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APPROXIMATING APPROXIMATE FIBRATIONS BY FIBRATIONS

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1. Introduction. A map $p: E \to B$ between metric spaces has the *approximate homotopy lifting property with respect to the space* X if given a cover $\bar{U}$ of B and maps $g: X \to E$ and $H: X \times [0, 1] \to B$ such that $H(x, 0) = pg(x)$ for all $x \in X$, then there exists a map $G: X \times [0, 1] \to E$ such that $G(x, 0) = g(x)$ and pG_t and H_t are $\bar{U}$-close for all $x \in X$ and $t \in [0, 1]$; i.e., given $(x, t) \in X \times [0, 1]$, there exists $U \in \bar{U}$ such that $pG(x, t)$ and $H(x, t)$ are elements of U. $p: E \to B$ is an *approximate fibration* if p has the approximate homotopy lifting property with respect to all spaces X. Coram and Duvall [**13**] introduced these concepts as a generalization of UV^{∞} maps and showed that approximate fibrations have many properties in common with Hurewicz fibrations if one uses shape theoretic concepts in place of their homotopy theoretic analogues (see, for example, Propositions 1 and 2). Hence, an approximate fibration may be regarded as the shape theoretic analogue of Hurewicz fibrations. They also showed that the uniform limit of a sequence of Hurewicz fibrations between two compact ANR's is an approximate fibration.

In [**25**], Lacher showed that a cell-like mapping between ANR's has the approximate homotopy lifting property with respect to polyhedra. By [**14**] (see Proposition 4), this implies that cell-like mappings are approximate fibrations. Armentrout [**2**], Siebenmann [**30**] and Finney [**17**] showed that cell-like mappings between manifolds of dimension $\neq 4$ are precisely those mappings which can be approximated by homeomorphisms. Hence, the natural question arises whether the approximate fibrations between manifolds are precisely those mappings which can be approximated by Hurewicz fibrations.

Recall that a map $F: E \to B$ is a *locally trivial fiber map* if for each $x \in B$ there exists a neighborhood U of x in B and a homeomorphism h of $F^{-1}(x) \times U$ onto $F^{-1}(U)$ such that $Fh(y, z) = z$ for all $(y, z) \in F^{-1}(x) \times U$. If B is paracompact and if F is a locally trivial fiber map, then F is a Hurewicz fibration [**31**; p. 96]. In this note the following results are proved.

THEOREM A. *Let E be a closed connected 3-manifold such that each inessential tame 2-sphere in E bounds a 3-cell and let B be a connected n-manifold, $n = 1, 2$. Let $f: E \to B$ be an approximate fibration and let $\epsilon > 0$ be given. If $n = 1$, then assume that $\check{\pi}_1(F) \neq Z_2$ where F is a fiber of f. Then there exists a locally trivial fiber map $g: E \to B$ such that $d(f, g) < \epsilon$.*

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The condition on inessential 2-spheres is related to the Poincaré conjecture and the condition on $\tilde{\pi}_1(F)$ is related to the conjecture that an h-cobordism bounded by projective planes is a product cobordism. A counterexample to either of these two conjectures would provide a counterexample to Theorem A if the corresponding hypothesis were removed.

THEOREM B. *Let $f: M \to S^1$ be an approximate fibration such that either* i) *M is a closed connected m-dimensional manifold, $m \geqq 6$, or* ii) *M is a compact connected Hilbert cube manifold. f can be approximated arbitrarily close by a locally trivial fiber map if and only if f is homotopic to a Hurewicz fibration.*

THEOREM C. *There exists a closed connected manifold M and an approximate fibration $f: M \to S^1$ such that f cannot be approximated arbitrarily close by a Hurewicz fibration. Let Q denote the Hilbert cube and let $\pi: M \times Q \to M$ denote the projection on the first factor. The map $f\pi: M \times Q \to S^1$ is an approximate fibration of a Hilbert cube manifold which cannot be approximated arbitrarily close by a Hurewicz fibration.*

Combining Theorem B with a result of R. D. Edwards [**16**], we have the following result for arbitrary compact ANR's.

COROLLARY D. *Let $f: M \to S^1$ be an approximate fibration of a compact ANR M onto S^1. f can be stably approximated arbitrarily close by a Hurewicz fibration if and only if f is stably homotopic to a Hurewicz fibration;* i.e., *$f\pi: M \times Q \to S^1$ can be approximated arbitrarily close by a Hurewicz fibration if and only if f is homotopic to a Hurewicz fibration.*

THEOREM E. *Let $f: M \to R^1$ be a proper approximate fibration onto the real numbers and suppose that either* i) *M is a connected m-dimensional manifold, $m \geqq 6$, or* ii) *M is a connected Hilbert cube manifold. f can be approximated arbitrarily close by a proper locally trivial fiber map if and only if f is properly homotopic to a Hurewicz fibration.*

Siebenmann [**29**] has determined necessary and sufficient conditions that a mapping of a closed connected m-dimensional manifold onto S^1, $m \geqq 6$, be homotopic to a locally trivial fiber map.

COROLLARY F. *Let $f: M \to S^1$ be an approximate fibration and suppose that M is a closed connected m-dimensional manifold, $m \geqq 6$. f can be approximated arbitrarily close by a locally trivial fiber map if and only if Siebenmann's obstruction $F(M)$ in the Whitehead group of $\pi_1 M$ vanishes.*

R. Goad [**18**] has obtained a higher-dimensional analogue of Theorem A for approximate fibrations between manifolds whose fibers have the shape of S^1. T. Chapman has informed the author that he and S. Ferry also have proved Theorem B in the case when M is a Hilbert cube manifold. The author expresses his gratitude to Z. Čerin and R. Daverman who read earlier versions of parts of this paper and pointed out errors and some improvements.

2. Preliminaries. We need the following four results from the work of Coram and Duvall [**13**; **14**]. Suppose that $f: E \to B$ is a proper mapping between locally compact ANR's.

PROPOSITION 1. *If f is an approximate fibration and if B is path-connected, then the fiber, $F = f^{-1}(x)$, is well-defined up to shape equivalence;* i.e., if $x, y \in B$, *then $f^{-1}(x)$ and $f^{-1}(y)$ have the same shape. F is a fundamental absolute neighborhood retract ($FANR$).*

Let $\check{\pi}_i(F, z)$ denote the ith shape homotopy group of F based at $z \in F$.

PROPOSITION 2. *Let f be an approximate fibration; then there exists a long exact sequence*

$$\ldots \to \check{\pi}_i(F, z) \xrightarrow{j_*} \pi_i(E, z) \xrightarrow{f_*} \pi_i(B, f(z)) \to \check{\pi}_{i-1}(F, z) \to \ldots$$

where j_* is the homomorphism (map when $i = 0$) induced by inclusion.

Suppose that $f: E \to B$ is a map; let $\bar{U}$ be a cover of B and let $g: X \to E$ and $H: X \times [0, 1] \to B$ be maps such that $H(x, 0) = fg(x)$, $x \in X$. If there exists $G: X \times [0, 1] \to E$ such that $G(x, 0) = g(x)$, fG and H are $\bar{U}$-close and whenever $H(x, t) = fg(x)$ for all t, $G(x, t) = fg(x)$ for all t, then f is said to have the *regular approximate homotopy lifting property with respect to X*.

PROPOSITION 3. *If f is an approximate fibration, then f has the regular approximate homotopy lifting property with respect to all spaces.*

PROPOSITION 4. *If f has the approximate homotopy lifting property for n-cells, $n \geqq 0$, then f is an approximate fibration.*

PROPOSITION 5. *If f is an approximate fibration and if $U \subseteq B$ is open, then $f|f^{-1}(U): f^{-1}(U) \to U$ is an approximate fibration.*

Proof. It is easily seen that $f|f^{-1}(U)$ has the approximate homotopy lifting property with respect to n-cells for all $n \geqq 0$. The result follows from Proposition 4.

We use the theory of ends (see [**28**]). If $f: E \to B$ is a proper monotone mapping between connected spaces, then f induces a bijection from the ends of E to the ends of B.

PROPOSITION 6. *Let $f: E \to B$ be a proper map of connected ANR's which is an approximate fibration with connected fiber. Let $\{U_i\}$ be a sequence of path-connected neighborhoods of the end ϵ of B such that*

6.1. *$U_{i+1} \subseteq U_i$ for all i.*

6.2. *The inclusion induced homomorphism $\pi_1 U_{i+1} \to \pi_1 U_i$ is an isomorphism for all i.*

6.3. $\bigcap U_i = \emptyset$.

6.4. *$\pi_2 U_i$ is trivial for all i.*

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Then, $\{f^{-1}(U_i)\}$ *is a sequence of neighborhoods of the end of* E *which corresponds to* ϵ *by the above-mentioned bijection such that*

6.5. $f^{-1}(U_{i+1}) \subseteq f^{-1}(U_i)$ *for all* i.

6.6. *The inclusion induced homomorphism* $\pi_1 f^{-1}(U_{i+1}) \to \pi_1 f^{-1}(U_i)$ *is an isomorphism for all* i.

6.7. $\bigcap f^{-1}(U_i) = \emptyset$.

Proof. Fix i and let $b \in U_{i+1}$. By Proposition 2, we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \bar{\pi}_1(f^{-1}(b)) & \to & \pi_1(f^{-1}(U_i)) & \xrightarrow{f_*} & \pi_1(U_i) & \to & 1 \\ & & \| & & \uparrow \alpha_* & & \uparrow \beta_* & & \\ 1 & \to & \bar{\pi}_1(f^{-1}(b)) & \to & \pi_1(f^{-1}(U_{i+1})) & \xrightarrow[f_*]{} & \pi_1(U_{i+1}) & \to & 1 \end{array}$$

where α_* and β_* are induced by inclusions and the rows are exact. The proposition follows by the five lemma.

PROPOSITION 7. *Let* f *be an approximate fibration and let* $x \in B$. *Then* $\bar{\pi}_1(f^{-1}(x), z)$ *is finitely presented.*

Proof. By Proposition 1, $f^{-1}(x)$ is an $FANR$ and, hence, $(f^{-1}(x), z)$ is fundamentally dominated by a finite polyhedron (P, p) [**3**]. By [**3**], $\bar{\pi}_1(f^{-1}(x), z)$ is isomorphic to a retract of $\pi_1(P, p)$ and the conclusion follows from Lemma 1.3 of [**33**].

PROPOSITION 8. *If Theorems A and B are true for approximate fibrations with connected fibers, then Theorems A and B are true for arbitrary approximate fibrations.*

Proof. Let $f: E \to B$ be an approximate fibration whose fiber F is not necessarily connected. Since F is fundamentally dominated by a finite polyhedron (see the proof of the previous proposition), F has a finite number of components. Let $m: E \to Y$ and $l: Y \to B$ be the monotone-light factorization of f.

We will now show that l is a covering map. Let $x \in B$ and let $U \subseteq B$ be a closed n-cell which is a neighborhood of x. Let V be a component of $l^{-1}(U)$. From the exact sequence

$$\pi_1(U, b) \to \pi_0(f^{-1}(b), e) \to \pi_0(f^{-1}(U), e)$$

we see that each component of $f^{-1}(b)$ lies in precisely one component of $f^{-1}(U)$. Hence $l|V$ is $1 - 1$ and, thus, is a homeomorphism. Therefore l is a covering map and Y is a n-manifold. Note that it is possible to put a metric $\tilde{d}$ on Y so that l is a local isometry; i.e., there exists $\epsilon_0 > 0$ such that if $\tilde{d}(x, y) < \epsilon_0$, then $\tilde{d}(x, y) = d(l(x), l(y))$.

We now claim that $m: E \to Y$ is an approximate fibration. Let $\epsilon > 0$, $g: X \to E$ and $H: X \times [0, 1] \to Y$ be given such that $H(x, 0) = mg(x)$ for all $x \in X$. We may assume that $\epsilon < \epsilon_0$. Let $G: X \times [0, 1] \to E$ be a map such that $G(x, 0) = g(x)$ and $d(fG(x, t), lH(x, t)) < \epsilon$ for all $x \in X$ and $t \in [0, 1]$. We now claim that $\tilde{d}(mG(x, t), H(x, t)) < \epsilon$; let $A_x = \{t \in [0, 1] | \tilde{d}(mG(x, t), H(x, t)) < \epsilon\}$. By using the facts that l is a covering map and a local isometry, it is straightforward to check that A_x is both open and closed in $[0, 1]$. Thus $m: E \to Y$ is an approximate fibration and the fiber of m is connected.

If we assume the hypotheses of Theorem A and assume that Theorem A is true for connected fibers, then we can find a locally trivial fiber map $\phi : E \to Y$ which approximates m. Then $l\phi : E \to B$ is a locally trivial fiber map which approximates f.

Now, suppose that Theorem B is true for approximate fibrations with connected fibers. By hypothesis, f is homotopic to a Hurewicz fibration f_0. Since l is a covering map, this homotopy can be lifted to a homotopy between m and m_0 where $lm_0 = f_0$. By [**31**], m_0 is also a Hurewicz fibration. Now we can apply Theorem B and proceed as above.

3. Proof of Theorem A. Suppose that $f: E \to B$ is an approximate fibration where E is a closed connected 3-manifold such that each inessential tame 2-sphere in E bounds a 3-cell and B is a connected n-manifold, $n = 1, 2$. By Proposition 8, it suffices to consider the case that $F = f^{-1}(b)$ is connected for some $b \in B$.

LEMMA 9. *If $n = 2$, then $\check{\pi}_1(F, e)$ is infinite.*

Proof. Suppose that $\check{\pi}_1(F, e)$ is finite. Let U be an open 2-cell in B and let $b \in U$; by Proposition 5, $f|f^{-1}(U): f^{-1}(U) \to U$ is an approximate fibration. It follows from Proposition 2 that $\pi_1(f^{-1}(U), e)$ is isomorphic to $\check{\pi}_1(F, e)$. If $f^{-1}(U)$ is not orientable, then let $\rho : W \to f^{-1}(U)$ be the oriented double covering. Note that, by covering space theory and Proposition 4, $\rho' = f\rho : W \to U$ is an approximate fibration whose fiber F' double covers F. It is straightforward to check that $\check{\pi}_1(F', e')$ is also finite. Thus, it suffices to consider the case when $f^{-1}(U)$ is orientable.

Let $b' \neq b$ be an element of U. Again, $f|f^{-1}(U - b')$ is an approximate fibration and it follows from Proposition 2 that $\pi_1(f^{-1}(U - b', e)$ is infinite. Since the integers is a homomorphic image of $\pi_1(f^{-1}(U - b'))$, $H_1(f^{-1}(U - b'))$ is also infinite. From the exact homology sequence of the pair $(f^{-1}(U), f^{-1}(U - b'))$, we see that $H_2(f^{-1}(U), f^{-1}(U - b'))$ is infinite. But, by duality, $H_2(f^{-1}(U), f^{-1}(U - b'))$ is isomorphic to $\check{H}^1(f^{-1}(b')) = \check{H}^1(F)$; however, since $\check{\pi}_1(F, e)$ is finite, $\check{H}^1(F)$ is also finite, a contradiction.

LEMMA 10. *If $U \subseteq B$ is an open subset with a finite number of ends, then $f^{-1}(U)$ is homeomorphic to the interior of a compact 3-manifold provided $\check{\pi}_1(F) \neq Z_2$, the cyclic group of order 2.*

Proof. Fix an end ϵ of U and let $\tilde{\epsilon}$ be the end of $f^{-1}(U)$ which h corresponds to ϵ. Let $\{U_i\}$ be a sequence of connected open sets in U such that each U_i is a neighborhood of the end ϵ, $U_i \supseteq U_{i+1}$ for each i and $\bigcap U_i = \emptyset$. If $n = 2$, we may assume that U_i is an open annulus for each i. It follows from Proposition 6 that $\{\pi_1(f^{-1}(U_i))\}$ is essentially constant [**28**] or, in the terminology of [**22**], π_1 is stable at the end $\bar{\epsilon}$ of $f^{-1}(U)$. The conclusion follows from [**22**].

Now, let $n = 2$.

LEMMA 11. *$\bar{\pi}_1(F)$ is isomorphic to the integers.*

Proof. Let $U \subseteq B$ be an open 2-cell and let $p \neq q$ be points of U. By Lemmas 9 and 10, $f^{-1}(U - \{p, q\})$ is homeomorphic to the interior of a compact 3-manifold R. By Proposition 2, we have the exact sequence

$$1 \to \bar{\pi}_1(F) \to \pi_1 f^{-1}(U - \{p, q\}) \to \pi_1(U - \{p, q\}) \to 1.$$

Since $\pi_1 R$ is isomorphic to $\pi_1 f^{-1}(U - \{p, q\})$, the conclusion follows from [**19**] and Proposition 7.

LEMMA 12. *If $U \subseteq B$ is an open 2-cell, then $f^{-1}(U)$ is homeomorphic to $S^1 \times \mathbf{R}^2$.*

Proof. By Lemma 10, $f^{-1}(U)$ is homeomorphic to the interior of a compact 3-manifold R. Let $\{U_i\}_{i=1}^{\infty}$ be open annuli in U as in the proof of Lemma 10. Since $\pi_1(\text{bdry } R)$ is isomorphic to the inverse limit of $\{\pi_1(f^{-1}(U_i))\}$, it follows from the previous lemma and the exact sequence

$$1 \to \bar{\pi}_1(F) \to \pi_1(f^{-1}(U_i)) \to \pi_1(U_i) \to 1$$

that bdry R is a torus or a Klein bottle. We will now show that the latter cannot occur; hence, to obtain a contradiction, suppose that bdry R is a Klein bottle. Let σ and τ denote generators of $\bar{\pi}_1(F)$ and $\pi_1(U_i)$, respectively. Since $\pi_1(f^{-1}(U_i))$ is isomorphic to $\pi_1(\text{bdry } R)$, a presentation for $\pi_1(f^{-1}(U_i))$ is $|\sigma', \tau' : \tau'\sigma'\tau'^{-1} = \sigma'^{-1}|$ where σ' is the image of σ and τ' is some preimage of τ. Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \bar{\pi}_1(F) & \to & \pi_1(f^{-1}(U_i)) & \to & \pi_1(U_i) & \to & 1 \\ & & \| & & \downarrow \gamma_* & & \downarrow & & \\ 1 & \to & \bar{\pi}_1(F) & \to & \pi_1(f^{-1}(U)) & \to & \pi_1(U) & \to & 1 \end{array}$$

where the vertical maps are induced by inclusion. Note that $\pi_1(f^{-1}(U))$ is generated by $\gamma_*(\sigma')$ and note that $\gamma_*(\tau') = 1$. From the presentation of $\pi_1(f^{-1}(U_i))$, it follows that $\pi_1(f^{-1}(U))$ is either trivial or cyclic of order 2; this contradicts the exactness of the last row and Lemma 11. Hence bdry R is a torus; since $\pi_1(f^{-1}(U))$ is infinite cyclic, R is homeomorphic to a solid torus [**27**] and the lemma is proved.

LEMMA 13. *Let $U \subseteq B$ be an open 2-cell and let $L \subseteq U$ be either a point or a closed 2-cell; then $f^{-1}(U - L)$ is homeomorphic to $S^1 \times S^1 \times \mathbf{R}$.*

Proof. Since $U - L$ has two ends, $f^{-1}(U - L)$ is homeomorphic to the interior of a compact 3-manifold R and, as in the proof of Lemma 12, bdry R is homeomorphic to the union of two tori. Since $\pi_1 f^{-1}(U - L)$ is isomorphic to $Z \oplus Z$ and the inclusion of each end induces an isomorphism on fundamental groups, it follows from [**15**] that R is homeomorphic to $S^1 \times S^1 \times [0, 1]$.

LEMMA 14. *Let U be an open 2-cell in B and let $\{U_1, U_2, \ldots, U_r\}$, $r \geqq 2$, be a collection of pairwise disjoint open 2-cells in U. Then there exists an embedding ϕ: $S^1 \times \Delta^2 \to f^{-1}(U)$ [Δ^2 is a closed 2-cell] and points $x_1, x_2, \ldots, x_r \in$ bdry Δ^2 such that*

(i) $\phi(S^1 \times \{x_i\}) \subseteq f^{-1}(U_i)$;

(ii) *the pair* $(f^{-1}(U), \phi(S^1 \times \{x_i\}))$ *is homeomorphic to* $(S^1 \times \mathbf{R}^2, S^1 \times \{0\})$;

(iii) *for each i, ϕ(bdry $(S^1 \times \Delta^2)) \cap f^{-1}(U_i) \subseteq \phi(S^1 \times W_i)$ where W_i is an open connected subset of* bdry Δ^2 *such that if $i \neq j$, then $W_i \cap W_j = \phi$.*

Proof. Let $p_i \in U_i$. Since $\pi_1(U - \{p_1, p_2, \ldots, p_r\})$ is isomorphic to F_r, the free group with r generators, $\pi_1(f^{-1}(U - \{p_1, p_2, \ldots, p_r\}))$ is isomorphic to the semi-direct product $Z \times_\alpha F_r$ for some action α of F_r on Z. By considering the following commutative diagram for each i,

$$\begin{array}{ccccccccc}
1 \to & \pi_1(F) & \to & \pi_1(f^{-1}(U - \{p_1, p_2, \ldots, p_r\})) & \to & \pi_1(U - \{p_1, p_2, \ldots, p_r\}) & \to 1 \\
& \Vert & & \downarrow & \swarrow & & \\
1 \to & \pi_1(F) & \to & \pi_1(f^{-1}(U - \{p_i\})) & \to & \pi_1(U - \{p_i\}) & \to 1.
\end{array}$$

and using Lemma 13, one can show that the action α of F_r on Z is trivial and, hence, $\pi_1(f^{-1}(U - \{p_1, p_2, \ldots, p_r\}))$ is isomorphic to the direct product of Z and F_r. By Lemma 10, there exists a compact manifold R whose interior is homeomorphic to $f^{-1}(U - \{p_1, p_2, \ldots, p_r\})$; we can assume that $R \subseteq f^{-1}(U - \{p_1, p_2, \ldots, p_r\})$ and bdry $R \subseteq \bigcup_{i=1}^{n} f^{-1}(U_i)$.

By [**19**], there exists a fibration $R \to S$ with the 1-sphere as fiber and which base S is a compact 2-manifold with $\pi_1(S) \cong F_r$. Since $\pi_1(R) \cong Z \times F_r$, R is an orientable S^1-bundle over S, i.e., the structural group which is the homeomorphisms of S^1 reduces to the group of orientation preserving homeomorphisms of S^1 [**21**]. Since S^1 is a deformation retract of the latter group, the classifying space for this group is simply-connected. Hence R is the trivial bundle over S, i.e., R is homeomorphic to $S^1 \times S$. Since bdry R has $(r + 1)$-components, bdry S also has $r + 1$ components and thus S is a punctured disk. Let Δ' be a closed 2-cell in S which meets each component of bdry S in a connected set. The restriction of the inverse of the above-mentioned homeomorphism from R to $S^1 \times S$ is the desired homeomorphism.

LEMMA 15. *Let U, V be open 2-cells in B, $U \subseteq V$ and let Σ be a 1-sphere in*

$f^{-1}(U)$. *The pair* $(f^{-1}(U), \Sigma)$ *is homeomorphic to the pair* $(S^1 \times \mathbf{R}^2, S^1 \times \{0\})$ *if and only if the pair* $(f^{-1}(V), \Sigma)$ *is homeomorphic to the pair* $(S^1 \times \mathbf{R}^2, S^1 \times \{0\})$,

Proof. Let $D \subseteq U$ be a closed 2-cell such that $f(\Sigma) \subseteq D$. By Lemma 13, there exist homeomorphisms h_1: $S^1 \times S^1 \times \mathbf{R} \to f^{-1}(U - D)$ and h_2: $S^1 \times S^1 \times \mathbf{R} \to f^{-1}(V - D)$; by [**15**], we may assume that $h_1(x, t) = h_2(x, t)$ for $x \in S^1 \times S^1$ and $t \geqq 0$. The conclusion of the lemma follows from another application of [**15**].

LEMMA 16. *Let* U_1, U_2, V *be open 2-cells in* B, $U_1 \cup U_2 \subseteq V$, $U_1 \cap U_2 = \phi$; *let* Σ_i *be a* 1-*sphere in* $f^{-1}(U_i)$, $i = 1, 2$, *such that the pair* $(f^{-1}(U_i), \Sigma_i)$ *is homeomorphic to the pair* $(S^1 \times \mathbf{R}^2, S^1 \times \{0\})$. *Then the triple* $(f^{-1}(V), \Sigma_1, \Sigma_2)$ *is homeomorphic to the triple* $(S^1 \times \mathbf{R}^2, S^1 \times \{p\}, S^1 \times \{q\})$.

Proof. Let $p_i \in U_i - f(\Sigma_i)$, $i = 1, 2,$. As in the proof of Lemma 14, there exists a compact genus zero surface S with three boundary components and a homeomorphism k of the interior of $S^1 \times S$ onto $f^{-1}(V - \{p_1, p_2\})$. Let S_0 be a component of bdry S such that the interior of some collar neighborhood of $S^1 \times S_0$ in $S^1 \times S$ is mapped onto a neighborhood of the end of $f^{-1}(V - \{p_1, p_2\})$ which is determined by $f^{-1}(p_1)$. Let S_1 be a 1-sphere in the interior of S such that there exists an annulus $A \subseteq S$ with bdry $A = S_0 \cup S_1$ and $\Sigma_1 \subseteq k(S^1 \times \operatorname{int} A) \subseteq k(S^1 \times (S_1 \cup \operatorname{int} A)) \subset f^{-1}(U_1)$.

Let $x \in S^1$ and consider $\gamma = k(\{x\} \times S_1)$. Note that γ is homotopically trivial in $f^{-1}(V)$ and hence in $f^{-1}(U_1)$. Since $(f^{-1}(U_1), \Sigma_1)$ is homeomorphic to $(S^1 \times \mathbf{R}^2, S^1 \times \{0\})$, by using [**15**], one can show that the triple $(f^{-1}(U_1), k(S^1 \times S_1), \Sigma_1)$ is homeomorphic to $(S^1 \times \mathbf{R}^2, S^1 \times S^1, S^1 \times \{0\})$. Hence, the closure of the component of $f^{-1}(V) - k(S^1 \times S_1)$ which contains Σ_1 is a solid torus in which γ is homotopically trivial. Hence γ bounds a disk in the latter solid torus which meets Σ_1 in precisely one point. Let $\hat{S}$ be the union of $S - \operatorname{int} A$ and the cone over S_1; it is straightforward to extend $k|S - \operatorname{int} A$ to $\hat{S}$. The lemma is proved by preforming a similar construction on the boundary component of S which corresponds to p_2.

LEMMA 17. *Let* K *be a triangulation of* B. *Then there exists a locally trivial fiber map* g: $E \to B$ *such that if* $x \in E$ *and* $\tau \in K$ *such that* $f(x) \in \tau$, *then* $g(x) \in N(\tau, K)$, *the simplicial neighborhood of* τ *in* K.

Proof. Let v be a vertex of K, let N be the star of v in K'', the second barycentric subdivision of K and let

$$\{v_1, v_2, \ldots, v_r\} = (\text{bdry } N) \cap \bigcup_{\sigma \in K_1} \sigma,$$

where K_1 is the 1-skeleton of K. Let N_i^0 be the star of v_i in $K^{(iv)}$ and let $N_i = N_i^0 \cap N$. By Lemma 14, there exists an embedding ϕ: $S^1 \times \Delta^2 \to f^{-1}(\operatorname{int} N)$ and points $x_1, x_2, \ldots, x_r \in \text{bdry } \Delta^2$ such that $\phi(S^1 \times \{x_i\}) \subseteq f^{-1}(\operatorname{int} N_i)$. Choose a homeomorphism λ: $\Delta^2 \to N$ so that $\lambda(x_i) = v_i$. Define g_1: image $\phi \to N$ by $g_1(\phi(x, y)) = \lambda(y)$ where $x \in S^1$, $y \in \Delta^2$. If we repeat this construction for each vertex v of K, we get a fiber map g_1 over a neighborhood of the 0-skeleton of K.

Let σ be a 1-simplex in K; bdry $\sigma = \{v, v'\}$. Let N and N' be the stars of v and v', respectively, in K'' and let $\sigma \cap$ bdry $N = \{v_1\}$ and $\sigma \cap$ bdry $N' = \{v_1'\}$. Let σ' be the subarc of σ such that bdry $\sigma' = \{v_1, v_1'\}$. Let ϕ and ϕ' be the embeddings of $S^1 \times \Delta^2$ into $f^{-1}(N)$ and $f^{-1}(N')$, respectively. Let V be the fourth derived simplicial neighborhood of σ'. Note that V does not meet the star (in $K^{(iv)}$) of v_i, $i > 1$ and the corresponding v_i', $i > 2$.

Let $\Sigma = \phi(S^1 \times \{x_1\})$ and $\Sigma' = \phi(S^1 \times \{x_1'\})$ where x_1, x_1' are chosen as above. By Lemmas 15 and 16, the triple $(f^{-1}(\text{int } V), \Sigma, \Sigma')$ is homeomorphic to the triple $(S^1 \times \mathbf{R}^2, S^1 \times \{p\}, S^1 \times \{q\})$. In particular, there exists an annulus $A \subseteq f^{-1}(\text{int } V)$ such that bdry $A = \Sigma \cup \Sigma'$. Isotope A, keeping $\Sigma \cup \Sigma'$ fixed, into general position in $f^{-1}(\text{int } V)$ with respect to $\Gamma \cup \Gamma'$ where $\Gamma = f^{-1}(\text{int } V) \cap$ bdry image ϕ and $\Gamma' = f^{-1}(\text{int } V) \cap$ bdry image ϕ. Hence we may assume that int $A \cap (\Gamma \cup \Gamma')$ is a finite number of simple closed curves and arcs. Note that these arcs must have their boundary in a component of bdry A.

Suppose that there exists a simple closed curve $\gamma \subseteq A \cap (\Gamma \cup \Gamma')$ such that γ bounds a disk D on A. By choosing a "minimal" curve having this property, we may assume that D contains no points of $\Gamma \cup \Gamma'$ in its interior. Suppose that $\gamma \subseteq \Gamma$. Since $\gamma \cap \Sigma = \phi$, γ has to be homotopically trivial and, hence, bounds a disk D' on bdry image ϕ. Note that $D \cup D'$ bounds a 3-cell C in $f^{-1}(N \cup V)$ and hence we can find an ambient isotopy h_t whose support lies in a small neighborhood of C in $f^{-1}(N \cup V)$ so that $h_1(\text{bdry image } \phi) \cap A \cap C = \phi$. If we choose the neighborhood of C sufficiently small, then the isotopy will keep $\phi(S^1 \times \{x_i\})$ fixed for all i. In order to avoid a plethora of notation, we will replace the symbols $h_1\phi$ and g_1h_1 by ϕ and g_1, respectively. By induction, we may assume that $A \cap (\Gamma \cup \Gamma')$ contain no simple closed curves which are homotopically trivial in A.

Suppose that int $A \cap (\Gamma \cup \Gamma')$ contains a simple closed curve γ; then γ and Σ bound an annulus in A. We can find simple closed curves γ_1 and γ_2 in $A \cap (\Gamma \cup \Gamma')$ such that $\gamma_1 \subseteq \Gamma$, $\gamma_2 \subseteq \Gamma'$ and the interior of the annulus $A_0 \subseteq A$ which is bounded by $\gamma_1 \cup \gamma_2$ does not meet any simple closed curve in $A \cap (\Gamma \cup \Gamma')$. We will replace A by A_0 and we need to make corresponding changes in g_1 and ϕ. If $\gamma_1 \neq \Sigma$, note that γ_1 and Σ are homotopic in bdry image ϕ; by condition (iii) of Lemma 14, γ_1 and Σ bound an annulus in bdry image ϕ which misses $\phi(S^1 \times \{x_i\})$, $i > 1$. Hence we can find an isotopy which takes Σ to γ_1 and whose support lies in a small neighborhood of this annulus which also misses $\phi(S^1 \times \{x_i\})$, $i > 1$. Similarly, if $\gamma_1' \neq \Sigma'$, then we make the corresponding adjustments. Hence, we may assume that int $A \cap (\Gamma \cup \Gamma')$ consists of at most arcs; this could occur if, for example, $\gamma_1 = \Sigma$.

Let τ_0 be an "innermost" arc which meets Σ; i.e., there exists an arc τ_1 on Σ such that bdry τ_0 = bdry τ_1 and $\tau_0 \cup \tau_1$ is a simple closed curve on A which bounds a disk δ_1 on A whose interior contains no point of int $A \cap (\Gamma \cup \Gamma')$

bounds a disk δ_2 on bdry image ϕ. By using the three cell bounded by $\delta_1 \cup \delta_2$ we make alterations similar to above and replace A by cl$(A - \delta_1)$. By induction, we may assume that int $A \cap (\Gamma \cup \Gamma') = \phi$. Note that $f(A_1) \subseteq$ int $(N \cup N' \cup V)$. It is easy to extend g_1: $\Sigma \cup \Sigma' \to$ bdry σ' to a locally trivial fiber map g_2: $A \to \sigma'$ .We repeat this construction for each 1-simplex of K.

Let λ be a 2-simplex in K and let W be the union of the stars of the vertices of λ in K'' and the derived neighborhoods V of the 1-cells used in the above construction. Let λ' be the boundary of the 2-cells λ_0 in λ which is the closure of the complement of the union of the stars of the vertices of λ. Note that $g_2^{-1}(\lambda')$ is a torus in $f^{-1}(W)$ and there exists a homeomorphism ξ: $S^1 \times \lambda' \to g_2^{-1}(\lambda')$ such that $g_2\xi(x, y) = x$ and $\xi(x \times \lambda')$ is homotopically trivial in $f^{-1}(W)$. It follows from Lemma 13 and [**4**] that $g_2^{-1}(\lambda')$ is the boundary of a solid torus T in $f^{-1}(W)$. It is easy to extend ξ to a homeomorphism of $S^1 \times \lambda_0$ to T and then to extend g_2 to a locally trivial fibre map g: $T \to \lambda_0$. We repeat this construction for each 2-simplex of K to get the desired fiber map g.

Now, let $n = 1$.

LEMMA 18. *If* $\pi_1(F) \neq Z_2$ *and* $U \subseteq B$ *is a proper connected open subset of* B, *then* $f^{-1}(U)$ *is homeomorphic to* $T \times \mathbf{R}$ *for some* 2-*manifold* T.

Proof. By Lemma 10, $f^{-1}(U)$ is homeomorphic to the interior of a compact manifold R with two boundary components R_1 and R_2. Note that from the proof of Lemma 10, $\pi_1(R_i)$ is isomorphic to $\bar{\pi}_1(F)$, $i = 1, 2$, and the inclusion induced map $\pi_1(R_i) \to \pi_1(R)$ is an isomorphism. By [**4**], R is homeomorphic to $R_1 \times [0, 1]$.

LEMMA 19. *Let* K *be a triangulation of* B *and suppose that* $\bar{\pi}_1(F) \neq Z_2$. *Then there exists a locally trivial fiber map* g: $E \to B$ *such that if* $x \in E$ *and* $\tau \in K$ *such that* $f(x) \in \tau$, *then* $g(x) \in N(\tau, K)$.

Proof. Let v be a vertex of K and let U be the open star of v in K''. By the previous lemma $f^{-1}(U)$ is homeomorphic to $T \times \mathbf{R}$; let T_v be the image of $T \times \{0\}$. Suppose that this construction is performed for each vertex of K. Let σ be a 1-simplex of K, bdry $\sigma = \{v, w\}$, and let V be the open simplicial neighborhood of σ in K''. By the previous lemma and [**4**], the connected submanifold W of $f^{-1}(V)$ whose boundary is $T_v \cup T_w$ is homeomorphic to $T \times [0, 1]$. Define $g(T_v) = v$, $g(T_w) = w$ and extend naturally over W to σ so that g is a locally trivial fiber map.

Theorem A now follows from Lemmas 17 and 19.

4. Proof of Theorem B. Suppose that f: $M \to S^1$ is an approximate fibration satisfying the hypotheses of Theorem B. Let F denote the fiber of f; by Proposition 1, F is an $FANR$ and, hence, has a finite number of components. By Proposition 2, the induced map f_* : $\pi_1 M \to \pi_1 S^1$ is nontrivial. By Proposition 8, it suffices to consider the case when F is connected and, hence, f_* is assumed onto.

Suppose that f is homotopic to the Hurewicz fibration g and let $\epsilon > 0$ be given. Let p: $\mathbf{R} \to S^1$ be the universal covering map and let q: $\tilde{M} \to M$ be the pullback of p using the map g. Let $\tilde{g}$: $\tilde{M} \to \mathbf{R}$ be the natural map such that $p\tilde{g} = gq$. From covering space theory, there exists a map $\tilde{f}$: $\tilde{M} \to \mathbf{R}$ such that $p\tilde{f} = fq$ and $\tilde{f}$ is homotopic to $\tilde{g}$.

Let π: $\tilde{M} \times Q \to \tilde{M}$ and π': $M \times Q \to M$ denote the projection along the first factor. Let $q_0 = q \times$ identity: $\tilde{M} \times Q \to M \times Q$ and let $\tilde{g}_0 = \tilde{g}\pi$, $g_0 = g\pi'$, $\tilde{f}_0 = \tilde{f}\pi$ and $f_0 = f\pi'$.

By Chapman and Ferry [**11**], $\tilde{g}_0$ and g_0 are locally trivial fibre maps whose fibres are compact Q-manifolds. Hence, if Y is a fiber of g, then there exists a homeomorphism λ: $Y \times Q \times \mathbf{R} \to \tilde{M} \times Q$ such that $\tilde{g}_0\lambda = \rho$ where ρ: $Y \times Q \times \mathbf{R} \to \mathbf{R}$ is the projection along the last factor. By Chapman [9], $Y \times Q$ has the homotopy type of a finite polyhedron P and, hence, $\tilde{M}$ has the homotopy type of P.

Let U be a proper connected open subset of S^1. By Proposition 5, $f|f^{-1}(U)$ is an approximate fibration. By Proposition 2 it follows that the inclusion of F into $f^{-1}(U)$ induces isomorphism $\bar{\pi}_i(F) \to \pi_i f^{-1}(U)$ for all i. Let V be an open subset of $\mathbf{R}$ such that $p|V$ is a homeomorphism of V onto U; then $q|\tilde{f}^{-1}(V)$: $\tilde{f}^{-1}(V) \to f^{-1}(U)$ is a homeomorphism. Note that $\tilde{f}$ and, hence, $\tilde{f}|\tilde{f}^{-1}(V)$ have the approximate homotopy lifting property for n-cells for all n. Again, by Propositions 2 and 4, the inclusion induced homomorphisms $\bar{\pi}_i(F) \to \pi_i(\tilde{f}^{-1}(V))$ and $\bar{\pi}_i(F) \to \pi_i(\tilde{M})$ are isomorphisms for all i. Thus, the inclusion of $\tilde{f}^{-1}(V)$ into $\tilde{M}$ also induces isomorphisms on all homotopy groups; since these spaces are homotopy equivalent to CW complexes [**9**] [**16**], this inclusion is a homotopy equivalence [**31**]. Thus we have the following.

LEMMA 20. *If U is a proper open connected subset of S^1, then $f^{-1}(U)$ is homotopy equivalent to the finite polyhedron P. If $U_0 \subseteq U$ is a connected open subset, then the inclusion of $f^{-1}(U_0)$ into $f^{-1}(U)$ is a homotopy equivalence.*

LEMMA 21. *Let U be a proper open connected subset of S^1. Then there exists a compact connected Q-manifold $Z_0 \subseteq f_0^{-1}(U)$ such that the inclusion is a homotopy equivalence, Z_0 separates the two ends of $f_0^{-1}(U)$ and Z_0 is collared in the closure of each component of $f_0^{-1}(U) - Z_0$.*

Proof. The proof is essentially contained in [**5**]; we shall sketch a proof indicating the necessary changes. Let V be an open subset of $\mathbf{R}$ such that $p|V$ is a homeomorphism of V onto U.

By using [**5**] and [**6**], we can find a compact connected Q-manifold $Z_1 \subseteq \tilde{f}_0^{-1}(V)$ such that Z_1 separates the ends of $\tilde{f}_0^{-1}(V)$ and Z_1 is collared in the closure of each of the two components, A and B, of $\tilde{f}_0^{-1}(V) - Z_1$. Let A^* and B^* be the closure of the components of $(\tilde{M} \times Q) - Z_1$ such that $A \subseteq A^*$ and $B \subseteq B^*$. Since $\tilde{M} \times Q$ is homeomorphic to $Y \times Q \times \mathbf{R}$, it is easy to show that

A^* and B^* have compact Q-submanifolds which are deformation retracts; hence A^* and B^* have the homotopy type of finite complexes [**9**].

We need the following proposition whose proof will be given at the end of the proof of Lemma 21.

LEMMA 22. *The inclusions* $A \hookrightarrow A^*$ *and* $B \hookrightarrow B^*$ *are homotopy equivalences.*

Thus A and B have the homotopy type of finite complex. Let α_1: $K \to A$ be a homotopy equivalence of a finite complex with A; α_1 may be assumed to be a Z-embedding and can be extended to an open embedding α_2: $K \times Q \times [0, 1] \to A$[**7**]. Using Z-set unknotting [**1**], it may be assumed that $\alpha_2(K \times Q \times \{0\})$ contains Z_1. Let $Z_2 = \alpha_2(K \times Q \times \{1/2\})$ and let A_1^* and B_1^* be the closure of the complements of $M \times Q - Z_2$ such that $A^* - A \subseteq A_1^*$. Let $A_1 = A_1^* \cap \tilde{f}_0^{-1}(V)$ and let B_1 be the closure of $\tilde{f}_0^{-1}(V) - B_1$.

As before, B_1^* has the homotopy type of a finite complex and the inclusion $B_1 \hookrightarrow B_1^*$ (see Lemma 22) is a homotopy equivalence. We now perform the same construction in B_1 to get a compact Q-manifold Z_3 as we did for finding Z_2 in A. Z_3 is the desired submanifold.

Proof of Lemma 22. Since A and A^* have the homotopy type of a CW-complex [**9**; **16**], it suffices to show that the inclusion induced homomorphism i_*: $\pi_k(A, x_0) \to \pi_k(A^*, x_0)$ is an isomorphism for all $k \geqq 0$.

Suppose that β: $(S^k, y_0) \to (A, x_0)$ represents an element $\pi_k(A, x_0)$ whose image under i_* is zero; thus β can be extended to a map β: $(D^{k+1}, y_0) \to (A^*, x_0)$ where D^{k+1} denotes the $(k + 1)$-cell. Let $z \in \mathbf{R}$ be the limit point of V such that $z \notin V$ and $\tilde{f}_0^{-1}(z) \subseteq A^*$; let us assume that z is an upper bound of V. Then there exists $y \in V$ such that $\tilde{f}_0^{-1}((y, z)) \subseteq A - \beta(S^k)$. Let h_t be a strong deformation retraction of $\mathbf{R}$ onto $(-\infty, (z + y)/2]$ with $h_0 =$ identity and $h_t((z + y)/2, +\infty)) = [(z + y)/2, +\infty)$ for all $t \in [0, 1]$. Since $\tilde{f}_0$ has the approximate homotopy lifting property, there exists a homotopy β_t: $D^{k+1} \to \tilde{M} \times Q$ such that $\beta_0 = \beta$ and $\tilde{f}_0\beta_t$ and $h_t\tilde{f}_0\beta$ are $(z - y)/2$-close for all t. Since $h_t\tilde{f}_0\beta(x) = \tilde{f}_0\beta(x)$ for $x \in \beta^{-1}\tilde{f}_0^{-1}((-\infty, (z + y)/2])$, $t \in [0, 1]$, $\beta_t(x) = \beta(x)$ for $x \in \alpha^{-1}f_0^{-1}((-\infty, (z + y)/2])$ for all t by Proposition 3. In particular, $\beta_t(x) = \beta(x)$ for all $x \in S^k$. Since $\tilde{f}_0\beta_t$ and $h_t\tilde{f}_0\beta$ are $(z - y)/2$ close, the image of β_t lies in A_* for all t and image of β_1 is contained in A.

The proof for showing that i_* is onto is similar.

LEMMA 23. *Suppose that* M *is a* Q-*manifold and let* U *be a proper open connected subset of* S^1. *Then there exists a compact connected* Q-*manifold* $Z_0 \subseteq f^{-1}(U)$ *such that the inclusion is a homotopy equivalence,* Z_0 *separates the two ends of* $f^{-1}(U)$ *and* Z_0 *is collared in the closure of each component of* $f^{-1}(U) - Z_0$.

Proof. This lemma follows from the previous lemma and the fact that π': $M \times Q \to M$ is a near homeomorphism [**8**].

LEMMA 24. *Suppose that M is finite-dimensional and let U be a proper open connected subset of S^1. Then there exists a closed connected codimension one locally flat submanifold $Z_0 \subseteq f^{-1}(U)$ such that the inclusion map is a homotopy equivalence.*

Proof. In order to prove this lemma, we will need the topological analogue of the proof of the main result of Siebenmann's thesis [**23**; **28**]; we shall assume familiarity with the definitions of [**28**]. Let E be an end of $f^{-1}(U)$; by using Proposition 6, it is easily shown that π_1 is stable at E. Let W be a 1-neighborhood of E. As in the proof of Lemma 21, we can show that $W \times Q$ has the homotopy type of a finite complex ($W \times Q$ corresponds to A in the proof). Thus W has the homotopy type of a finite complex and Siebenmann's obstruction for putting a boundary on $f^{-1}(U)$ vanishes. The boundary of the $(n-2)$-neighborhood is the desired submanifold.

Choose a triangulation K of S^1 such that the mesh of K is $\epsilon/2$. Let the vertices $\{v_i\}_{i=1}^m$ of K be indexed so that v_i and v_{i+1} form the boundary of a 1-simplex in K, $i = 1, 2, \ldots, n-1$. Let U_i be the open star of v_i in the first barycentric subdivision K' of K; let W_i be the open star in K' of the 1-simplex whose boundary is $\{v_i, v_{i+1}\}$ for $i \neq n$ and $\{v_n, v_1\}$ for $i = n$.

Let N_i be the submanifold of $f^{-1}(U_i)$ given by Lemmas 23 and 24. Since the inclusion of $f^{-1}(U_1)$ and $f^{-1}(U_2)$ into $f^{-1}(W_1)$ are homotopy equivalences by Lemma 20, the inclusions of N_1 and N_2 into C_1 are also homotopy equivalences where C_1 is the compact submanifold of $f^{-1}(W_1)$ whose frontier in $f^{-1}(W_1)$ is $N_1 \cup N_2$. In general, C_1 is not homeomorphic to $N_1 \times [0, 1]$; the vanishing of the Whitehead torsion [**10**] of the inclusion map $N_1 \hookrightarrow C_1$ is a necessary and sufficient condition for the existence of such a homeomorphism. We will replace N_2 by N_2' which will satisfy this condition; first, we need the following result from [**12**].

PROPOSITION 25. *If N is a Q-manifold and $\mu \in$ Wh $\pi_1(N)$, then there is a decomposition $N \times [0, 1] = N^1 \cup N^2$ such that*

(1) *the N^i's are compact Q-manifolds and $N^1 \cap N^2$ is a bicollared Q-manifold;*

(2) *$N \times \{0\} \subseteq$ int N^1 and $N \times \{1\} \subseteq$ int N^2;*

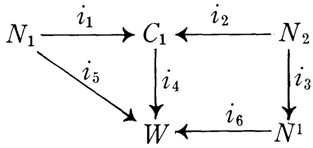
(3) *$N \times \{0\} \hookrightarrow N^1$ is a homotopy equivalence and μ is the Whitehead torsion of this inclusion.*

LEMMA 26. *The submanifold N_2' of $f^{-1}(U_2)$ given by Lemmas* 23 *and* 24 *can be chosen such that there exists a homeomorphism ξ of $N_1 \times [0, 1]$ onto C_1' such that $\xi(N_1 \times \{0\}) = N_1$ and $\xi(N_1 \times \{1\}) = N_2'$.*

Proof. If M is finite-dimensional, then $C_1 - N_2$ is homeomorphic to $N_1 \times [0, 1)$ by [**32**] and the result follows by choosing the image of $N_1 \times \{t\}$ for t sufficiently close to 1.

Suppose that M is a Q-manifold. Let ξ_0; $N_2 \times [0, 1] \to f^{-1}(U_2)$ be an embedding such that $\xi_0|N_2 \times [0, 1)$ is an open embedding, $\xi_0(x, 0) = x$ for

$x \in N_2$ and image $\xi_0 \cap C_1 = N_2$. Let $N^1 \cup N^2$ = image ξ_0 be the decomposition of image ξ_0 given by Proposition 25 such that $\mu = i_{2*}^{-1}(-\tau(i_1))$ where i_1: $N_1 \to C_1$ and i_2: $N_2 \to C_1$ are inclusions. Let $W = C_1 \cup N^1$ and consider the following diagram where all maps are inclusion maps.



By construction, $\tau(i_3) = i_{3*}(\mu)$. By excision [**10**],

$$\tau(i_4) = i_{6*}\tau(i_3) = i_{6*}i_{3*}(\mu) = i_{4*}i_{2*}(\mu) = i_{4*}(-\tau(i_1)).$$

By the composition formula, $\tau(i_5) = \tau(i_4 i_1) = \tau(i_4) + i_{4*}\tau(i_1) = 0$. Hence the inclusion of N_1 into W is a simple homotopy equivalence; by [**6**], there exists a homeomorphism ξ_1 from $N_1 \times [0, 1]$ onto W. By using the Z-set unknotting theorem [**1**], we may assume that $\xi_1(x, 0) = x$ for $x \in N_1$. If j: $W \to N^1 \cap N^2$ denotes the homotopy inverse of $i_6|N^1 \cap N^2$, then it is easily seen that ji_5: $N_1 \to N^1 \cap N^2$ is also a simple homotopy equivalence. By using [**1**] and [**6**], we again may assume that $\xi_1(N_1 \times \{1\}) = N^1 \cap N^2$. $N_2' = N^1 \cap N^2$ is our desired submanifold.

By induction, we can replace $N_3, N_4, \ldots, N_n$ by $N_3', N_4', \ldots, N_n'$ respectively so that there exists a homeomorphism ξ of $N_1 \times [1, n]$ into M such that $\xi(N_1) \times \{i\} = N_i'$, $i = 2, \ldots, n$ and $\xi(N_1) \times \{1\} = N_1$. Let C_0 be the closure of the complement of image ξ in M; note that the frontier of C_0 in M is $\xi(N_1 \times \{1, n\})$ and the inclusion of each component of the latter set into C_0 is a homotopy equivalence.

LEMMA 27. *If there exists a homeomorphism* μ: $N_1 \times [0, 1] \to C_0$ *such that* $\mu(N_1 \times \{0, 1\}) = \xi(N_1 \times \{1, n\})$, *then there exists a locally trivial fiber map* $\bar{f}$: $M \to S^1$ *such that* f *and* $\bar{f}$ *are* ϵ*-close and, hence Theorem B is proved.*

Proof. There is no loss of generality in assuming that $\mu(N_1 \times \{1\}) = \xi(N_1 \times \{1\})$ and the covering map p: $\mathbf{R} \to S^1$ is the epimorphism whose kernel is the integers. Define $\bar{f}$: $M \to S^1$ by $\bar{f}(z) = p(t/n)$ where $z = \xi(x, t)$ or $\mu(x, t)$. $\bar{f}$ is the desired function.

LEMMA 28. *The homeomorphism* μ *exists if the inclusion of* $N_0 = \xi(N_1 \times \{1\})$ *into* C_0 *is a simple homotopy equivalence.*

Proof. If M is finite dimensional, then this lemma is a consequence of the s-cobordism theorem in the topological category [**23**]. If M is a Q-manifold, then this was essentially proved in the proof of Lemma 26.

Consider $C_0 \times Q \subseteq M \times Q$; let Φ: $C_0 \times Q \to \tilde{M} \times Q$ be an embedding such that $q_0\Phi$ = identity. Let T be the generator of the covering transformation group of q_0: $\tilde{M} \times Q \to M \times Q$ such that $\lambda^{-1}T\lambda(Y \times Q \times \{t\}) = Y \times$

$Q \times \{t+1\})$ for all $t \in \mathbf{R}$. (Recall that p: $\mathbf{R} \to S^1$ is the epimorphism whose kernel is the integers.) Pick integers x_0 and x_1 such that

$$\lambda^{-1}[\Phi(C_0 \times Q) \cup T\Phi(C_0 \times Q)] \subseteq Y \times Q \times (x_0 + 1, x_1 - 1).$$

Let W_1 be the closure of the component of $Y \times Q \times [x_0, x_1] - \lambda^{-1}\Phi(N_0 \times Q)$ which contains $Y \times Q \times \{x_0\}$ and let W_2 be the closure of the component of $Y \times Q \times [x_0, x_1] - \lambda^{-1}[\Phi(N_0 \times Q) \cup T\Phi(N_0 \times Q)]$ which misses $Y \times Q \times \{x_0, x_1\}$.

For $t \in \mathbf{R}$, define S_t: $Y \times Q \to Y \times Q$ by $\lambda^{-1}T\lambda(y, t) = (S_t(y), t+1)$ for $y \in Y \times Q$ and define S from $Y \times Q \times [x_0, x_1]$ onto itself by

$$S(y, t) = \begin{cases} (S_t(y), x_0 + 2(t - x_0)) & t \in [x_0, x_0 + 1] \\ (S_t(y), t + 1) & t \in [x_0 + 1, x_1 - 2] \\ (S_t(y), \tfrac{1}{2}(x_1 + t)) & t \in [x_1 - 2, x_1 + 1]. \end{cases}$$

Note that S is a homeomorphism of $Y \times Q \times [x_0, x_1]$ onto itself such that $S(W_1) = W_1 \cup W_2$.

LEMMA 29. *There exists a homeomorphism* κ: $N_0 \times Q \times [0, 1] \to W_2$ *such that* $\kappa(N_0 \times Q \times \{0\}) = \lambda^{-1}\Phi(N_0 \times Q)$ *and* $\kappa(N_0 \times Q \times \{1\}) = \lambda^{-1}T\lambda(N_0 \times Q)$.

Proof. Let E be the closure of the component of $Y \times Q \times \mathbf{R} - \lambda^{-1}\Phi(N_0 \times Q)$ which contains $Y \times Q \times \{x_0\}$. The inclusion of $\lambda^{-1}\Phi(N_0 \times Q)$ into E is a homotopy equivalence. As in the proof of Lemma 26, there exists an embedding δ: $N_0 \times Q \times [0, 1] \to E$ such that

$$\delta(N_0 \times Q \times \{0\}) = \lambda^{-1}\Phi(N_0 \times Q)$$

and

$$Y \times Q \times \{x_0\} \subseteq \delta(N_0 \times Q \times (0, 1)).$$

Let W_0 be the closure of (image δ) $- W_1$; note that image $\delta = W_0 \cup W_1$. Since $W_0 \cap W_1$ is a Z-set in W_0, $S|W_1$ can be extended to a homeomorphism of $W_0 \cup W_1$ onto $W_0 \cup W_1 \cup W_2$ [**1**]. Since $W_0 \cup W_1$ is homeomorphic to $N_0 \times Q \times [0, 1]$ and $(W_0 \cup W_1) \cap W_2$ is a Z-set in W_2, $W_0 \cup W_1 \cup W_2$ is homeomorphic to W_2 [**7**]. The composition of these three homeomorphisms gives the desired homeomorphism.

LEMMA 30. *The inclusion of* N_0 *into* C_0 *is a simple homotopy equivalence.*

Proof. First, note by construction that the pair $(C_0 \times Q, N_0 \times Q)$ is homeomorphic to the pair $(W_2, \lambda^{-1}\Phi(N_0 \times Q))$. By the previous proposition, the Whitehead torsion of the inclusion of $N_0 \times Q$ into $C_0 \times Q$ is trivial; hence, the Whitehead torsion of the inclusion of N_0 into C_0 is also trivial and, thus, this map is a simple homotopy equivalence.

By Lemma 28, the proof of Theorem B is completed.

5. Proof of Theorem C. Let X be a nontrivial h-cobordism such that if bdry $X = X_0 \cup X_1$, then the component X_0 is homeomorphic to X_1; such h-cobordisms have been constructed by Milnor [**26**]. Let $\Phi: X_0 \times [0, 1) \to X - X_1$ be a homeomorphism such that $\Phi(x, 0) = x$ for all $x \in X_0$ [**32**]. Define $f': X \to [0, 1]$ by

$$f'(y) = \begin{cases} t & \text{if } y = \Phi(x, t),\ (x, t) \in X_0 \times [0, 1) \\ 1 & \text{if } y \in X_1. \end{cases}$$

Note that f' is a continuous map.

Let M be the decomposition space obtained from X by identifying X_0 with X_1 by means of some homeomorphism of X_0 onto X_1. f' induces a map $f: M \to S^1$ where S^1 is obtained from $[0, 1]$ by identifying 0 with 1. For each $x \in S^1$, $f^{-1}(x)$ is homeomorphic to X_0; by [**14**], f is an approximate fibration. If f were homotopic to a Hurewicz fibration, then it would follow as in the proof of Theorem B that the inclusion of X_0 into X is a simple homotopy equivalence contradicting the fact that X is a non-trivial h-cobordism.

We leave the proof of Theorem E to the reader.

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