad (n \geq 2, 1 \leq k \leq \ell').$$

The sequence $\delta_{(n,k)}$, indexed by the couples of integers (n, k) with $1 \leq k \leq \ell'$ in lexicographical order, increases and tends to infinity. We denote again by $(n, k) + 1$ the successor of (n, k) relatively to the lexicographic order. Notice that (5.10) may actually be written in the form

$$M(\underline{P}_{n-1,k}) \gg\ll \frac{1}{\delta_{(n,k)+1}}, \quad (1 \leq k \leq \ell').$$

Let ϵ be a positive real number which will be selected later to be sufficiently small. Write $\underline{\Delta} = (r, s, t)$ and put $h = \|\underline{\Delta}\|$. Assuming h is large enough, there exists a

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unique index (n,k) such that $\epsilon\delta_{(n,k)} < h \le \epsilon\delta_{(n,k)+1}$. A similar splitting of cases occurs as in Lemma 6.1.

Suppose first that $1 \le k \le \ell' - 1$. Then we bound from above

$$\begin{aligned} |ra_{n-1,k} + sb_{n-1,k} + tc_{n-1,k}| &\le |c_{n-1,k}|L(\underline{\Delta}) + 2hM(\underline{P}_{n-1,k}) \\ &\ll q_{n-1,k}L(\underline{\Delta}) + \epsilon, \\ |ra_{n-1,k+1} + sb_{n-1,k+1} + tc_{n-1,k+1}| &\ll q_{n-1,k+1}L(\underline{\Delta}) + \epsilon. \end{aligned} \tag{6.3}$$

If we suppose that $L(\underline{\Delta}) \le \epsilon q_{n-1,k+1}^{-1}$ and ϵ is small enough, the left-hand sides of both inequalities (6.3) must vanish, since these are integers. Then the line Δ, which contains the two points $P_{n-1,k}$ and $P_{n-1,k+1}$, coincides with $\Delta_{n,0}$, in contradiction with our assumptions. Therefore the lower bounds

$$L(\underline{\Delta}) > \epsilon q_{n-1,k+1}^{-1} = 16\epsilon h_n^{-\sigma_{k+1}} \quad \text{and} \quad h > \epsilon\delta_{(n,k)} = \frac{\epsilon}{16} h_n^{\sigma_k - 1}$$

hold, so that

$$L(\underline{\Delta}) \gg \epsilon (h/\epsilon)^{-\sigma_{k+1}/(\sigma_k - 1)} \gg \epsilon (h/\epsilon)^{-\mu},$$

bounding $\sigma_{k+1} \le \sigma/\tau_0$ and $\sigma_k - 1 \ge w - 1$, since $k \ge 1$.

Consider now the case $k = \ell'$. Then h belongs to the interval

$$\frac{\epsilon}{16} h_{n+1}^{\sigma - \tau_0} = \frac{\epsilon q_{n,0}}{h_n} < h \le \frac{\epsilon q_{n,1}}{h_{n+1}} = \frac{\epsilon}{16} h_{n+1}^{w-1}. \tag{6.4}$$

Arguing as in (6.3), we use here the single inequality

$$|ra_{n,0} + sb_{n,0} + tc_{n,0}| \ll q_{n,0}L(\underline{\Delta}) + \epsilon. \tag{6.5}$$

If Δ does not pass through the point $P_{n,0}$, the left-hand side of (6.5) is ≥ 1, and noting that $\sigma \ge w\tau_0$, we obtain the required lower bound

$$L(\underline{\Delta}) \gg q_{n,0}^{-1} \gg (h/\epsilon)^{-\sigma/(\sigma - \tau_0)} \gg (h/\epsilon)^{-\mu},$$

provided ϵ is small enough. It remains to deal with lines Δ containing the point $P_{n,0}$. Since $\Delta_{n+1,0}$ is the line joining $P_{n,0}$ and $P_{n+1,0}$, we may apply formula (4.1) to find

$$\frac{1}{h}\left| r\frac{a_{n+1,0}}{c_{n+1,0}} + s\frac{a_{n+1,0}}{c_{n+1,0}} + t \right| = d(P_{n+1,0}, \Delta) = d(\Delta, \Delta_{n+1,0}) d(P_{n,0}, P_{n+1,0}). \tag{6.6}$$

It readily follows from (6.4) and (3.1) that

$$h > \frac{\epsilon}{16} h_{n+1}^{\sigma - \tau_0} \ge \frac{\epsilon}{16} h_{n+1}^{(w-1)\tau_0} \ge \frac{\epsilon}{16} h_{n+1}^{\tau_0} = \frac{\epsilon}{16} h_n.$$

Accordingly, we may define k' as the largest integer $k \le \ell$ such that $h > \epsilon h_{n,k}/16$. Suppose first that $k' \le \ell - 1$, so that $\epsilon h_{n,k'}/16 < h \le \epsilon h_{n,k'+1}/16$. Then the Liouville inequality, applied to the pencil of lines passing through $P_{n,0}$, yields the lower bound

$$d(\Delta, \Delta_{n,k'}) \gg \frac{q_{n,0}}{h_{n,k'}\, h} \gg \epsilon^{-1} \frac{q_{n,0}}{h_{n,k'} h_{n,k'+1}}.$$

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On the other hand, (5.6) gives the upper bound

$$d(\Delta_{n,k'}, \Delta_{n+1,0}) \ll \frac{q_{n,0}}{h_{n,k'}\, h_{n,k'+1}}.$$

Using now the triangle inequality, the two above inequalities imply the lower bound

$$d(\Delta, \Delta_{n+1,0}) \gg \frac{q_{n,0}}{h_{n,k'}\, h}, \tag{6.7}$$

provided ϵ is small enough. Notice that (6.7) follows directly from the Liouville inequality when $k' = \ell$. Next, combining (5.9), (6.6) and (6.7), we find the lower bound

$$\left| r\frac{a_{n+1,0}}{c_{n+1,0}} + s\frac{b_{n+1,0}}{c_{n+1,0}} + t \right| \gg \frac{h_{n+1}}{h_{n,k'} q_{n,1}} \gg h_{n+1}^{-(w-1+\tau_{k'})}. \tag{6.8}$$

On the other hand, it follows from (6.4) and (5.8) that

$$h\, d(P_{n+1,0}, \Theta) \ll \epsilon \frac{q_{n,1}}{h_{n+1}} \frac{h_{n+2}}{q_{n+1,0} q_{n+1,1}} = 16\epsilon h_{n+1}^{w-1-(w-1+\sigma)/\tau_0} \leq 16\epsilon h_{n+1}^{-w},$$

noting that $\sigma \geq w\tau_0$ and $0 < \tau_0 < 1$. Therefore in the left-hand side of (6.8) we can replace the coefficients $a_{n+1,0}/c_{n+1,0}$ and $b_{n+1,0}/c_{n+1,0}$ by their limits α and β, to obtain the lower bound

$$|r\alpha + s\beta + t| \gg h_{n+1}^{-(w-1+\tau_{k'})} \gg (h/\epsilon)^{-(w-1+\tau_{k'})/\tau_{k'}} \gg (h/\epsilon)^{-\mu},$$

since $h > \epsilon h_{n,k'}/16 = \epsilon h_{n+1}^{\tau_{k'}}/16$. Fixing ϵ sufficiently small, we have proved the required lower bound $L(\underline{\Delta}) \gg h^{-\mu}$ for any integer triple $\underline{\Delta}$ which is not a multiple of some $\underline{\Delta}_{n,k}$.

Finally we prove the second part of Lemma 6.2. Let $\underline{\Delta}$ be a non-zero integer triple with norm $h \leq \epsilon h_{n+1}$. The previous inequality (6.5) remains valid. When $P_{n,0}$ does not lie on Δ, we thus obtain the stronger lower bound

$$L(\underline{\Delta}) \gg q_{n,0}^{-1} \gg h_{n+1}^{-\sigma} \gg h_{n+1}^{-w}.$$

Suppose now that Δ passes through $P_{n,0}$. Notice that Δ cannot be equal to $\Delta_{n+1,0}$, since the norm h of $\underline{\Delta}$ is smaller than the height $H(\Delta_{n+1,0}) \geq h_{n+1}$ of the line $\Delta_{n+1,0}$. Then, we use the Liouville inequality to bound from below

$$d(\Delta, \Delta_{n+1,0}) \gg \frac{q_{n,0}}{h\, h_{n+1}}.$$

Taking again the argumentation leading to (6.8) with $k' = \ell$, we find the lower bound

$$\left| r\frac{a_{n+1,0}}{c_{n+1,0}} + s\frac{b_{n+1,0}}{c_{n+1,0}} + t \right| \gg q_{n,1}^{-1} \gg h_{n+1}^{-w}. \tag{6.9}$$

As before, we may substitute in (6.9) the coordinates of the point $P_{n+1,0}$ by those of Θ, to obtain the required estimate

$$L(\underline{\Delta}) = |r\alpha + s\beta + t| \gg h_{n+1}^{-w}. \qquad \blacksquare$$

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We easily deduce from the assumptions (3.1) that the strict upper bounds

$$\lambda = \max\Big(\frac{1}{w-1+\tau_0}, \frac{\sigma-\tau_0}{\sigma}\Big) < \frac{w-1}{\sigma},$$
$$\mu = \max\Big(\frac{\sigma}{(w-1)\tau_0}, \frac{w-1+\tau_0}{\tau_0}\Big) = \frac{w-1+\tau_0}{\tau_0} < \frac{w-1+\tau_1}{\tau_0},$$

hold. Then Lemmas 6.1 and 6.2, together with (5.12), show that the exponents of approximation $\omega(\Theta)$ and $\omega({}^t\Theta)$ are reached respectively on the set of integer triples $(\underline{\Delta}_{n,k})_{n\geq 1, 0\leq k\leq \ell}$ and $(\underline{P}_{n,k})_{n\geq 1,0\leq k\leq \ell'}$. Now the estimates (5.10) and (5.11) give the equalities

$$\omega({}^t\Theta) = \max_{0\leq k\leq \ell'-1}\Big(\frac{\sigma_{k+1}-1}{\sigma_k}\Big) = \frac{w-1}{\sigma}$$
$$\omega(\Theta) = \max_{0\leq k\leq \ell-1}\Big(\frac{w-1+\tau_{k+1}}{\tau_k}\Big) = \frac{w-1+\tau_1}{\tau_0}.$$

The second parts of Lemmas 6.1 and 6.2 provide us with the upper bounds

$$\hat{\omega}({}^t\Theta) \leq \frac{w-1}{w} \quad \text{and} \quad \hat{\omega}(\Theta) \leq w.$$

Taking into account the lower bounds (5.12), this concludes the proof of our proposition. Notice that Lemma 6.2, together with (5.11), yields a fine measure of linear independence over $\mathbf{Q}$ of the numbers $1, \alpha, \beta$, which are obviously $\mathbf{Q}$-linearly independent as required by the theorem.

7 Infinite Exponents

The basic construction considered in Section 5 may be greatly extended by the introduction of variable exponents $\tau_{n,k}$ and $\sigma_{n,k}$ depending on n instead of the fixed exponents τ_k and σ_k occurring in (5.4). At each step n, we may also allow ℓ and ℓ' to vary (observe that Lemmas 4.1 and 4.2 are valid for any positive integers ℓ and ℓ'). We take advantage of this flexibility to complete the proof of the theorem in the remaining cases where $v = +\infty$. Our intention here is not to repeat the whole argument, but to briefly indicate below some specific choices of parameters $\tau_{n,k}$ and $\sigma_{n,k}$ leading to any quadruple of the form

$$(+\infty, v', w, \frac{w-1}{w}) \quad \text{where} \quad 2 \leq w \leq +\infty,\ w-1 \leq v' \leq +\infty.$$

Notice however that it might be useful to display more general constructions in order to compute the Hausdorff dimension of subsets of points $\Theta \in \mathbf{R}^2$ for which the quadruple of exponents $\Omega(\Theta)$ belongs to various parts [3] of $\mathbf{R}^4$.

[3] As an example, the precise value of the Hausdorff dimension of the set $\{(\alpha,\beta) \in \mathbf{R}^2; \hat{\omega}(\alpha,\beta) \geq w\}$, for a given real number $w > 2$, remains unknown. See [1, 4, 19] for estimates of that dimension in terms of w.

Let w and v' be real numbers with $w \geq 2$ and $v' \geq w-1$. Denote $\sigma = (w-1)/v'$ and for any integer $n > w/\sigma$, set

$$\tau_{n,0} = \frac{1}{n}, \quad \tau_{n,1} = 1, \quad \sigma_{n,0} = \sigma, \quad \sigma_{n,1} = w.$$

We first extend the increasing sequence $\sigma_{n,0} < \sigma_{n,1}$ using an arithmetical progression with n terms

$$w = \sigma_{n,1} < \cdots < \sigma_{n,n} = n\sigma,$$

whose step $(n\sigma - w)/(n-1)$ is < 1. Then, the properties (5.1)–(5.3), with $\ell = 1$ and $\ell' = n$, remain true with our present choice of parameters. The assumption $\sigma \leq 1$ yields the fundamental upper bound $\sigma_{n,0} \leq \tau_{n,0} + \tau_{n,1}$ occurring in (3.1). Next we fix two increasing sequences of positive real numbers $(h_n)_{n>w/\sigma}$ and $(q_{n,k})_{n>w/\sigma, 0\leq k\leq n}$, satisfying the recurrence relations

$$h_{n+1} = h_n^n = h_n^{1/\tau_{n,0}}, \qquad q_{n,k} = h_{n+1}^{\sigma_{n,k}}/16, \quad (0 \leq k \leq n).$$

The compatibility relations $q_{n+1,0} = q_{n,n}$ hold for any n. Going again through the construction described in Section 5, we obtain a point $\Theta = (\alpha, \beta)$ with

$$\omega(\Theta) = \limsup_{n\to+\infty} \Big(\frac{\sigma_{n,1} - 1 + \tau_{n,1}}{\tau_{n,0}} \Big) = +\infty,$$

$$\omega({}^t\Theta) = \limsup_{n\to+\infty} \max_{0\leq k\leq n-1} \Big(\frac{\sigma_{n,k+1} - 1}{\sigma_{n,k}} \Big) = \frac{w-1}{\sigma} = v',$$

$$\hat\omega(\Theta) = \liminf_{n\to+\infty} \Big(\frac{\sigma_{n,1} - 1 + \tau_{n,1}}{\tau_{n,1}} \Big) = w,$$

$$\hat\omega({}^t\Theta) = \liminf_{n\to+\infty} \min_{0\leq k\leq n-1} \Big(\frac{\sigma_{n,k+1} - 1}{\sigma_{n,k+1}} \Big) = \frac{w-1}{w}.$$

We omit the details of the proof, which follows *mutatis mutandis* the same lines as for the proposition. Notice that Lemma 6.1 remains actually valid with the exponent $\lambda = 1$.

When $v' = +\infty$, we make use of sequences $(\sigma_{n,0})_n$ tending to 0. If $w \geq 2$ is a real number, take $\ell = \ell' = 1$ and set

$$\tau_{n,0} = \frac{1}{n}, \quad \tau_{n,1} = 1, \quad \sigma_{n,0} = \frac{w}{n}, \quad \sigma_{n,1} = w, \quad (n \geq w). \tag{7.1}$$

Then we obtain a point Θ such that

$$\omega(\Theta) = \omega({}^t\Theta) = +\infty \quad \text{and} \quad \hat\omega(\Theta) = w, \quad \hat\omega({}^t\Theta) = \frac{w-1}{w}.$$

If moreover $w = +\infty$, substitute (for example) $\sqrt{n}$ for w in the formulas (7.1). In that case, the construction produces a point $\Theta = (\alpha, \beta)$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$, such that

$$\omega(\Theta) = \omega({}^t\Theta) = \hat\omega(\Theta) = +\infty \quad \text{and} \quad \hat\omega({}^t\Theta) = 1.$$

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