Canad. Math. Bull. Vol. **50** (2), 2007 pp. 268–283

On the Lack of Inverses to C^*-Extensions Related to Property T Groups

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Abstract. Using ideas of S. Wassermann on non-exact C^*-algebras and property T groups, we show that one of his examples of non-invertible C^*-extensions is not semi-invertible. To prove this, we show that a certain element vanishes in the asymptotic tensor product. We also show that a modification of the example gives a C^*-extension which is not even invertible up to homotopy.

1 Introduction

The Brown–Douglas–Fillmore theory of C^*-extensions [2] works nicely for nuclear C^*-algebras because an extension of a nuclear C^*-algebra is always invertible in the extension semi-group. As a steadily growing number of examples show, this is not the case for general extensions [1, 6, 9, 13, 17–19]. In contrast, besides all its other merits, the E-theory of Connes and Higson [3] provides a framework which incorporates arbitrary extensions of C^*-algebras, and in previous work we have clarified the way in which this happens [10, 11]. Specifically, in the E-theory setting the notion of triviality of extensions must be weakened, at least so far as to consider an extension of C^*-algebras

$$0 \longrightarrow B \longrightarrow E \xrightarrow{q} A \longrightarrow 0 \tag{1.1}$$

to be trivial when it is asymptotically split, by which we mean that there is an asymptotic homomorphism [3] $\varphi = (\varphi_t)_{t\in[0,\infty)} \colon A \to E$ such that $q \circ \varphi_t = \mathrm{id}_A$ for each $t \in [0, \infty)$. When the quotient C^*-algebra A is a suspension, *i.e.,* is of the form $C_0(\mathbb{R}) \otimes D$, this is the only change which is needed to ensure that E-theory becomes a complete analogue of the BDF theory for nuclear C^*-algebras. Specifically, when the quotient C^*-algebra is a suspension and the ideal is stable, every extension is semi-invertible, by which we mean that it is invertible in the sense corresponding to the weakened notion of triviality, *i.e.,* one can add an extension to it so that the result is asymptotically split. Furthermore, a given extension will represent 0 in E-theory if and only if it can be made asymptotically split by adding an asymptotically split extension to it. One purpose of the present paper is to show by example that this nice situation does not persist when the quotient C^*-algebra is not a suspension. We will show that an extension considered by S. Wassermann [19], and shown by him to be non-invertible, is not semi-invertible either. By slightly modifying Wassermann's

Received by the editors February 2, 2005.
AMS subject classification: Primary: 19K33; secondary: 46L06, 46L80, 20F99.
Keywords: C^*-algebra extension, property T group, asymptotic tensor C^*-norm, homotopy.

example, we also obtain an extension which is not even invertible up to homotopy, giving us the first example of a C^*-algebra for which the semi-group of homotopy classes of extensions by a stable C^*-algebra, *in casu* the algebra of compact operators, is not a group. The conclusion is that the E-theory approach to C^*-extensions does not completely save us from the unpleasantness of extensions without inverses. But unlike the BDF theory, as shown in [11], in E-theory they can be eliminated at the cost of a single suspension.

The method we adopt in order to show that Wassermann's example from [19] is not semi-invertible is basically the same as his, although the verification is somewhat more complicated since it uses the asymptotic tensor norm, which was introduced in [12], in place of the minimal tensor norm. To show that a suitably modified version of the extension is not even invertible up to homotopy we proceed quite differently in order to bring a K-theoretical obstruction to bear.

2 Wassermann's Extension Is Not Semi-Invertible

2.1 Wassermann's Example

Let G be an infinite countable discrete group with the property T of Kazhdan [14]. It is a result of Wang [16], *cf.* [19], that there are at most a countable number of unitary equivalence classes of finite-dimensional unitary representations of G. As in [19], we assume that there actually are infinitely many equivalence classes of such representations. This is the case, for example, when $G = SL_3(\mathbb{Z})$. We then fix a sequence $\pi_k, k = 1, 2, 3, \ldots,$ of inequivalent finite-dimensional irreducible unitary representations of G which contains a representative for each equivalence class of such representations. Consider the direct sum $\pi = \bigoplus_{k=1}^{\infty} \pi_k$ of these representations, acting on the Hilbert space H, and let B be the C^*-subalgebra of $\mathbb{L}(H)$ generated by $\{\pi(g) : g \in G\}$. The C^*-subalgebra of $\mathbb{L}(H)$ generated by B and the ideal $\mathbb{K} = \mathbb{K}(H)$ of compact operators on H will be denoted by E. Then $\mathbb{K}$ is an ideal in E and we denote the quotient $E/\mathbb{K}$ by A. It was shown in [19] that the extension

$$0 \longrightarrow \mathbb{K} \longrightarrow E \xrightarrow{q} A \longrightarrow 0 \tag{2.1}$$

is not invertible, or not semi-split. We are going to prove that it is also not semi-invertible.

Theorem 2.1 *The extension* (2.1) *is not semi-invertible.*

We shall elaborate a little on Wassermann's argument, so let us therefore first outline it. He shows that (2.1) does not admit a completely positive section for the quotient map (*i.e.,* is not invertible) because the sequence

$$\mathbb{K} \otimes B \longrightarrow E \otimes_{\min} B \longrightarrow A \otimes_{\min} B \tag{2.2}$$

is not exact. That (2.2) is not exact he deduces as follows: The representation $G \ni g \mapsto \pi(g) \otimes \pi(g)$ of G in $E \otimes_{\min} B$ extends by the universal property of $C^*(G)$ to

https://doi.org/10.4153/CMB-2007-027-1 Published online by Cambridge University Press

a $*$-homomorphism $\Delta\colon C^*(G) \to E \otimes_{\min} B$. By [8, Theorem 1.10], G is finitely generated. Let $g_1, g_2, \dots, g_n$ be a set of generators of G. We assume that this set contains the neutral element and is symmetric, *i.e.,* contains g_i^{-1} for all $i = 1, \dots, n$. Wassermann shows [19, proof of Theorem 6] that there is $\delta > 0$ such that the spectrum of the image of the element

$$\Delta\Big(\frac{1}{n}\sum_{i=1}^{n} g_i\Big) \in E \otimes_{\min} B$$

in the quotient $E \otimes_{\min} B/\mathbb{K} \otimes B$ lies in $[-1, 1-\delta] \cup \{1\}$ and contains 1, while the spectrum of its image in $A \otimes_{\min} B$ under the quotient map of (2.2) lies in $[-1, 1-\delta]$. Thus (2.2) is clearly not exact, and it follows that (2.1) is not invertible.

In order to adopt this approach in the asymptotic setting, it is crucial that Wassermann's proof of Theorem 6 in [19] gives a tiny bit of additional information. Recall, [7, 14], that 1 is an isolated point in the spectrum of $\frac{1}{n}\sum_{i=1}^{n} g_i$ in $C^*(G)$, and that the corresponding spectral projection p is the support projection of the trivial representation. In particular, $p = h\big(\frac{1}{n}\sum_{i=1}^{n} g_i\big)$ for an appropriately chosen continuous function h on $[-1, 1]$, and then it is clear that Wassermann's argument for Theorem 6 in [19] proves that the image of p is non-zero in $E \otimes_{\min} B/\mathbb{K} \otimes B$, but zero in $A \otimes_{\min} B$. It is this fact, that the non-invertibility of (2.1) can be detected by the non-vanishing of a projection in a certain C^*-algebra, which makes it possible to adopt it to the asymptotic case.

2.2 Left Asymptotic Tensor C^*-Norm

Let us review the construction of the left asymptotic tensor norm [12]. Let $\varphi = (\varphi_t)_{t\in[1,\infty)}\colon A \to \mathbb{L}(H_1)$ be an asymptotic homomorphism from A to the C^*-algebra of bounded operators $\mathbb{L}(H_1)$ of some Hilbert space H_1, referred to in the following as *an asymptotic representation* of A, and let $\pi\colon B \to \mathbb{L}(H_2)$ be a (genuine) representation of B. Then φ and π define in the natural way two commuting $*$-homomorphisms,

$$A \to C_b\big([1,\infty), \mathbb{L}(H_1 \otimes H_2)\big)\,/C_0\big([1,\infty), \mathbb{L}(H_1 \otimes H_2)\big)\,,$$

$$B \to C_b\big([1,\infty), \mathbb{L}(H_1 \otimes H_2)\big)\,/C_0\big([1,\infty), \mathbb{L}(H_1 \otimes H_2)\big)\,,$$

giving rise to a $*$-homomorphism

$$\varphi \odot \pi\colon A \odot B \to C_b\big([1,\infty), \mathbb{L}(H_1 \otimes H_2)\big)\,/C_0\big([1,\infty), \mathbb{L}(H_1 \otimes H_2)\big)\,.$$

The left asymptotic tensor norm $\|\cdot\|_\lambda$, defined on $A \odot B$, is

$$\|c\|_\lambda = \sup_{\varphi,\pi} \|\varphi \odot \pi(c)\|,$$

where the supremum is taken over all asymptotic representations of A and all representations of B. On a linear combination, $c = \sum_{i=1}^{m} a_i \otimes b_i$, of simple tensors,

$$\|c\|_\lambda = \sup_{\varphi,\pi}\Big(\limsup_{t\to\infty}\Big\|\sum_{i=1}^{m} \varphi_t(a_i) \otimes \pi(b_i)\Big\|\Big)\,.$$

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Let $A \otimes_\lambda B$ be the completion of $A \odot B$ with respect to the norm $\|\cdot\|_\lambda$.

It is a convenient fact that the left asymptotic tensor norm can be calculated using only a single representation of B.

Lemma 2.2 *Let $\pi'\colon B \to \mathbb{L}(H')$ be a faithful representation of B such that $\pi'(B) \cap \mathbb{K}(H') = \{0\}$. Then $\|c\|_\lambda = \sup_\varphi \|\varphi \odot \pi'(c)\|$ for all $c \in A \odot B$.*

Proof Let $\rho\colon B \to \mathbb{L}(H_2)$ be an arbitrary representation of B. We claim that

$$\|\varphi \odot \rho(c)\| \leq \|\varphi \odot \pi'(c)\| \tag{2.3}$$

for every asymptotic representation φ of A and every $c \in A \odot B$. To prove this, let $\varepsilon > 0$ and write $c = \sum_{i=1}^m a_i \otimes b_i$. By Voiculescu's non-commutative Weyl–von Neumann theorem [15], there is an isometry $V\colon H_2 \to H'$ such that

$$\sum_{i=1}^m \|a_i\|\|\rho(b_i) - V^*\pi'(b_i)V\| \leq \varepsilon.$$

Since $\limsup_{t\to\infty} \|\varphi_t(a_i)\| \leq \|a_i\|$, it follows that

$$\begin{aligned}
\|\varphi \odot \rho(c)\| &\leq \limsup_{t\to\infty} \Big\| \sum_{i=1}^m \varphi_t(a_i) \otimes V^*\pi'(b_i)V \Big\| + \varepsilon \\
&\leq \limsup_{t\to\infty} \Big\| (1 \otimes V^*) \Big(\sum_{i=1}^m \varphi_t(a_i) \otimes \pi'(b_i) \Big) (1 \otimes V^*) \Big\| + \varepsilon \\
&\leq \limsup_{t\to\infty} \Big\| \sum_{i=1}^m \varphi_t(a_i) \otimes \pi'(b_i) \Big\| + \varepsilon,
\end{aligned}$$

proving (2.3) and hence the lemma. ■

We now show how the asymptotic tensor norm can be used in proving non-semi-invertibility of an extension. Note that, thanks to the exactness of the maximal tensor product, $E \otimes_{\min} B/\mathbb{K} \otimes B$ is a quotient of $A \otimes_{\max} B$. On the other hand, $A \otimes_{\min} B$ is the quotient of $E \otimes_{\min} B/\mathbb{K} \otimes B$. Therefore $A \odot B$ is a dense subspace in $E \otimes_{\min} B/\mathbb{K} \otimes B$. We denote the norm on $A \odot B$ inherited from $E \otimes_{\min} B/\mathbb{K} \otimes B$ by $\|\cdot\|_E$. Since this norm is a cross-norm, we may view $E \otimes_{\min} B/\mathbb{K} \otimes B$ as a tensor product of A and B and write $E \otimes_{\min} B/\mathbb{K} \otimes B = A \otimes_E B$. Recall that $A \odot B$ is dense in $A \otimes_\lambda B$ as well.

Lemma 2.3 *Suppose that there exists $c \in A \odot B$ such that $\|c\|_E > \|c\|_\lambda$. Then the extension* (2.1) *is not semi-invertible.*

Proof The idea of the proof is borrowed from [17]. Suppose the contrary, *i.e.,* that (2.1) is semi-invertible. Then there exists an extension $0 \longrightarrow \mathbb{K} \longrightarrow E' \xrightarrow{q'} A \longrightarrow 0$

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and an asymptotic splitting $s = (s_t)_{t\in[0,\infty)}\colon A \to C$, where $D \subset M_2(\mathbb{L}(H))$ is the C^*-subalgebra

$$D = \left\{ \begin{pmatrix} e & b_1 \\ b_2 & e' \end{pmatrix} : b_1, b_2 \in \mathbb{K}, e \in E, e' \in E', q(e) = q'(e') \right\}.$$

By definitition of the left asymptotic tensor norm there is an asymptotic homomorphism $s_t \otimes_\lambda \mathrm{id}_B\colon A \otimes_\lambda B \to D \otimes_{\min} B$ with the property that

$$\lim_{t\to\infty} s_t \otimes_\lambda \mathrm{id}_B\,(a \odot b) - s_t(a) \odot b = 0$$

on simple tensors. Let $d\colon D \to E$ be the completely positive contraction given by $d\begin{pmatrix} e & b_1 \\ b_2 & e' \end{pmatrix} = e$. Then the map $d \otimes \mathrm{id}_B\colon D \otimes_{\min} B \to E \otimes_{\min} B$ is a well-defined contraction. Let $q_B\colon E \otimes_{\min} B \to E \otimes_{\min} B/\mathbb{K} \otimes B = A \otimes_E B$ be the quotient map. Consider the composition

$$r_t = q_B \circ (d \otimes \mathrm{id}_B) \circ (s_t \otimes_\lambda \mathrm{id}_B)\colon A \otimes_\lambda B \to A \otimes_E B.$$

The maps q_B and $d \otimes \mathrm{id}_B$ are contractions and the family $(s_t \otimes_\lambda \mathrm{id}_B)_{t\in[1,\infty)}$ is asymptotically contractive, so the family $(r_t)_{t\in[1,\infty)}$ is asymptotically contractive. Since $\lim_{t\to\infty} r_t(c) - c = 0$, it follows that $\|c\|_E = \limsup_{t\to\infty} \|r_t(c)\|_E \le \|c\|_\lambda$. The contradiction to $\|c\|_E > \|c\|_\lambda$ completes the proof. ■

Let $f(t)$ be a polynomial $f(t) = \frac{1}{4}t^2 + \frac{1}{2}t + \frac{1}{4}$, and set $x = f\left(\frac{1}{n}\sum_{i=1}^n g_i\right) \in C^*(G)$. Then $0 \le x \le 1$ and 1 is an isolated point in the spectrum of x. Put $\Delta(x) \in A \odot B$. As pointed out above, Wassermann has shown that the spectrum of the element $\Delta(\frac{1}{n}\sum_{i=1}^n g_i) \in A \odot B$ in the quotient $E \otimes_{\min} B/\mathbb{K} \otimes B$ contains 1, and it follows that $\|\Delta(x)\|_E = 1$. By Lemma 2.3, Theorem 2.1 will follow if we show that

$$\|\Delta(x)\|_\lambda < 1. \tag{2.4}$$

Let $\Delta_\lambda\colon C^*(G) \to A \otimes_\lambda B$ be the $*$-homomorphism determined by the condition that $\Delta_\lambda(g) = q\left(\pi(g)\right) \otimes \pi(g), g \in G$. The desired conclusion (2.4) is then equivalent to $\|\Delta_\lambda(p)\|_\lambda = 0$, because 1 is isolated in the spectrum of x.

2.3 Calculation of $\|\Delta_\lambda(p)\|_\lambda$

Lemma 2.4 *One has* $\|\Delta_\lambda(p)\|_\lambda = 0$.

Proof Set $H' = \bigoplus_{i=1}^\infty H$ and let $i_\infty\colon B \to \mathbb{L}(H')$ be the infinite sum of copies of the inclusion $B \subseteq \mathbb{L}(H)$. Then $\|c\|_\lambda = \sup_\varphi \|\varphi \odot i_\infty(c)\|$ for all $c \in A \odot B$ by Lemma 2.2. Let $\varepsilon \in (0, \frac{1}{100})$. There is then an asymptotic representation $\varphi\colon A \to \mathbb{L}(H_1)$ and an equi-continuous asymptotic representation $\Phi\colon A \otimes_\lambda B \to \mathbb{L}(H_1 \otimes H')$ such that

$$\limsup_{t\to\infty} \|\Phi_t(\Delta_\lambda(p))\| \ge \|\Delta_\lambda(p)\|_\lambda - \varepsilon, \tag{2.5}$$

$$\lim_{t\to\infty} \|\Phi_t\,(c) - \sum_{i=1}^m \varphi_t(a_i) \otimes i_\infty(b_i)\| = 0 \tag{2.6}$$

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for all $c = \sum_{i=1}^m a_i \otimes b_i \in A \odot B$. For each k, let q_k' be the orthogonal projection onto the support in H of the representation π_k, and let $q_k = \prod_{i=1}^\infty q_k' \in \mathbb{L}(H')$ be the infinite repeat of q_k'. Note that each $1_{H_1} \otimes q_k$ commutes with $\sum_{i=1}^m \varphi_t(a_i) \otimes i_\infty(b_i)$ for all t and all $c = \sum_{i=1}^m a_i \otimes b_i \in A \odot B$. By approximating $\Delta_\lambda(p)$ with elements from $A \odot B$, we can find an element $z = \sum_{i=1}^m a_i \otimes b_i \in A \odot B$ such that

$$\limsup_{t\to\infty} \|\Phi_t(\Delta_\lambda(p)) - \sum_{i=1}^m \varphi_t(a_i) \otimes i_\infty(b_i)\| < \varepsilon. \tag{2.7}$$

To simplify notation, set $z_t = \sum_{i=1}^m \varphi_t(a_i) \otimes i_\infty(b_i)$, and $y_t = \frac{1}{2}(z_t + z_t^*)$. Since Φ is an asymptotic homomorphism and $\Delta_\lambda(p)$ a projection, it follows from (2.7) that for some $T > 0$,

$$\|y_t^2 - y_t\| \leq 5\varepsilon \tag{2.8}$$

when $t \geq T$. It follows that

$$\left\| ((1_{H_1} \otimes q_k)y_t)^2 - (1_{H_1} \otimes q_k)y_t \right\| \leq 5\varepsilon$$

for all $t > T$. Since $5\varepsilon < 1/4$, we find that the characteristic function $h = 1_{[1/2,\infty)}$ is continuous on the spectrum of y_t and on the spectrum of each $(1_{H_1} \otimes q_k)y_t$ when $t > T$. It follows that $h(y_t)$ and $h((1_{H_1} \otimes q_k)y_t)$ are projections for all k and all $t > T$. We claim that

$$h(y_t) = 0 \tag{2.9}$$

for all $t > T$. If not, there is some $t_0 > T$ such that $h(y_{t_0}) \neq 0$. There must then be a k, which we now fix, such that $h((1_{H_1} \otimes q_k)y_{t_0}) \neq 0$ since $\sum_i 1_{H_1} \otimes q_i = 1$. But then

$$\|h((1_{H_1} \otimes q_k)y_t)\| = 1 \tag{2.10}$$

for all $t > T$, since $h(t(1_{H_1} \otimes q_k)y_t)$ varies norm-continuously with t and is a projection for all $t > T$. Let $\rho_k\colon B \to C^*(\pi_k(G))$ be the finite-dimensional representation of B obtained by restricting the elements of B to the subspace of H supporting the representation π_k of G. There is then a representation $\mu\colon C^*(\pi_k(G)) \to \mathbb{L}(H')$ such that

$$\mu \circ \rho_k(b) = q_k i_\infty(b) \tag{2.11}$$

for all $b \in B$. Furthermore, there is an equi-continuous asymptotic homomorphism $\psi\colon A \otimes C^*(\pi_k(G)) \to \mathbb{L}(H_1 \otimes H')$ such that

$$\lim_{t\to\infty} \left\| \psi_t(c) - \sum_{i=1}^m \varphi_t(a_i) \otimes \mu(x_i) \right\| = 0 \tag{2.12}$$

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for all $c = \sum_{i=1}^{m} a_i \otimes x_i \in A \odot C^*(\pi_k(G))$. Note that $\mathrm{id}_A \otimes \rho_k \colon A \odot B \to A \odot C^*(\pi_k(G))$ extends to a $*$-homomorphism $\kappa \colon A \otimes_\lambda B \to A \otimes C^*(\pi_k(G))$. It follows from (2.6), (2.11) and (2.12) that

$$\lim_{t\to\infty} \|\psi_t \circ \kappa(d) - (1_{H_1} \otimes q_k)\Phi_t(d)\| = 0 \tag{2.13}$$

for all $d \in A \otimes_\lambda B$. Since κ factors through $A \otimes_{\min} B$, we know from [19] that $\|\kappa((\Delta_\lambda(p)))\| = 0$. It follows therefore from (2.13) that

$$\limsup_{t\to\infty} \left\| (1_{H_1} \otimes q_k)\Phi_t\big((\Delta_\lambda(p))\big) \right\| = 0,$$

and then by use of (2.7) that

$$\limsup_{t\to\infty} \|(1_{H_1} \otimes q_k)y_t\| \leq \varepsilon.$$

Since $\varepsilon < 1/2$, this contradicts (2.10), and we conclude that (2.9) must hold. Combined with (2.8) we find that the spectrum of y_t is contained in $[-1/2, 1/2]$, and hence that $\|y_t\| \leq 1/2$. It follows then from (2.7) that

$$\limsup_{t\to\infty} \|\Phi_t(\Delta_\lambda(p))\| \leq 1/2 + \varepsilon < 1.$$

Since $\Delta_\lambda(p)$ is a projection, we deduce first that $\limsup_{t\to\infty} \|\Phi_t(\Delta_\lambda(p))\| = 0$, and then from (2.5) that $\|\Delta_\lambda(p)\|_\lambda = 0$. ∎

2.4 Some Remarks

Theorem 2.1 means that it is not possible to add an extension of A by $\mathbb{K}$ to (2.1) such that the resulting extension admits an asymptotic homomorphism consisting of sections for the quotient map. In particular, the extension (2.1) itself does not admit such a family of sections. This fact may seem slightly surprising because the extension is clearly quasi-diagonal and there is an obvious sequence $s_n \colon A \to E$, $n = 1, 2, \dots$ of maps, each of which is a section for the quotient map such that they form a *discrete* asymptotic homomorphism. It was therefore no coincidence that the connectedness of the parameter space $[0, \infty)$ was used at a crucial point in the proof above.

In [12] we raised the question, if the left asymptotic tensor product is associative. It follows from Lemma 2.4 that the answer is negative.

We have looked through all examples known to us of non-invertible extensions to check if they are semi-invertible or not. Kirchberg's examples [9]are semi-invertible by results of [11]. Another example by Wassermann [18] can be shown not to be semi-invertible by the same method as here. Unfortunately, we know nothing about semi-invertibility of other examples.

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3 Homotopy Non-Invertibility

3.1 A Modification of the Wassermann's Extension

To give an example of an extension which is not only not semi-invertible, but also not even homotopy invertible, we modify the extension (2.1) as follows. Let d_i be the dimension of the Hilbert space H_i on which the representation π_i acts. Let n_i be a sequence of integers such that $\lim_{i\to\infty} \frac{n_i}{d_i} = \infty$. For each i we let $n_i \cdot \pi_i$ be the direct sum of n_i copies of the representation π_i, and let $\pi' = \bigoplus_{i=1}^{\infty} n_i \cdot \pi_i$ be the direct sum of the resulting sequence of representations acting on the Hilbert space H. Let E' be the C^*-subalgebra of $\mathbb{L}(H)$ generated by $\{\pi'(g) \colon g \in G\}$ and by $\mathbb{K}$, the compact operators on H. Set $A' = E'/\mathbb{K}$. Note that A' is isomorphic to the C^*-algebra A from the Wassermann's example, as both can be defined as completions of the group ring $\mathbb{C}[G]$ with respect to the same (semi)norm $\|\cdot\| = \limsup_{i\to\infty} \|\pi_i(\cdot)\|$.

By $\mathrm{Ext}_h(A, B)$ we denote the semigroup of homotopy classes of extensions of the form (1.1).

Theorem 3.1 *The extension*

$$0 \longrightarrow \mathbb{K} \longrightarrow E' \longrightarrow A' \longrightarrow 0 \tag{3.1}$$

is not invertible in $\mathrm{Ext}_h(A', \mathbb{K})$.

Proof To show that (3.1) is not invertible up to homotopy, let $\varphi\colon A' \to Q(\mathbb{K})$ be the Busby invariant of (3.1), and assume to reach a contradiction that $\psi\colon A' \to Q(\mathbb{K})$ is an extension such that $\varphi \oplus \psi$ is homotopic to 0. Let V_1, V_2 be isometries in $\mathbb{L}(H)$ such that $V_1V_1^* + V_2V_2^* = 1$, and set $\lambda(a) = \mathrm{Ad}\, q(V_1) \circ \varphi(a) + \mathrm{Ad}\, q(V_2) \circ \psi(a)$, $a \in A'$, where $q\colon \mathbb{L}(H) \to Q(\mathbb{K})$ is the quotient map. There is then a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathcal{E} & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{ev}_1} & & \uparrow & & \| & & \\
0 & \longrightarrow & I\mathbb{K} & \longrightarrow & \mathcal{E}' & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{ev}_0} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{K} & \longrightarrow & \mathbb{K} \oplus A' & \longrightarrow & A' & \longrightarrow & 0,
\end{array} \tag{3.2}$$

where λ is a the Busby invariant of the upper extension $I\mathbb{K} = C[0,1] \otimes \mathbb{K}$ and $\mathrm{ev}_s\colon I\mathbb{K} \to \mathbb{K}$ is evaluation at $s \in [0,1]$. Set $D = \prod_{k=1}^{\infty} \mathbb{L}(H_k)$. By tensoring with

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D we obtain from (3.2) the commuting diagram

(3.3)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K}\otimes D & \longrightarrow & \mathcal{E}\otimes_{\min} D & \longrightarrow & \mathcal{E}\otimes_{\min} D/\mathbb{K}\otimes D & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle \mathrm{ev}_1} & & \uparrow & & \uparrow{\scriptstyle p_0} & & \\
0 & \longrightarrow & \mathrm{I}\mathbb{K}\otimes D & \longrightarrow & \mathcal{E}'\otimes_{\min} D & \longrightarrow & \mathcal{E}'\otimes_{\min} D/\mathrm{I}\mathbb{K}\otimes D & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{ev}_0} & & \downarrow & & \downarrow{\scriptstyle p_1} & & \\
0 & \longrightarrow & \mathbb{K}\otimes D & \longrightarrow & (\mathbb{K}\otimes D)\oplus(A'\otimes_{\min} D) & \longrightarrow & A'\otimes_{\min} D & \longrightarrow & 0.
\end{array}$$

Let $\overline{\mathrm{ev}}_s\colon M(\mathrm{I}\mathbb{K}) \to M(\mathbb{K})$ and $\widehat{\mathrm{ev}}_s\colon Q(\mathrm{I}\mathbb{K}) \to Q(\mathbb{K})$ be the $*$-homomorphisms induced by ev_s. Denote $\mathcal{E}\otimes_{\min} D/\mathbb{K}\otimes D$ and $\mathcal{E}'\otimes_{\min} D/\mathrm{I}\mathbb{K}\otimes D$ by $A'\otimes_{\mathcal{E}} D$ and $A'\otimes_{\mathcal{E}'} D$, respectively. The Busby invariant of the middle extension of (3.3) is a $*$-homomorphism $\varphi'\colon A'\otimes_{\mathcal{E}'} D \to Q(\mathrm{I}\mathbb{K}\otimes D)$ such that $\widehat{\mathrm{ev}}_1\circ\varphi' = \mu\circ p_0$, where $\mu\colon A'\otimes_{\mathcal{E}} D \to Q(\mathrm{I}\mathbb{K}\otimes D)$ is the Busby invariant of the upper extension in (3.3), while $\widehat{\mathrm{ev}}_0\circ\varphi' = 0$.

By the Bartle–Graves selection theorem there are continuous sections

$$\chi\colon Q(\mathrm{I}\mathbb{K}\otimes D)\to M(\mathrm{I}\mathbb{K}\otimes D) \quad \text{and} \quad \chi_k\colon Q(\mathrm{I}\mathbb{K}\otimes D_k)\to M(\mathrm{I}\mathbb{K}\otimes D_k)$$

for the quotient maps $M(\mathrm{I}\mathbb{K}\otimes D)\to Q(\mathrm{I}\mathbb{K}\otimes D)$ and $M(\mathrm{I}\mathbb{K}\otimes D_k)\to Q(\mathrm{I}\mathbb{K}\otimes D_k)$, respectively, for all $k = 1,2,3,\dots$. We can choose these maps to be self-adjoint and such that $\|\chi(x)\| \le 2\|x\|$, $x \in Q(\mathrm{I}\mathbb{K}\otimes D)$, and $\|\chi_k(y)\| \le 2\|y\|$, $y \in Q(\mathrm{I}\mathbb{K}\otimes D_k)$, for all k. Set $D_k = \mathbb{L}(H_k)$, and let

$$p_k\colon D = \prod_{k=1}^{\infty} D_k \to D_k$$

be the canonical projection. The map $\mathrm{id}_A\otimes p_k\colon A'\odot D \to A'\odot D_k$ extends to a $*$-homomorphism $\mathrm{id}_A\otimes p_k\colon A'\otimes_{\mathcal{E}'} D \to A'\otimes D_k$. Let

$$\overline{\mathrm{id}_{\mathrm{I}\mathbb{K}}\otimes p_k}\colon M(\mathrm{I}\mathbb{K}\otimes D)\to M(\mathrm{I}\mathbb{K}\otimes D_k)$$

be the unique $*$-homomorphism extending $\mathrm{id}_{\mathrm{I}\mathbb{K}}\otimes p_k\colon\ \mathrm{I}\mathbb{K}\otimes D\to \mathrm{I}\mathbb{K}\otimes D_k$, and

$$\widehat{\mathrm{id}_{\mathrm{I}\mathbb{K}}\otimes p_k}\colon Q(\mathrm{I}\mathbb{K}\otimes D)\to Q(\mathrm{I}\mathbb{K}\otimes D_k)$$

the resulting $*$-homomorphism. Let $\Phi\colon A' \to Q(\mathrm{I}\mathbb{K})$ be the Busby invariant of the middle extension of (3.2).

We denote by $\Phi\widehat{\otimes}\,\mathrm{id}_{D_k}$ the $*$-homomorphism

$$\Phi\widehat{\otimes}\,\mathrm{id}_{D_k}\colon A'\otimes D_k \to Q(\mathrm{I}\mathbb{K}\otimes D_k)$$

obtained by composing $\Phi\otimes\mathrm{id}_{D_k}\colon A'\otimes D_k \to Q(\mathrm{I}\mathbb{K})\otimes D_k$ with the canonical embedding $Q(\mathrm{I}\mathbb{K})\otimes D_k \subseteq Q(\mathrm{I}\mathbb{K}\otimes D_k)$. By checking on simple tensors one finds that

$$\widehat{\mathrm{id}_{\mathrm{I}\mathbb{K}}\otimes p_k}\circ\varphi' = \big(\Phi\widehat{\otimes}\,\mathrm{id}_{D_k}\big)\circ(\mathrm{id}_{A'}\otimes p_k),$$

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which implies that

$$(3.4)\qquad \big(\overline{\mathrm{id}_{\mathrm{I}\mathbb{K}}\otimes p_k}\big)\circ\chi\circ\varphi'(x)-\chi_k\circ(\Phi\widehat{\otimes}\,\mathrm{id}_{D_k})\circ(\mathrm{id}_{A'}\otimes p_k)(x)\in \mathrm{I}\mathbb{K}\otimes D_k$$

for all k and all $x\in A'\otimes_{\mathcal{E}'} D$. Let $\overline{\pi}_i$ be the representation of G contragredient to π_i. The representation

$$g\mapsto q\left(\pi'(g)\right)\otimes\Big(\prod_{k=1}^{\infty}\overline{\pi}_k\Big)(g)$$

of G into $A'\odot D$ gives rise to a $*$-homomorphism $\Delta'\colon C^*(G)\to A'\otimes_{\mathcal{E}'} D$. Set $Q=\Delta'(p)$, where p, as above, is the spectral projection of the element $\frac{1}{n}\sum_{i=1}^{n} g_i$ corresponding to the set $\{1\}$. Since π_i is inequivalent to π_k for $i\neq k$ and π' contains only finitely many copies of each π_k, it follows from [19, Lemma 1] that

$$(3.5)\qquad \mathrm{id}_{A'}\otimes p_k(Q)=0\quad\text{for all } k.$$

It follows from (3.4) and (3.5) that

$$(3.6)\qquad (\overline{\mathrm{id}_{\mathrm{I}\mathbb{K}}\otimes p_k})\circ\chi\circ\varphi'(Q)\in \mathrm{I}\mathbb{K}\otimes D_k$$

for all k. Let P_I and P denote the $*$-homomorphisms

$$P_I=\prod_{k=1}^{\infty}\overline{\mathrm{id}_{\mathrm{I}\mathbb{K}}\otimes p_k}\colon M(\mathrm{I}\mathbb{K}\otimes D)\to\prod_{k=1}^{\infty}M(\mathrm{I}\mathbb{K}\otimes D_k),$$

$$P=\prod_{k=1}^{\infty}\overline{\mathrm{id}_{\mathbb{K}}\otimes p_k}\colon M\left(\mathbb{K}\otimes D\right)\to\prod_{k=1}^{\infty}M\left(\mathbb{K}\otimes D_k\right),$$

respectively. Put

$$N_I=P_I^{-1}\Big(\prod_{k=1}^{\infty}\mathrm{I}\mathbb{K}\otimes D_k\Big)\subset M(\mathrm{I}\mathbb{K}\otimes D),$$

$$N=P^{-1}\Big(\prod_{k=1}^{\infty}\mathbb{K}\otimes D_k\Big)\subset M(\mathbb{K}\otimes D).$$

It follows from (3.6) that $\chi\circ\varphi'(Q)\in N_I$. Note that $\mathrm{I}\mathbb{K}\otimes D$ is an ideal in N_I and $\mathbb{K}\otimes D$ is an ideal in N. We denote the quotients $N_I/\,\mathrm{I}\mathbb{K}\otimes D$ and $N/\mathbb{K}\otimes D$ by R_I and R, respectively. Note that $R_I\subseteq Q\,(\mathrm{I}\mathbb{K}\otimes D)$ and that $\varphi'(Q)\in R_I$. Evaluation at $s\in[0,1]$ induces a $*$-homomorphism $E_s\colon M(\mathrm{I}\mathbb{K}\otimes D)\to M(\mathbb{K}\otimes D)$ with the property that $E_s(N_I)=N$, so we get a $*$-homomorphism $\widehat{E}_s\colon R_I\to R$ induced by E_s for each $s\in[0,1]$. To proceed with the proof, we need some calculations in K-theory.

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3.2 K-Theory Calculations

Consider the extensions

$$0 \longrightarrow \mathrm{I}\mathbb{K} \otimes D \longrightarrow N_I \longrightarrow R_I \longrightarrow 0, \tag{3.7}$$

$$0 \longrightarrow \mathbb{K} \otimes D \longrightarrow N \longrightarrow R \longrightarrow 0. \tag{3.8}$$

The map

$$\prod_{k=1}^{\infty} p_{k*} : K_0(\mathrm{I}\mathbb{K} \otimes D) \to \prod_{k=1}^{\infty} K_0(\mathrm{I}\mathbb{K} \otimes D_k)$$

is injective by [4, Lemma 3.2] and the map

$$\prod_{k=1}^{\infty} p_{k*} : K_1(\mathrm{I}\mathbb{K} \otimes D) \to \prod_{k=1}^{\infty} K_1(\mathrm{I}\mathbb{K} \otimes D_k)$$

is injective by [4, Lemma 3.3]. In particular, $K_1(\mathrm{I}\mathbb{K} \otimes D) = 0$. Therefore the extension (3.7) gives us a commuting diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_0(\mathrm{I}\mathbb{K} \otimes D) & = & K_0(\mathrm{I}\mathbb{K} \otimes D) \\
\downarrow & & \downarrow \\
K_0(N_I) & \xrightarrow{p_{I*}} & \prod_{k=1}^{\infty} K_0(\mathrm{I}\mathbb{K} \otimes D_k) \\
\downarrow & & \downarrow \\
K_0(R_I) & \longrightarrow & \left(\prod_{k=1}^{\infty} K_0(\mathrm{I}\mathbb{K} \otimes D_k)\right) / K_0(\mathrm{I}\mathbb{K} \otimes D) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

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Throwing away the interval, we also get a commuting diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_0(\mathbb{K} \otimes D) & = & K_0(\mathbb{K} \otimes D) \\
\downarrow & & \downarrow \\
K_0(N) & \xrightarrow{P_*} & \prod_{k=1}^{\infty} K_0(\mathbb{K} \otimes D_k) \\
\downarrow & & \downarrow \\
K_0(R) & \longrightarrow & \left(\prod_{k=1}^{\infty} K_0(\mathbb{K} \otimes D_k)\right) / K_0(\mathbb{K} \otimes D) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Evaluation at any $s \in [0, 1]$ induces an isomorphism

$$\left(\prod_{k=1}^{\infty} K_0(\mathrm{I}\mathbb{K} \otimes D_k)\right) / K_0(\mathrm{I}\mathbb{K} \otimes D) \to \left(\prod_{k=1}^{\infty} K_0(\mathbb{K} \otimes D_k)\right) / K_0(\mathbb{K} \otimes D)$$

in the obvious way, and the diagram

$$\begin{array}{ccc}
K_0(R_I) & \xrightarrow{\widehat{E}_{s*}} & K_0(R) \\
\downarrow & & \downarrow \\
\left(\prod_{k=1}^{\infty} K_0(\mathrm{I}\mathbb{K} \otimes D_k)\right) / K_0(\mathrm{I}\mathbb{K} \otimes D) & \longrightarrow & \left(\prod_{k=1}^{\infty} K_0(\mathbb{K} \otimes D_k)\right) / K_0(\mathbb{K} \otimes D)
\end{array}$$

commutes for every $s \in [0, 1]$. Let x be the image in

$$\left(\prod_{k=1}^{\infty} K_0(\mathbb{K} \otimes D_k)\right) / K_0(\mathbb{K} \otimes D)$$

of the element $[\varphi'(Q)] \in K_0(R_I)$. Since $\widehat{\mathrm{ev}}_0 \circ \varphi' = 0$, we get that $\widehat{E}_0(\varphi'(Q)) = 0$, which leads to the conclusion that

$$x = 0. \tag{3.9}$$

As we shall see, we get a different result when we consider the case $s = 1$. Let $r\colon N \to R$ be the quotient map. Set $W_j = V_j \otimes 1_D \in M(\mathbb{K} \otimes D)$, $j = 1, 2$. Then

$$\widehat{E}_1 \circ \varphi'(Q) = r\left(W_1 e W_1^* + W_2 a W_2^*\right), \tag{3.10}$$

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where e is the spectral projection of $\frac{1}{n}\sum_{i=1}^{n}\pi'(g_i)\otimes\left(\bigoplus_{k=1}^{\infty}\overline{\pi}_k\right)(g_i)$ corresponding to $\{1\}$, and $a \geq 0$ is some lift in N of a projection in $R \subset Q(\mathbb{K}\otimes D)$. Since $W_jN \subset N$ and $W_j^*N \subset N$, the W_j's define multipliers, first of N, and then of R. It follows therefore from (3.10) that $[\widehat{E}_1 \circ \varphi'(Q)] = [r(e)] + [r(a)]$ in $K_0(R)$.

Consider the extension

$$0 \longrightarrow \mathbb{K}\otimes D \longrightarrow N^+ \xrightarrow{r^+} R^+ \longrightarrow 0 \tag{3.11}$$

obtained from the extension (3.8) by unitalizing. It follows from [5, Lemma 9.6] that there are natural numbers n, m such that $r(a) \oplus 1_n \oplus 0_m \in M_{1+n+m}(R^+)$ can be lifted to a projection $f_1 \in M_{1+n+m}(N^+)$. Note that the image of f_1 in $M_{1+n+m}(\mathbb{C})$ under the canonical surjection $M_{1+n+m}(N^+) \to M_{1+n+m}(\mathbb{C})$ is a projection of rank n. (We use here that the proof in [5] works equally well when the assumption that the ideal is AF is replaced by the weaker assumption, valid in (3.11), that the ideal has trivial K_1-group.) There is a commuting diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K_0(\mathbb{K}\otimes D) & = & K_0(\mathbb{K}\otimes D) \\
\downarrow & & \downarrow \\
K_0(N^+) & \xrightarrow{P_*^+} & \prod_{k=1}^{\infty} K_0\left((\mathbb{K}\otimes D_k)^+\right) \\
\downarrow & & \downarrow \\
K_0(R^+) & \longrightarrow & \left(\prod_{k=1}^{\infty} K_0\left((\mathbb{K}\otimes D_k)^+\right)\right)/K_0(\mathbb{K}\otimes D) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

with exact columns. Thus the image of x in $\left(\prod_{k=1}^{\infty} K_0\left((\mathbb{K}\otimes D_k)^+\right)\right)/K_0(\mathbb{K}\otimes D)$ under the inclusion

$$\left(\prod_{k=1}^{\infty} K_0(\mathbb{K}\otimes D_k)\right)/K_0(\mathbb{K}\otimes D) \subseteq \left(\prod_{k=1}^{\infty} K_0((\mathbb{K}\otimes D_k)^+)\right)/K_0(\mathbb{K}\otimes D)$$

is also the image of $P_*^+\left([e]+[f_1]-[1_n]\right) \in \prod_{k=1}^{\infty} K_0((\mathbb{K}\otimes D_k)^+)$ under the quotient map

$$\prod_{k=1}^{\infty} K_0((\mathbb{K}\otimes D_k)^+) \to \left(\prod_{k=1}^{\infty} K_0\left((\mathbb{K}\otimes D_k)^+\right)\right)/K_0(\mathbb{K}\otimes D).$$

Write $P^+(f_1) = (h_k)$, where $h_k \in M_{1+n+m}\left((\mathbb{K} \otimes D_k)^+\right)$ for each k is a projection whose image in $M_{1+n+m}(\mathbb{C})$ under the canonical surjection $M_{1+n+m}\left((\mathbb{K} \otimes D_k)^+\right) \to M_{1+n+m}(\mathbb{C})$ is a projection of rank n. Since $\bigcup_j M_{1+n+m}((M_j(\mathbb{C}) \otimes D_k)^+)$ is dense in $M_{1+n+m}((\mathbb{K} \otimes D_k)^+)$, there is a $j \in \mathbb{N}$ and a projection $f_2^k \in M_{1+n+m}((M_j(\mathbb{C}) \otimes D_k)^+)$ which is unitarily equivalent to h_k. Since

$$M_{1+n+m}((M_j(\mathbb{C}) \otimes D_k)^+) = M_{1+n+m}\left(M_j(\mathbb{C}) \otimes D_k\right) \oplus M_{1+n+m}(\mathbb{C}),$$

we get $f_2^k = f_3^k + f_4^k$, with f_3^k and f_4^k orthogonal projections in $M_{1+n+m}((\mathbb{K} \otimes D_k)^+)$, $f_3^k \in M_{1+n+m}(\mathbb{K} \otimes D_k)$, and $[f_4^k] = [1_n]$ in $K_0((\mathbb{K} \otimes D_k)^+)$. It follows that

$$P_*^+\left([e] + [f_1] - [1_n]\right) = P_*[e] + \left([f_3^k]\right)_{k=1}^{\infty} \in \prod_{k=1}^{\infty} K_0\left(\mathbb{K} \otimes D_k\right).$$

Now we identify $K_0(\mathbb{K} \otimes D_k)$ with $\mathbb{Z}$ as ordered groups, and consequently

$$\prod_{k=1}^{\infty} K_0\left(\mathbb{K} \otimes D_k\right)$$

with $\prod_{k=1}^{\infty} \mathbb{Z}$. Then $P_*([e]) = (a_k)_{k=1}^{\infty}$ and $\left([f_3^k]\right)_{k=1}^{\infty} = (b_k)_{k=1}^{\infty}$, where $b_k \geq 0$ for all k, and a_k is greater or equal to the multiplicity of the trivial representation of G in $\pi' \otimes \overline{\pi}_k$ which equals n_k. Since $K_0(\mathbb{K} \otimes D)$ is the subgroup of $\prod_{k=1}^{\infty} \mathbb{Z}$ consisting of the sequences (c_k) in $\mathbb{Z}$ for which $\sup_k \left|\frac{c_k}{d_k}\right| < \infty$ by [4, Lemma 3.2], we conclude that $x \neq 0$ because $\lim_{k\to\infty} \frac{n_k}{d_k} = \infty$. This contradicts (3.9). ∎

3.3 The Connes–Higson Construction Is Not Faithful

Let $E(\cdot\,,\,\cdot)$ denote E-theory group of Connes and Higson [3]. The Connes–Higson construction produces asymptotic homomorphisms out of extensions and defines an additive map $CH\colon \operatorname{Ext}_h(A, B) \to E(SA, B)$, where $SA = C_0(0, 1) \otimes A$ is the suspension of A. It was shown in [11] that the map CH is an isomorphism when A is already a suspension.

Lemma 3.2 *The image of the extension* (3.1) *under the Connes–Higson construction is homotopy trivial.*

Proof For the extension (3.1), consider the six-term exact sequence in E-theory,

$$\begin{array}{ccccc} E(A', \mathbb{K}) & \longrightarrow & E(E', \mathbb{K}) & \longrightarrow & E(\mathbb{K}, \mathbb{K}) \\ \uparrow & & & & \downarrow \partial \\ E(S\mathbb{K}, \mathbb{K}) & \longleftarrow & E(SE', \mathbb{K}) & \longleftarrow & E(SA', \mathbb{K}) \end{array}$$

Let $\varphi\colon A' \to Q(\mathbb{K})$ be the Busby invariant of the extension (3.1). Then the image of the class $[CH(\varphi)] \in E(SA', \mathbb{K})$ under the map $E(SA', \mathbb{K}) \to E(SE', \mathbb{K})$ is zero because the composition $E' \to A' \to Q(\mathbb{K})$ of φ and of the quotient map defines a split (hence trivial) extension of E'. Thus, $[\varphi] \in \operatorname{im}(\partial)$.

To show that $\partial = 0$, one has to show that the map $E(E', \mathbb{K}) \to E(\mathbb{K}, \mathbb{K}) = \mathbb{Z}$ (or, equivalently, the map $\operatorname{Ext}_h(SE', \mathbb{K}) \to \operatorname{Ext}_h(S\mathbb{K}, \mathbb{K})$) is surjective. Let us construct an extension $SE' \to Q(\mathbb{K} \otimes \mathbb{K})$, whose restriction onto $S\mathbb{K}$ generates $\operatorname{Ext}_h(S\mathbb{K}, \mathbb{K})$.

Let H_i be the Hilbert space of the representation $\pi_i' = n_i \cdot \pi_i$. We can assume that π_1' is the one-dimensional trivial representation of G of multiplicity 1. Put $H_i^m = H_i$ for $m \in \mathbb{N}$ and $H = \bigoplus_{i,m} H_i^m$. Let $g \in G$ act on H as $\bigoplus_i \pi_i'(g)$ on each $\bigoplus_i H_i^m$, $m \in \mathbb{N}$. If we identify $\mathbb{K}$ with $\mathbb{K}(\bigoplus_i H_i^m)$, let $k \in \mathbb{K}$ act similarly on H as $\bigoplus_m k_m$, where $k_m = k$ acts on $\bigoplus_i H_i^m$. This gives us a $*$-homomorphism from E' to $B(H)$. Let us also define a Fredholm operator F on H by setting $F|_{H_i^m} = \mathrm{id}$ for all i, m except $i = 1$ and $F(H_1^m) = H_1^{m+1}$ (shift with respect to the superscript). This operator defines a map from $C(\mathbb{T})$ to $B(H)$, which is a $*$-homomorphism modulo compacts. Since the two maps, from E' and from $C(\mathbb{T})$, commute modulo compacts, we have a map from $C(\mathbb{T}) \otimes E'$ to $B(H)$, which becomes a $*$-homomorphism after composing it with the quotient map $B(H) \to Q(\mathbb{K})$. Restrict this map to SE' and denote the resulting extension by Φ, $\Phi \in \operatorname{Ext}_h(SE', \mathbb{K})$. Let also $\iota\colon \mathbb{C} \to \mathbb{K}$ be the standard homomorphism given by $\iota(1) = e_{11}$. Then the diagram

$$\begin{array}{ccc} & \operatorname{Ext}_h(S\mathbb{C}, \mathbb{K}) & \\ {}^{j^*}\nearrow & & \nwarrow^{\iota^*} \\ \operatorname{Ext}_h(SE', \mathbb{K}) & \xrightarrow{\quad r \quad} & \operatorname{Ext}_h(S\mathbb{K}, \mathbb{K}) \end{array}$$

commutes, where arrows are induced by inclusions. One easily sees that $j^*(\Phi)$ coincides with the Toeplitz extension. Since ι^* is an isomorphism, r is surjective, hence $\partial = 0$. Since $[CH(\varphi)]$ lies in the image of ∂, one has $[CH(\varphi)] = 0$. ■

Note that φ is not trivial in $\operatorname{Ext}_h(A', \mathbb{K})$ (otherwise it would be invertible, which is not true). Thus we can conclude that the assertion of Connes and Higson in [3] that "La E-théorie est ainsi le quotient par homotopie de la théorie des extensions" should not be taken too literally. In fact, the exact relation between C^*-algebra extensions and asymptotic homomorphisms is still not fully uncovered, although our previous work on the subject contains much detailed information.

Acknowledgement The first named author is grateful to S. Wassermann for sending him a copy of his book [20].

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