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Deformations of G_2 and Spin(7) Structures

Spiro Karigiannis

Abstract. We consider some deformations of G_2-structures on 7-manifolds. We discover a canonical way to deform a G_2-structure by a vector field in which the associated metric gets "twisted" in some way by the vector cross product. We present a system of partial differential equations for an unknown vector field w whose solution would yield a manifold with holonomy G_2. Similarly we consider analogous constructions for Spin(7)-structures on 8-manifolds. Some of the results carry over directly, while others do not because of the increased complexity of the Spin(7) case.

1 Introduction

1.1 Cross Product Structures

An additional structure that can be imposed on a smooth Riemannian manifold M of dimension n is that of an r-fold *cross product*. This is an *alternating* r-linear smooth map

$$B\colon \underbrace{TM \times \cdots \times TM}_{r \text{ copies}} \to TM$$

that is compatible with the metric in the sense that

$$|B(e_1, \dots, e_r)|^2 = |e_1 \wedge \cdots \wedge e_r|^2$$
$$\langle B(e_1, \dots, e_r), e_j \rangle = 0 \quad 1 \leq j \leq r$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric. Such a cross product also gives rise to an $(r+1)$-form α given by

$$\alpha(e_1, \dots, e_r, e_{r+1}) = \langle B(e_1, \dots, e_r), e_{r+1} \rangle.$$

Cross products on real vector spaces were classified by Brown and Gray in [2]. Global vector cross products on manifolds were first studied by Gray in [12]. They fall into four categories:

(1) When $r = n - 1$ and α is the volume form of the manifold. Under the metric identification of vector fields and one forms, this cross product corresponds to the *Hodge star* operator on $(n-1)$-forms. This is not an extra structure beyond that given by the metric.

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(2) When $n = 2m$ and $r = 1$, we can have a one-fold cross product $J\colon TM \to TM$. Such a map satisfies $J^2 = -I$ and is an *almost complex structure*. The associated 2-form is the *Kähler form* ω.

(3) The first of two *exceptional cases* is a 2-fold cross product on a 7-manifold. Such a structure is called a *G_2-structure*, and the associated 3-form φ is called a *G_2-form*.

(4) The second exceptional case is a 3-fold cross product on an 8-manifold. This is called a Spin(7)*-structure*, and the associated 4-form Φ is called a Spin(7)*-form*.

In cases 2–4 the existence of these structures is a topological condition on M given in terms of characteristic classes (see [12, 22, 25]). One can also study the restricted sub-class of such manifolds where the associated differential form α is *parallel* with respect to the Levi–Civita connection ∇. In case (1), the volume form is always parallel. For the almost complex structures J of case (2), $\nabla J = 0$ if and only if the manifold is Kähler, which is equivalent to: $d\omega = 0$ and the almost complex structure is *integrable*. In this case, the Riemannian holonomy of the manifold is a subgroup of $U(m)$. For cases (3) and (4), the condition that the differential form be parallel is a non-linear differential equation. Manifolds with parallel G_2-structures have holonomy a subgroup of G_2 and manifolds with parallel Spin(7)-structures have holonomy a subgroup of Spin(7), hence their names. One can also show (see [1]) that such manifolds are all *Ricci-flat*.

There is a sub-class of the Kähler manifolds which are Ricci-flat. Such manifolds possess a global non-vanishing holomorphic *volume form* Ω in addition to the Kähler form ω, and these two forms satisfy some relation. These manifolds are called *Calabi–Yau* manifolds as their existence was demonstrated by Yau's proof of the Calabi conjecture [26]:

Theorem 1.1.1 (Calabi–Yau, 1978) *Let M be a compact complex manifold with vanishing first Chern class $c_1 = 0$. Then if ω is a Kähler form on M, there exists a unique Ricci-flat Riemannian metric g on M whose associated Kähler form is in the same cohomology class as ω.*

This theorem characterises those manifolds admitting Calabi–Yau metrics in terms of certain topological information. The equivalence is demonstrated by writing the Ricci-flat condition as a partial differential equation and proving existence and uniqueness of solutions. Calabi–Yau manifolds have holonomy a subgroup of $SU(m)$ and are characterized by *two* parallel forms, ω and Ω. In fact, they posses two parallel cross products: a 1-fold cross product J, and a complex analogue of case (1) above, where Ω plays the role of the volume form and the $(m-1)$-fold cross product is a *complex* Hodge star.

Calabi–Yau manifolds (at least in complex dimension 3) have long been of interest in string theory. More recently, manifolds with holonomy G_2 and Spin(7) have also been studied. (See, for example, [3, 4, 19, 20, 21, 17, 18, 15]). It would be useful to have an analogue of the Calabi–Yau theorem, or something similar, in the G_2 and Spin(7) cases. There is a significant difference, however, which makes G_2 and Spin(7) manifolds much more difficult to study.

An almost complex structure J does not by itself determine a metric. If we also have a Riemannian metric, then together the compatibility requirement yields the Kähler form $\omega(u, v) = g(Ju, v)$. In contrast, a 2-fold or 3-fold cross product structure *does* determine the metric uniquely, and thus also determines the associated 3-form φ or 4-form Φ. Because the metric and complex structure are "uncoupled" in the Calabi–Yau case, we can start with a *fixed* integrable complex structure J, and then look for different metrics (which correspond to different Kähler forms for the same J) which are Ricci-flat and make J parallel. As J is already integrable, it is parallel precisely when ω is closed, so we can simply look at different metrics which all correspond to closed Kähler forms, and from that set look for a Ricci-flat metric. Hence we can restrict ourselves to starting with a Kähler manifold, and looking at other Kahler metrics which could be Ricci-flat. The Calabi–Yau theorem then says that there exists precisely one such metric in each cohomology class which contains at least one Kähler metric.

In the G_2 and $\mathrm{Spin}(7)$ cases, however, we cannot fix a cross product structure and then vary the metric to make it parallel. For a given cross product, the metric is determined. In the Calabi–Yau case, we can start with $U(m)$ holonomy and describe the conditions for being able to obtain $\mathrm{SU}(m)$ holonomy. For G_2 and $\mathrm{Spin}(7)$, there is no intermediate starting class. A crucial ingredient in the proof of the Calabi–Yau theorem is the $\partial\bar{\partial}$ lemma, which allows us to write the difference of any two Kähler forms in terms of an unknown *function* f. Therefore as a first step towards an analogous result in the G_2 and $\mathrm{Spin}(7)$ cases, we would like to determine the simplest data required to describe the relations between any two G_2 or $\mathrm{Spin}(7)$ forms.

1.2 Overview of New Results

If we start with only a G_2-structure, not necessarily parallel, this gives us a 3-form which satisfies some "positive-definiteness" property, since it determines a Riemannian metric. In [9], Fernández and Gray classified such manifolds by looking at the decomposition of $\nabla\varphi$ into G_2-irreducible components. There are 16 such classes, with various inclusion relations between them. There is a similar decomposition in [14] of almost complex manifolds into subclasses. Some of these classes are: integrable (complex), symplectic, almost Kähler, and nearly Kähler. Thus these 16 subclasses of manifolds with a G_2-structure are analogues of these "weaker than Kähler" conditions. Similar studies by Fernández in [7] of the $\mathrm{Spin}(7)$ case yield 4 subclasses of manifolds with a $\mathrm{Spin}(7)$-structure.

As a first step in trying to determine an analogue for the Calabi conjecture in the G_2 case, we can study these various weaker subclasses and their deformations. If we start in one class, and change the 3-form φ in some way (which changes the metric too) we would like to know under what conditions this subclass is preserved, or more generally what subclass the new G_2-structure now belongs to. The space of 3-forms on a manifold with a G_2-structure decomposes into a direct sum of irreducible G_2-representations:

$$\bigwedge^3 = \bigwedge^3_1 \oplus \bigwedge^3_7 \oplus \bigwedge^3_{27}$$

where $\bigwedge^3_k$ is a k-dimensional vector space at each point on M. This decomposition depends on our initial 3-form φ_o, however. This again is in stark contrast to the decomposition on a complex manifold into forms of type (p, q), which depends only on the complex structure and does not change as we vary the Kähler (or metric) structure. We can consider a deformation $\tilde{\varphi} = \varphi_0 + \eta$ of the G_2-structure, for $\eta \in \bigwedge^3_k$ and determine conditions on φ_o and η which preserve the subclass or change it in an interesting way.

If $\eta \in \bigwedge^3_1$, this corresponds to a conformal scaling of the metric, and one can explicitly describe which of the 16 classes are conformally invariant. (These results were already known to Fernández and Gray but here they are reproduced in a different way.) A new result in this case is the following:

Theorem 1.2.1 *Let $\theta_o = *_o(*_o d\varphi_o \wedge \varphi_o)$ be the canonical* 1*-form arising from a G_2-structure φ_o. Then if $\tilde{\varphi} = f^3\varphi_o$ for some non-vanishing function f, the new canonical* 1*-form $\tilde{\theta}$ differs from the old θ_o by an exact form:*

$$\tilde{\theta} = -12d\big(\log(f)\big) + \theta_o.$$

Thus in the classes where θ is closed, (there are some and they are conformally invariant classes), we get a well-defined cohomology class in $H^1(M)$, invariant under conformal changes of metric. A similar result also holds in the Spin(7) *case.*

If, however, we deform φ_o by an element $\eta \in \bigwedge^3_7$, then $\eta = w \lrcorner *_o \varphi_o$ for some vector field w, and in Section 3.2 we prove the following:

Theorem 1.2.2 *Under such a deformation $\tilde{\varphi}$ is again a G_2-structure and the new*

$$\langle v_1, v_2 \rangle_{\sim} = \frac{1}{(1 + |w|_o^2)^{\frac{2}{3}}} (\langle v_1, v_2 \rangle_o + \langle w \times v_1, w \times v_2 \rangle_o)$$

where $\times$ is the vector cross product associated to the original G_2-structure φ_o.

From this one can write down non-trivial differential equations on the vector field w for certain subclasses to be preserved. It would be interesting to solve some of these equations for the unknown vector field w. This would mean that there were certain distinguished vector fields on some classes of manifolds with G_2-structures. The important result here, however, is that the new 3-form $\tilde{\varphi}$ is *always* positive-definite. That is, it always corresponds to a G_2-structure. This gives information about the structure of the open set $\bigwedge^3_+(M)$ of positive definite 3-forms on M.

If instead we deform φ in the $\bigwedge^3_7$ direction infinitesmally by the flow equation

$$\frac{\partial}{\partial t}\varphi_t = w \lrcorner *_t \varphi_t$$

then we show in Section 3.3 that the metric g *does not change* and also:

Theorem 1.2.3 *The solution is given by*

$$\varphi(t) = \varphi_0 + \frac{1-\cos(|w|t)}{|w|^2}\big(w \lrcorner * (w \lrcorner * \varphi_0)\big) + \frac{\sin(|w|t)}{|w|}(w \lrcorner * \varphi_0).$$

Hence the solution exists for all time and is a closed path in $\bigwedge^3(M)$*. Also, the path only depends on the unit vector field* $\pm\frac{w}{|w|}$*, and the norm* $|w|$ *only affects the speed of travel along this path.*

In [4] the fact that the space of G_2-structures which correspond to the *same* metric as a fixed G_2-structure yields an $\mathbb{RP}^7$ bundle over M is mentioned. This is the content of the above theorem, and we provide an explicit description of these G_2-structures in terms of vector fields on M. In addition, in the special cases of $M = N \times S^1$, where N is a Calabi–Yau 3-fold, we show that this closed path of G_2-structures corresponds to the freedom of changing the *phase* of the holomorphic volume form $\Omega \mapsto e^{it}\Omega$ on N. Thus this theorem can be seen as a generalization of this situation.

The same kind of analysis can be done in the Spin(7) case. Similar but more complicated results hold in this case and are presented in Section 5. Here there are only 4 subclasses but the decomposition of $\bigwedge^4$ into irreducible Spin(7)-representations is more complicated:

$$\bigwedge^4 = \bigwedge^4_1 \oplus \bigwedge^4_7 \oplus \bigwedge^4_{27} \oplus \bigwedge^4_{35}.$$

In this case it is the space $\bigwedge^4_7$ which infinitesmally gives a closed path of Spin(7)-structures all corresponding to the same metric. However, perhaps initially somewhat surprisingly, this time non-infinitesmal deformations in the $\bigwedge^4_7$ direction *do not* yield a new Spin(7)-structure. This is explained in detail in Section 5.2. Much of the construction does indeed carry over, however, and it may be possible to alter it somehow to make it work.

1.3 Notation and Conventions

Many of the calculations that follow use various relations between the interior product $\lrcorner$, the exterior product $\wedge$, and the Hodge star operator $*$ as well as some identities involving determinants. Readers unfamiliar with this can refer to Appendix A.

In much of the computations there are two metrics present: an old metric g_o and a new metric $\tilde{g}$. Their associated volume forms, induced metrics on differential forms, and Hodge star operators are also identified by a subscript $_o$ for old or a $\tilde{}$ for new. We also often use the metric isomorphism between vector fields and one-forms, and denote this isomorphism by $w^\flat$ for the one-form associated to the vector field w and $\alpha^\sharp$ for the vector field associated to the one-form α. In the presence of two metrics, this isomorphism is *always* only used for the old metric g_o.

Finally, since many of the computations are extremely lengthy but similar, many of the explicit details have been omitted in this published version. See [23] for a longer version of this paper complete with all the details.

2 Manifolds With a G_2-Structure

2.1 G_2-Structures

Let M be an oriented 7-manifold with a global 2-fold cross product structure. Such a structure will henceforth be called a G_2-*structure*. Its existence is a topological condition, given by the vanishing of the second Stiefel–Whitney class $w_2 = 0$. (See [12, 22, 25] for details.) This cross product $\times$ gives rise to an associated *Riemannian metric* g and an alternating 3-form φ which are related by

$$\varphi(u, v, w) = g(u \times v, w). \tag{2.1}$$

This should be compared to the relation between a Kähler metric ω and a compatible almost complex structure J:

$$\omega(u, v) = g(Ju, v).$$

Note that in the Kähler case, the metric and the almost complex structure can be prescribed independently. This is *not* true in the case of manifolds with a G_2-structure, and this leads to some complications (and the inherit non-linearity of the problem). For a G_2-structure φ, near a point $p \in M$ we can choose local coordinates $x^1, \dots, x^7$ so that *at the point* p, we have:

$$\varphi_p = dx^{123} - dx^{167} - dx^{527} - dx^{563} + dx^{415} + dx^{426} + dx^{437} \tag{2.2}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$. In these coordinates the metric at p is the standard Euclidean metric $g_p = \sum_{k=1}^7 dx^k \otimes dx^k$ and the Hodge star dual $*\varphi$ of φ is

$$(*\varphi)_p = dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} + dx^{2637} + dx^{1537} + dx^{1526}. \tag{2.3}$$

Remark 2.1.1 Different conventions exist in the literature for (2.2) and (2.3), which may or may not differ from our choice by renumbering of coordinates and/or a change of orientation.

The 3-forms on M that arise from a G_2-structure are called *positive* 3-forms or *non-degenerate*. We will denote this set by $\bigwedge^3_{\text{pos}}$. The subgroup of SO(7) that preserves φ_p is the exceptional Lie group G_2. This can be found for example in [3, 16]. Hence at each point p, the set of G_2-structures at p is isomorphic to $\mathrm{GL}(7, \mathbb{R})/G_2$, which is $49 - 14 = 35$ dimensional. Since $\bigwedge^3(\mathbb{R}^7)$ is also 35 dimensional, the set $\bigwedge^3_{\text{pos}}(p)$ of positive 3-forms at p is an *open subset* of $\bigwedge^3_p$. We will determine some new information about the structure of $\bigwedge^3_{\text{pos}}$ in Section 3.2.

Remark 2.1.2 Note that in the Spin(7) case the situation is very different. The set of 4-forms on an 8-manifold M that determine a Spin(7)-structure is *not* an open subset of $\bigwedge^4(M)$. This is discussed in Section 4.1.

2.2 Decomposition of $\bigwedge^*(M)$ Into Irreducible G_2-representations

The group G_2 acts on $\mathbb{R}^7$, and hence acts on the spaces $\bigwedge^*$ of differential forms on M. One can decompose each space $\bigwedge^k$ into irreducible G_2-representations. The results of this decomposition are presented below (see [9, 22, 25]). The notation $\bigwedge^k_l$ refers to an l-dimensional irreducible G_2-representation which is a subspace of $\bigwedge^k$. Also, "vol" will denote the volume form of M (determined by the metric g), and w is a vector field on M.

$$\begin{aligned}
\bigwedge^0_1 &= \{f \in C^\infty(M)\}, & \bigwedge^1_7 &= \{\alpha \in \Gamma(\bigwedge^1(M))\},\\
\bigwedge^2 &= \bigwedge^2_7 \oplus \bigwedge^2_{14}, & \bigwedge^3 &= \bigwedge^3_1 \oplus \bigwedge^3_7 \oplus \bigwedge^3_{27},\\
\bigwedge^4 &= \bigwedge^4_1 \oplus \bigwedge^4_7 \oplus \bigwedge^4_{27}, & \bigwedge^5 &= \bigwedge^5_7 \oplus \bigwedge^5_{14},\\
\bigwedge^6_7 &= \{w \lrcorner \operatorname{vol}\}, & \bigwedge^7_1 &= \{f \operatorname{vol};\, f \in C^\infty(M)\}.
\end{aligned}$$

Since $G_2 \subset \mathrm{SO}(7)$, the decomposition respects the *Hodge star* $*$ operator, and $*\bigwedge^k_l = \bigwedge^{7-k}_l$. In addition, taking wedge product with φ or $*\varphi$ is either zero or an isomorphism onto its image for each irreducible summand, by Schur's Lemma.

Proposition 2.2.1 *If α is a 1-form, we have the following identities:*

$$*\big(\varphi \wedge *(\varphi \wedge \alpha)\big) = -4\alpha, \tag{2.4}$$

$$*\varphi \wedge *\big(\varphi \wedge \alpha\big) = 0,$$

$$*\big(*\varphi \wedge *(*\varphi \wedge \alpha)\big) = 3\alpha, \tag{2.5}$$

$$\varphi \wedge *(*\varphi \wedge \alpha) = 2(*\varphi \wedge \alpha), \tag{2.6}$$

$$|\varphi \wedge \alpha|^2 = 4|\alpha|^2, \tag{2.7}$$

$$|*\varphi \wedge \alpha|^2 = 3|\alpha|^2. \tag{2.8}$$

Proof Since the statements are pointwise, it is enough to check them in local coordinates using (2.2) and (2.3). This is tedious but straightforward. ■

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We now explicitly describe the decomposition for $k = 2, 3$.

$$\textstyle\bigwedge^2_7 = \{ w \lrcorner \varphi; w \in \Gamma(T(M)) \} = \Big\{ \beta \in \bigwedge^2; *(\varphi \wedge \beta) = 2\beta \Big\}, \tag{2.9}$$

$$\textstyle\bigwedge^2_{14} = \Big\{ \beta \in \bigwedge^2; *\varphi \wedge \beta = 0 \Big\} = \Big\{ \beta \in \bigwedge^2; *(\varphi \wedge \beta) = -\beta \Big\}, \tag{2.10}$$

$$\textstyle\bigwedge^3_1 = \{ f\varphi; f \in C^\infty(M) \}, \tag{2.11}$$

$$\textstyle\bigwedge^3_7 = \Big\{ *(\varphi \wedge \alpha); \alpha \in \bigwedge^1_7 \Big\} = \{ w \lrcorner *\varphi; w \in \Gamma(T(M)) \}, \tag{2.12}$$

$$\textstyle\bigwedge^3_{27} = \Big\{ \eta \in \bigwedge^3; \varphi \wedge \eta = 0 \text{ and } *\varphi \wedge \eta = 0 \Big\}. \tag{2.13}$$

2.3 The Metric of a G_2-Structure

From Proposition 2.2.1, we can obtain a formula for determining the metric g from the 3-form φ:

Proposition 2.3.1 *If v is a vector field on M, then*

$$(v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 6|v|^2 \operatorname{vol}. \tag{2.14}$$

Proof From Lemma A and (2.6) we have

$$v \lrcorner \varphi = *(v^\flat \wedge *\varphi)$$

and

$$(v \lrcorner \varphi) \wedge \varphi = 2(v^\flat \wedge *\varphi).$$

Thus we obtain

$$(v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 2|v^\flat \wedge *\varphi|^2 \operatorname{vol} = 6|v|^2 \operatorname{vol}$$

where we have used (2.8). ■

By polarizing (2.14) in v, we obtain the relation:

$$(v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi = 6\langle v, w \rangle \operatorname{vol}.$$

We can now give the expression for the metric in terms of the 3-form φ.

Theorem 2.3.2 *Let v be a tangent vector at a point p and let $e_1, e_2, \ldots, e_7$ be any basis for T_pM. Then the length $|v|$ of v is given by*

$$|v|^2 = 6^{\frac{2}{9}} \frac{\big((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi\big)(e_1, e_2, \ldots, e_7)}{\Big(\det\Big(\big((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi\big)(e_1, e_2, \ldots, e_7)\Big)\Big)^{\frac{1}{9}}}. \tag{2.15}$$

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Proof We work in local coordinates at the point p. In this notation $g_{ij} = \langle e_i, e_j \rangle$ with $1 \le i, j \le 7$. Let $\det(g)$ denote the determinant of (g_{ij}). We have from (2.14) that

$$\begin{aligned}\big((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi\big) &= 6 g_{ij} \,\mathrm{vol} \\ &= 6 g_{ij} \sqrt{\det(g)} e^1 \wedge e^2 \wedge \cdots \wedge e^7,\end{aligned}$$

$$\begin{aligned}\det\Big(\big((e_i \lrcorner \varphi) \wedge (e_j \lrcorner \varphi) \wedge \varphi\big) (e_1, e_2, \dots, e_7) \Big) &= 6^7 \det(g) \det(g)^{\frac{7}{2}} \\ &= 6^7 \det(g)^{\frac{9}{2}},\end{aligned}$$

and since

$$\begin{aligned}(v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 6|v|^2 \,\mathrm{vol} \\ = 6|v|^2 \sqrt{\det(g)} e^1 \wedge e^2 \wedge \cdots \wedge e^7,\end{aligned}$$

$$\big((v \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi\big)(e_1, e_2, \dots, e_7) = 6|v|^2 \det(g)^{\frac{1}{2}},$$

these two expressions can be combined to yield (2.15). ■

2.4 The Cross Product of a G_2-Structure

In this section we describe the cross product operation on a manifold with a G_2-structure in terms of the 3-form φ.

Definition 2.4.1 Let u and v be vector fields on M. The *cross product*, denoted $u \times v$, is a vector field on M whose associated 1-form under the metric isomorphism satisfies:

$$(u \times v)^\flat = v \lrcorner u \lrcorner \varphi. \tag{2.16}$$

Notice that this immediately yields the relation between $\times$, φ, and the metric g:

$$g(u \times v, w) = (u \times v)^\flat (w) = w \lrcorner v \lrcorner u \lrcorner \varphi = \varphi(u, v, w). \tag{2.17}$$

Another characterization of the cross product comes from Lemma A:

$$(u \times v)^\flat = v \lrcorner u \lrcorner \varphi = *(u^\flat \wedge v^\flat \wedge *\varphi). \tag{2.18}$$

Now since $u^\flat \wedge v^\flat$ is a 2-form, we can write it as $\beta_7 + \beta_{14}$, with $\beta_j \in \bigwedge^2_j$. Then we have, using (2.9) and (2.10):

$$\begin{aligned}(u \times v)^\flat \wedge *\varphi &= *(\beta_7 \wedge *\varphi) \wedge *\varphi \\ &= 3 * \beta_7.\end{aligned} \tag{2.19}$$

Taking the norm of both sides, and using (2.8):

$$|(u \times v)^\flat \wedge *\varphi|^2 = 3|(u \times v)^\flat|^2 = 3|u \times v|^2 = 9|\beta_7|^2,$$

from which we obtain

$$|\beta_7|^2 = \frac{1}{3}|u \times v|^2. \tag{2.20}$$

Lemma 2.4.2 *Let u and v be vector fields. Then*

$$|u \times v|^2 = |u \wedge v|^2. \tag{2.21}$$

Proof With $\beta = u^\flat \wedge v^\flat$, we have from (2.9) and (2.10):

$$\begin{aligned}\beta \wedge \varphi &= 2 * \beta_7 - *\beta_{14},\\ \beta \wedge \beta \wedge \varphi &= 2|\beta_7|^2 \operatorname{vol} - |\beta_{14}|^2 \operatorname{vol}\\ &= 0,\end{aligned}$$

since $\beta = u^\flat \wedge v^\flat$ is decomposable. So $|\beta_{14}|^2 = 2|\beta_7|^2$ and finally we obtain from (2.20):

$$|u \times v|^2 = 3|\beta_7|^2 = |\beta_7|^2 + |\beta_{14}|^2 = |\beta|^2 = |u \wedge v|^2. \qquad \blacksquare$$

More identities involving the cross product are given in [23]. The following lemma will be used in Section 3.2 to determine how the metric changes under a deformation in the $\bigwedge^3_7$ direction.

Lemma 2.4.3 *The following identity holds for v and w vector fields:*

$$(v \lrcorner w \lrcorner * \varphi) \wedge (v \lrcorner w \lrcorner \varphi) \wedge *\varphi = 2|v \wedge w|^2 \operatorname{vol}. \tag{2.22}$$

Proof We start with Lemma A to rewrite

$$\begin{aligned}v \lrcorner w \lrcorner * \varphi &= *\big(v^\flat \wedge *(w \lrcorner * \varphi)\big)\\ &= - * (v^\flat \wedge w^\flat \wedge \varphi) = -2\beta_7 + \beta_{14},\end{aligned}$$

using the notation as above. From equations (2.16) and (2.19) we have

$$(v \lrcorner w \lrcorner \varphi) \wedge *\varphi = -3 * \beta_7.$$

Combining these two equations and (2.20),

$$\begin{aligned}(v \lrcorner w \lrcorner * \varphi) \wedge (v \lrcorner w \lrcorner \varphi) \wedge *\varphi &= (-2\beta_7 + \beta_{14}) \wedge (-3 * \beta_7)\\ &= 6|\beta_7|^2 \operatorname{vol} = 2|u \wedge v|^2 \operatorname{vol},\end{aligned}$$

which completes the proof. ■

Finally, we prove a theorem which will be useful in Section 3.2 where we will use it to show that to first order, deforming a G_2-structure by an element of $\bigwedge^3_7$ does not change the metric.

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Theorem 2.4.4 *Let u, v, w be vector fields. Then*

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) = 0.$$

Note that in terms of the decompositions in (2.9) *and* (2.12), *this theorem says that the wedge product map*

$$\bigwedge^2_7 \times \bigwedge^2_7 \times \bigwedge^3_7 \to \bigwedge^7_1$$

is the zero map.

Proof Since it is an 8-form,

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge * \varphi = 0.$$

Taking the interior product with w and rearranging,

$$\begin{aligned}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) = &-(w \lrcorner u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge * \varphi \\ &- (u \lrcorner \varphi) \wedge (w \lrcorner v \lrcorner \varphi) \wedge * \varphi.\end{aligned}$$

Now since $* \varphi \wedge (w \lrcorner \varphi) = 3 * w^\flat$, we get

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) = -3(w \lrcorner u \lrcorner \varphi) \wedge * v^\flat - 3(w \lrcorner v \lrcorner \varphi) \wedge * u^\flat.$$

Finally, from (A.7), we have

$$\begin{aligned}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge (w \lrcorner * \varphi) &= -3(u \lrcorner \varphi) \wedge *(w^\flat \wedge v^\flat) - 3(v \lrcorner \varphi) \wedge *(w^\flat \wedge u^\flat) \\ &= -3\varphi \wedge *(u^\flat \wedge w^\flat \wedge v^\flat) - 3\varphi \wedge *(v^\flat \wedge w^\flat \wedge u^\flat) \\ &= 0.\end{aligned}$$

■

2.5 The 16 Classes of G_2-Structures

According to the classification of Fernández and Gray in [9], a manifold with a G_2-structure has holonomy a subgroup of G_2 if and only if $\nabla \varphi = 0$, which they showed to be equivalent to

$$d\varphi = 0 \text{ and } d * \varphi = 0.$$

They established this equivalence by decomposing the space W that $\nabla \varphi$ belongs to into irreducible G_2-representations, and identifying the invariant subspaces of W with isomorphic subspaces of $\wedge^*(M)$. This space W decomposes as

$$W = W_1 \oplus W_7 \oplus W_{14} \oplus W_{27},$$

where the subscript k denotes the dimension of the irreducible representation W_k. Now $d\varphi \in \bigwedge^3_1 \oplus \bigwedge^3_7 \oplus \bigwedge^3_{27}$ and $d * \varphi \in \bigwedge^5_7 \oplus \bigwedge^5_{14}$. Up to isomorphism, the projections $\pi_k(d\varphi)$ and $\pi_k(d * \varphi)$ are non-zero constant multiples of $\pi_k(\nabla \varphi)$. Therefore in the following we will consider $d\varphi$ and $d * \varphi$ instead of $\nabla \varphi$. Since both of these have a component in a 7-dimensional representation, they are multiples:

Lemma 2.5.1 *The following identity holds:*

$$\mu = *d\varphi \wedge \varphi = - * d * \varphi \wedge *\varphi, \tag{2.23}$$

where we have defined the 6-form μ by the above two equal expressions. They are the components $\pi_7(d\varphi)$ and $\pi_7(d * \varphi)$ transferred to the isomorphic space $\bigwedge^6_7$.

Proof See [3] for a proof. ■

We prefer to work with the associated 1-form, $\theta = *\mu$. We will see that in some subclasses this 1-form is closed or at least "partially closed."

Now we say a G_2-structure is in the class $W_i \oplus W_j \oplus W_k$ with i, j, k distinct where $\{i, j, k\} \subset \{1, 7, 14, 27\}$ if only the component of $d\varphi$ or $d * \varphi$ in the l-dimensional representation vanishes. Here $\{l\} = \{1, 7, 14, 27\} \setminus \{i, j, k\}$. Similarly the G_2-structure is in the class $W_i \oplus W_j$ if the k and l-dimensional components vanish, and in the class W_i if the other three components are zero. In this way we arrive at 16 classes of G_2-structures on a manifold. In Table 2.1 we describe the classes in terms of differential equations on the form φ. This classification first appeared in [9] and then in essentially this form in [5].

In Table 2.1, the function $h = \frac{1}{7} * (\varphi \wedge d\varphi)$ is the image of $\pi_1(d\varphi)$ in $\bigwedge^0$ under the isomorphism $\bigwedge^4_1 \cong \bigwedge^0_1$. The abbreviation "LC" stands for *locally conformal to* and means that for those classes, we can (at least locally) conformally change the metric to enter a strictly smaller subclass. This will be explained in Section 3.1.

We now prove the closedness or partial closedness of θ in the various classes as given in the final column of Table 2.1. The closedness of θ in the classes $W_1 \oplus W_7$ and $W_7 \oplus W_{14}$ was originally shown using a different approach by Cabrera in [5].

Lemma 2.5.2 *If φ is in the classes W_7, $W_7 \oplus W_{14}$, or $W_1 \oplus W_7$, then $d\theta = 0$. Furthermore, if φ is in the classes $W_7 \oplus W_{27}$ or $W_1 \oplus W_7 \oplus W_{27}$ then $\pi_7(d\theta) = 0$.*

Proof We begin by showing that if φ satisfies $d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$, then $d\theta = 0$, and if φ satisfies $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$, then $\pi_7(d\theta) = 0$.

Suppose $d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$. We differentiate this equation to obtain:

$$d\theta \wedge \varphi = \theta \wedge d\varphi = \theta \wedge \left(-\frac{1}{4}\theta \wedge \varphi\right) = 0.$$

But wedge product with φ is an isomorphism from $\bigwedge^2$ to $\bigwedge^5$, so $d\theta = 0$. Now suppose $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$. Differentiating this equation yields

$$d\theta \wedge *\varphi = \theta \wedge d * \varphi = \theta \wedge \left(-\frac{1}{3}\theta \wedge *\varphi\right) = 0.$$

But wedge product with $*\varphi$ is an isomorphism from $\bigwedge^2_7$ to $\bigwedge^6_7$, so $\pi_7(d\theta) = 0$.

Thus by comparing with Table 2.1, we have shown that in the classes $W_7 \oplus W_{14}$ and W_7, we have $d\theta = 0$. Also, in the classes $W_1 \oplus W_7 \oplus W_{27}$ and $W_7 \oplus W_{27}$ we

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Class	Defining Equations	Name	$d\theta$
$W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$	no relation on $d\varphi$, $d * \varphi$.		
$W_7 \oplus W_{14} \oplus W_{27}$	$d\varphi \wedge \varphi = 0$		$d\theta =?$
$W_1 \oplus W_{14} \oplus W_{27}$	$\theta = 0$		$\theta = 0$
$W_1 \oplus W_7 \oplus W_{27}$	$d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$ or $\varphi \wedge (*d * \varphi) = -2d * \varphi$	"integrable"	$\pi_7(d\theta) = 0$
$W_1 \oplus W_7 \oplus W_{14}$	$d\varphi + \frac{1}{4}\theta \wedge \varphi - h * \varphi = 0$		$d\theta =?$
$W_{14} \oplus W_{27}$	$d\varphi \wedge \varphi = 0$ and $\theta = 0$		$\theta = 0$
$W_7 \oplus W_{27}$	$d\varphi \wedge \varphi = 0$ and $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$		$\pi_7(d\theta) = 0$
$W_7 \oplus W_{14}$	$d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$	LC almost G_2	$d\theta = 0$
$W_1 \oplus W_{27}$	$d * \varphi = 0$	semi-G_2	$\theta = 0$
$W_1 \oplus W_{14}$	$d\varphi - h * \varphi = 0$		$\theta = 0$
$W_1 \oplus W_7$	$d\varphi + \frac{1}{4}\theta \wedge \varphi - h * \varphi = 0$ and $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$	LC nearly G_2	$d\theta = 0$
W_{27}	$d\varphi \wedge \varphi = 0$ and $d * \varphi = 0$		$\theta = 0$
W_{14}	$d\varphi = 0$	almost G_2	$\theta = 0$
W_7	$d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$ and $d\varphi + \frac{1}{4}\theta \wedge \varphi = 0$	LC G_2	$d\theta = 0$
W_1	$d\varphi - h * \varphi = 0$ and $d * \varphi = 0$	nearly G_2	$\theta = 0$
$\{0\}$	$d\varphi = 0$ and $d * \varphi = 0$	G_2	$\theta = 0$

Table 2.1: The 16 classes of G_2-structures

have $\pi_7(d\theta) = 0$. We still have to show that θ is closed in the class $W_1 \oplus W_7$. We already have that $\pi_7(d\theta) = 0$, so we need only show that $\pi_{14}(d\theta) = 0$ in this case. We differentiate $d\varphi + \frac{1}{4}\theta \wedge \varphi - h * \varphi = 0$ to obtain

$$\begin{aligned} 0 &= \frac{1}{4} d\theta \wedge \varphi - \frac{1}{4}\theta \wedge d\varphi - dh \wedge *\varphi - hd * \varphi \\ &= \frac{1}{4} d\theta \wedge \varphi - \frac{1}{4}\theta \wedge \Big(-\frac{1}{4}\theta \wedge \varphi + h * \varphi\Big) - dh \wedge *\varphi - h\Big(-\frac{1}{3}\theta \wedge *\varphi\Big) \\ &= \frac{1}{4} d\theta \wedge \varphi + \alpha \wedge *\varphi, \end{aligned}$$

for some 1-form α, where we have used the fact that $d * \varphi + \frac{1}{3}\theta \wedge *\varphi = 0$ in this class. But $\alpha \wedge *\varphi$ is in $\bigwedge^5_7$, and since wedge product with φ is an isomorphism from $\bigwedge^2_k$ to $\bigwedge^5_k$ for $k = 7, 14$, this shows that $\pi_{14}(d\theta) = 0$. ■

The inclusion relations among these various subclasses are analyzed in [9, 5, 6, 8, 3, 4, 19, 20, 25]. For all but one case, examples can be found of manifolds which are in a particular class but *not* in a strictly smaller subclass. For example, a manifold in the class W_{14} which does *not* have holonomy G_2 appears in [8]. There is one case of an inclusion in Table 2.1 which is *not* strict. This is given by the following result, which first appeared in [5].

Proposition 2.5.3 *The class $W_1 \oplus W_{14}$ equals $W_1 \cup W_{14}$ exactly.*

Proof In the class $W_1 \oplus W_{14}$, we have $d\varphi - h * \varphi = 0$ (and by consequence $\theta = 0$). Differentiating this equation,

$$dh \wedge *\varphi = -hd * \varphi.$$

If $h \neq 0$, then by dividing by h and using Proposition 2.2.1, we see that $d * \varphi \in \bigwedge^5_7$, so $\pi_{14}(d * \varphi) = 0$. But since we already have that $\theta = 0$, this means $d * \varphi = 0$ and hence φ is actually of class W_1 (nearly G_2). If $h = 0$ then $d\varphi = 0$ and φ is of class W_{14} (almost G_2). ■

Remark 2.5.4 Note that in the proof of the above proposition, we see that if φ is of class W_1 (nearly G_2), then $dh \wedge *\varphi = 0$, and so $dh = 0$ by Proposition 2.2.1. Therefore in the nearly G_2 case, the function h is *locally constant*, or constant if the manifold M is connected. In [13] Gray showed that all nearly G_2 manifolds are actually *Einstein*.

In [10, 11], Fernández and Ugarte show that for manifolds with a G_2-structure in the classes $W_1 \oplus W_7 \oplus W_{27}$ ("integrable") or $W_7 \oplus W_{14}$, there exists a subcomplex of the deRham complex. They then show how to define analogues of Dolbeault cohomology of complex manifolds in these two cases, including analogues of $\bar{\partial}$-harmonic forms. They derive properties of these cohomology theories and topological restrictions on the existence of G_2-structures in some strictly smaller subclasses.

3 Deformations of a Fixed G_2-Structure

Let us begin with a fixed G_2-structure on a manifold M in a certain class. We are interested in how deforming the form φ affects the class. In other words, we are interested in what kinds of deformations preserve which classes of G_2-structures. Now since $\varphi \in \bigwedge^3_1 \oplus \bigwedge^3_7 \oplus \bigwedge^3_{27}$, there are three canonical ways to deform φ. For example, since $\bigwedge^3_1 = \{f\varphi\}$, adding to φ an element of $\bigwedge^3_1$ amounts to conformally scaling φ. This preserves the decomposition into irreducible representations in this case. However, since the decomposition does depend on φ (unlike the decomposition of forms into (p, q) types on a Kähler manifold) in general if we add an element of $\bigwedge^3_7$ or $\bigwedge^3_{27}$ the decomposition does change. So deforming in those two directions a priori only makes sense infinitesmally. However, we shall see that adding an element of $\bigwedge^3_7$ in fact *does* yield a new G_2-structure.

3.1 Conformal Deformations of G_2-Structures

Let f be a smooth, nowhere vanishing function on M. For notational convenience, we will conformally scale φ by f^3. Let the new form $\tilde{\varphi} = f^3\varphi_o$. We first compute the new metric $\tilde{g}$ and the new volume form $\mathrm{vol}_\sim$ in the following lemma.

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Lemma 3.1.1 *The metric g_o on vector fields, the metric g_o^{-1} on one forms, and the volume form* vol_o *transform as follows:*

$$\mathrm{vol}_\sim = f^7 \,\mathrm{vol}_o, \quad \tilde{g} = f^2 g_o, \quad \tilde{g}^{-1} = f^{-2} g_o^{-1}.$$

Proof Using Proposition 2.3.1, we have in a local coordinate chart:

$$\begin{aligned}\tilde{g}(u,v)\,\mathrm{vol}_\sim &= \frac{1}{6}(u \lrcorner \tilde{\varphi}) \wedge (v \lrcorner \tilde{\varphi}) \wedge \tilde{\varphi} \\ &= f^9 g_o(u,v)\,\mathrm{vol}_o, \\ \tilde{g}(u,v)\sqrt{\det(\tilde{g})}dx^1 \cdots dx^7 &= f^9 g_o(u,v)\sqrt{\det(g_o)}dx^1 \cdots dx^7.\end{aligned}$$

Thus, taking determinants of the coefficients of both sides,

$$\begin{aligned}\det(\tilde{g})^{\frac{7}{2}} \det(\tilde{g}) &= f^{63} \det(g_o)^{\frac{7}{2}} \det(g_o), \\ \sqrt{\det(\tilde{g})} &= f^7 \sqrt{\det(g_o)}.\end{aligned}$$

This gives $\mathrm{vol}_\sim = f^7 \,\mathrm{vol}_o$, from which we see that $\tilde{g} = f^2 g_o$ and $\tilde{g}^{-1} = f^{-2} g_o$. ■

Corollary 3.1.2 *If α is a k-form, then $\tilde{*}\alpha = f^{7-2k} *_o \alpha$. Furthermore, the new 3-form $\tilde{\varphi}$ satisfies $\tilde{*}\tilde{\varphi} = f^4 *_o \varphi_o$.*

Proof This follows easily from Lemma 3.1.1 since the new metric on k-forms is $\langle\, , \rangle_\sim = f^{-2k}\langle\, , \rangle_o$. ■

Combining these results yields:

Lemma 3.1.3 *We have the following relations:*

$$\begin{aligned} d\tilde{\varphi} &= 3f^2 df \wedge \varphi_o + f^3 d\varphi_o, \\ d\tilde{*}\tilde{\varphi} &= 4f^3 df \wedge *_o\varphi_o + f^4 d *_o \varphi_o, \\ \tilde{*} d\tilde{\varphi} &= 3f *_o (df \wedge \varphi_o) + f^2 *_o d\varphi_o, \\ \tilde{*} d\tilde{*}\tilde{\varphi} &= 4 *_o (df \wedge *_o\varphi_o) + f *_o (d *_o \varphi_o).\end{aligned}$$

Proof This follows from Corollary 3.1.2. ■

Using these results, we can determine which classes of G_2-structures are conformally invariant. We can also determine what happens to the 6-form μ from equation (2.23) as well as the associated 1-form $\theta = *\mu$. This is all given in the following theorem:

Theorem 3.1.4 *Under the conformal deformation $\tilde{\varphi} = f^3\varphi_o$, we have:*

$$d\tilde{*}\tilde{\varphi} + \frac{1}{3}\tilde{\theta} \wedge \tilde{*}\tilde{\varphi} = f^4\big(d *_o \varphi_o + \frac{1}{3}\theta_o \wedge *_o\varphi_o\big)\,, \tag{3.1}$$

$$d\tilde{\varphi} + \frac{1}{4}\tilde{\theta} \wedge \tilde{\varphi} = f^3\big(d\varphi_o + \frac{1}{4}\theta_o \wedge \varphi_o\big)\,, \tag{3.2}$$

$$d\tilde{\varphi} \wedge \tilde{\varphi} = f^6(d\varphi_o \wedge \varphi_o), \tag{3.3}$$

$$d\tilde{\varphi} + \frac{1}{4}\tilde{\theta} \wedge \tilde{\varphi} - \tilde{h}\tilde{*}\tilde{\varphi} = f^3\big(d\varphi_o + \frac{1}{4}\theta_o \wedge \varphi_o - h_o *_o \varphi_o\big)\,, \tag{3.4}$$

$$\tilde{\mu} = -12f^4 *_o df + f^5\mu_o, \tag{3.5}$$

$$\tilde{\theta} = -12d\big(\log(f)\big) + \theta_o. \tag{3.6}$$

Hence, we see (from Table 2.1*) that the classes which are conformally invariant are exactly $W_7 \oplus W_{14} \oplus W_{27}$, $W_1 \oplus W_7 \oplus W_{27}$, $W_1 \oplus W_7 \oplus W_{14}$, $W_7 \oplus W_{27}$, $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, and W_7. These are precisely the classes which have a W_7 component. (This conclusion was originally observed in* [9] *using a different method.)*

Additionally, (3.6) *shows that since θ changes by an exact form, in the classes where $d\theta = 0$, we have a well defined cohomology class $[\theta]$ which is unchanged under a conformal scaling. These are the classes $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, and W_7.*

Proof We begin by using Lemma 3.1.3 and (2.23) to compute $\tilde{\mu}$ and $\tilde{\theta}$:

$$\begin{aligned}
\tilde{\mu} &= \tilde{*}d\tilde{\varphi} \wedge \tilde{\varphi} \\
&= \big(3f *_o (df \wedge \varphi_o) + f^2 *_o d\varphi_o\big) \wedge f^3\varphi_o \\
&= 3f^4\varphi_o \wedge *_o(\varphi_o \wedge df) + f^5\mu_o \\
&= -12f^4 *_o df + f^5\mu_o,
\end{aligned}$$

where we have used (2.4) in the last step. Now from Corollary 3.1.2, we get:

$$\tilde{\theta} = \tilde{*}\tilde{\mu} = -12f^{-1}df + \theta_o = -12d\big(\log(f)\big) + \theta_o.$$

Now using the above expression for $\tilde{\theta}$, we have:

$$\begin{aligned}
d\tilde{*}\tilde{\varphi} + \frac{1}{3}\tilde{\theta} \wedge \tilde{*}\tilde{\varphi} &= 4f^3df \wedge *_o\varphi_o + f^4d *_o \varphi_o + \frac{1}{3}(-12f^{-1}df + \theta_o) \wedge f^4 *_o \varphi_o \\
&= f^4\big(d *_o \varphi_o + \frac{1}{3}\theta_o \wedge *_o\varphi_o\big)\,,
\end{aligned}$$

$$\begin{aligned}
d\tilde{\varphi} + \frac{1}{4}\tilde{\theta} \wedge \tilde{\varphi} &= 3f^2df \wedge \varphi_o + f^3d\varphi_o + \frac{1}{4}(-12f^{-1}df + \theta_o) \wedge f^3\varphi_o \\
&= f^3\big(d\varphi_o + \frac{1}{4}\theta_o \wedge \varphi_o\big)\,.
\end{aligned}$$

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and finally, since $\varphi_o \wedge \varphi_o = 0$,

$$d\tilde{\varphi} \wedge \tilde{\varphi} = (3f^2 df \wedge \varphi_o + f^3 d\varphi_o) \wedge f^3\varphi_o = f^6(d\varphi_o \wedge \varphi_o).$$

Finally, since $h = \frac{1}{7} * (\varphi \wedge d\varphi)$, we have

$$\begin{aligned}\tilde{h}\tilde{*}\tilde{\varphi} &= \frac{1}{7}\tilde{*}(\tilde{\varphi} \wedge d\tilde{\varphi}) f^4 *_o \varphi_o \\ &= \frac{1}{7} f^{-7} *_o (f^6 \varphi_o \wedge d\varphi_o) f^4 *_o \varphi_o \\ &= f^3 h_o *_o \varphi_o,\end{aligned}$$

which yields (3.4) when combined with (3.2). This completes the proof. ■

These results now enable us to give necessary and sufficient conditions for obtaining a closed or co-closed $\tilde{\varphi}$ by conformally scaling the original φ_o.

Theorem 3.1.5 *Let φ_o be a positive 3-form (associated to a G_2-structure). Under the conformal deformation $\tilde{\varphi} = f^3\varphi_o$, the new 3-form $\tilde{\varphi}$ satisfies*

- $d\tilde{\varphi} = 0 \Leftrightarrow \varphi_o$ *is at least class* $W_7 \oplus W_{14}$ *and* $12d\log(f) = \theta_o$.
- $d *_o \varphi_o = 0 \Leftrightarrow \varphi_o$ *is at least class* $W_1 \oplus W_7 \oplus W_{27}$ *and* $12d\log(f) = \theta_o$.

Note that in both cases, in order to have $\tilde{\varphi}$ be closed or co-closed after conformal scaling, the original 1-form θ_o has to be exact. In particular if the manifold is simply-connected or more generally $H^1(M) = 0$ then this will always be the case if φ_o is in the classes $W_7 \oplus W_{14}$, $W_1 \oplus W_7$, or W_7, where $d\theta_o = 0$.

Proof From Lemma 3.1.3, for $d\tilde{\varphi} = 0$, we need

$$d\tilde{\varphi} = 3f^2 df \wedge \varphi_o + f^3 d\varphi_o = 0 \quad \Rightarrow \quad d\varphi_o = -3d\log(f) \wedge \varphi_o,$$

which says that $d\varphi_o \in \bigwedge^4_7$ by Proposition 2.2.1. Hence $\pi_1(d\varphi_o)$ and $\pi_{27}(d\varphi_o)$ both vanish and φ_o must be already at least of class $W_7 \oplus W_{14}$. Then to make $d\tilde{\varphi} = 0$, we need to eliminate the W_7 component, which requires $12d\log(f) = \theta_o$ by Theorem 3.1.4. Similarly, to make $d\tilde{*}\tilde{\varphi} = 0$, Lemma 3.1.3 gives

$$d\tilde{*}\tilde{\varphi} = 4f^3 df \wedge *_o\varphi_o + f^4 d *_o \varphi_o = 0 \quad \Rightarrow \quad d *_o \varphi_o = -4d\log(f) \wedge *_o\varphi_o,$$

which says $d *_o \varphi_o \in \bigwedge^5_7$ and $\pi_{14}(d *_o \varphi_o) = 0$ by Proposition 2.2.1. Thus φ_o must already be at least class $W_1 \oplus W_7 \oplus W_{27}$ and we need to choose f by $12d\log(f) = \theta_o$ to scale away the W_7 component. ■

Remark 3.1.6 We have shown that the transformation $\tilde{\varphi} = f^3\varphi_o$ stays in a particular subclass as long as there is a W_7 component to that class. If there is, and the original θ_o is exact, then we can choose f to scale away the W_7 component and enter a stricter subclass. Conversely, Theorem 3.1.4 shows that a conformal scaling by a non-constant f will always generate a non-zero W_7 component if we started with none. Hence, if we are trying to construct metrics of holonomy G_2 on a simply-connected manifold, it is enough to construct a metric in the class W_7, since we can then conformally scale (uniquely) to obtain a metric of holonomy G_2. This is why the class W_7 is called *locally conformal* G_2.

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3.2 Deforming φ by an Element of $\bigwedge^3_7$

The type of deformation of φ that is next in line in terms of increasing complexity is to add an element of $\bigwedge^3_7$. This space is isomorphic to $\bigwedge^1_7 \cong \Gamma\big(T(M)\big)$, so we can think of this process as deforming φ by a vector field. In fact, an element $\eta \in \bigwedge^3_7$ is of the form $w \lrcorner * \varphi$ for some vector field w, by (2.12). Let $\tilde{\varphi} = \varphi_o + t w \lrcorner *_o \varphi_o$, for $t \in \mathbb{R}$. We will develop formulas for the new metric $\tilde{g}$, the new Hodge star $\tilde{*}$, and other expressions entirely in terms of the old φ_o, the old $*_o$, and the vector field w. Note in this case the background decomposition into irreducible G_2-representations changes, and in Section 3.3 we will linearize by taking $\frac{d}{dt}|_{t=0}$ of our results.

Lemma 3.2.1 *In the expression*

$$6|v|^2_\sim \operatorname{vol}_\sim = (v \lrcorner \tilde{\varphi}) \wedge (v \lrcorner \tilde{\varphi}) \wedge \tilde{\varphi}$$

which is a cubic polynomial in t, the linear and cubic terms both vanish, and the coefficient of the quadratic term is

$$6|v \wedge w|^2_o \operatorname{vol}_o .$$

Proof The coefficient of t^3 is:

$$(v \lrcorner w \lrcorner *_o \varphi_o) \wedge (v \lrcorner w \lrcorner *_o \varphi_o) \wedge (w \lrcorner *_o \varphi_o).$$

This expression is zero because it arises by taking the interior product with w of the 8-form

$$(v \lrcorner w \lrcorner *_o \varphi_o) \wedge (v \lrcorner w \lrcorner *_o \varphi_o) \wedge *_o \varphi_o = 0.$$

The coefficient of t is:

$$(v \lrcorner \varphi_o) \wedge (v \lrcorner \varphi_o) \wedge (w \lrcorner *_o \varphi_o) + 2(v \lrcorner \varphi_o) \wedge (v \lrcorner w \lrcorner *_o \varphi_o) \wedge \varphi_o.$$

Using (A.8) on the second term and rearranging, this coefficient becomes

$$3(v \lrcorner \varphi_o) \wedge (v \lrcorner \varphi_o) \wedge (w \lrcorner *_o \varphi_o),$$

which vanishes by Theorem 2.4.4.

The coefficient of t^2 is:

$$(v \lrcorner w \lrcorner *_o \varphi_o) \wedge \Big((v \lrcorner w \lrcorner *_o \varphi_o) \wedge \varphi_o + 2(v \lrcorner \varphi_o) \wedge (w \lrcorner *_o \varphi_o) \Big). \tag{3.7}$$

Applying (A.8) twice and rearranging, this coefficient becomes

$$3(v \lrcorner w \lrcorner *_o \varphi_o) \wedge (v \lrcorner w \lrcorner \varphi_o) \wedge *_o \varphi_o.$$

The statement now follows from Lemma 2.4.3. ■

Before we can use Lemma 3.2.1 to obtain the new metric, we have to extract the new volume form.

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Proposition 3.2.2 *With $\tilde{\varphi} = \varphi_o + w \lrcorner *_o \varphi_o$, the new volume form is*

$$\mathrm{vol}_{\sim} = (1 + |w|_o^2)^{\frac{2}{3}} \, \mathrm{vol}_o \, . \tag{3.8}$$

Proof We work in local coordinates. Let $e_1, e_2, \dots, e_7$ be a basis for the tangent space, with $w = w^j e_j$, $g_{ij} = \langle e_i, e_j \rangle_o$ and $\tilde{g}_{ij} = \langle e_i, e_j \rangle_{\sim}$. Then Lemma 3.2.1 says that

$$|v|_{\sim}^2 \sqrt{\det(\tilde{g})} = \left(|v|_o^2 + |v \wedge w|_o^2 \right) \sqrt{\det(g)}.$$

Polarizing this equation, we have:

$$\langle v_1, v_2 \rangle_{\sim} \sqrt{\det(\tilde{g})} = \left(\langle v_1, v_2 \rangle_o + \langle v_1, v_2 \rangle_o |w|_o^2 - \langle v_1, w \rangle_o \langle v_2, w \rangle_o \right) \sqrt{\det(g)},$$
$$\tilde{g}_{ij} \sqrt{\det(\tilde{g})} = \left(g_{ij} + \langle e_i \wedge w, e_j \wedge w \rangle_o \right) \sqrt{\det(g)},$$

with $v_1 = e_i$ and $v_2 = e_j$. Now substituting $w = w^k e_k$ in the second term,

$$\langle e_i \wedge w, e_j \wedge w \rangle_o = |w|_o^2 g_{ij} - w_i w_j .$$

Thus we have

$$\tilde{g}_{ij} \sqrt{\det(\tilde{g})} = \left(g_{ij}(1 + |w|_o^2) - w_i w_j \right) \sqrt{\det(g)}.$$

We take determinants of both sides of this equation, and use the fact that they are 7×7 matrices, to obtain

$$\left(\det(\tilde{g})\right)^{\frac{9}{2}} = \left(\det(g)\right)^{\frac{7}{2}} \det\left(g_{ij}(1 + |w|_o^2) - w_i w_j \right). \tag{3.9}$$

Using Lemma A.4, the determinant on the right is

$$(1 + |w|_o^2)^7 \det(g) - |w|_o^2 (1 + |w|_o^2)^6 \det(g) = (1 + |w|_o^2)^6 \det(g). \tag{3.10}$$

Substituting this result into equation (3.9), we obtain

$$\left(\det(\tilde{g})\right)^{\frac{9}{2}} = \left(\det(g)\right)^{\frac{7}{2}} (1 + |w|_o^2)^6 \det(g),$$
$$\sqrt{\det(\tilde{g})} = (1 + |w|_o^2)^{\frac{2}{3}} \sqrt{\det(g)},$$

which completes the proof. ■

Now letting $t = 1$, with $\tilde{\varphi} = \varphi_o + w \lrcorner *_o \varphi_o$, Lemma 3.2.1 and Proposition 3.2.2 yield

$$|v|_{\sim}^2 \, \mathrm{vol}_{\sim} = (|v|_o^2 + |v \wedge w|_o^2) \, \mathrm{vol}_o,$$
$$\langle v, v \rangle_{\sim} = \frac{1}{(1 + |w|_o^2)^{\frac{2}{3}}} \left(\langle v, v \rangle_o + |v|_o^2 |w|_o^2 - \langle v, w \rangle_o^2 \right).$$

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Polarizing this equation, we obtain:

$$\langle v_1, v_2\rangle_{\sim} = \frac{1}{(1+|w|_o^2)^{\frac{2}{3}}}\left(\langle v_1, v_2\rangle_o + \langle v_1, v_2\rangle_o |w|_o^2 - \langle v_1, w\rangle_o \langle v_2, w\rangle_o\right), \tag{3.11}$$

which by (2.21) can also be written as

$$\langle v_1, v_2\rangle_{\sim} = \frac{1}{(1+|w|_o^2)^{\frac{2}{3}}}\left(\langle v_1, v_2\rangle_o + \langle w\times v_1, w\times v_2\rangle_o\right). \tag{3.12}$$

Note that in the above expression $\times$ refers to the vector cross product associated to the initial G_2-structure φ_o. Later we will describe this metric geometrically.

In local coordinates with $w = w^i e_i$, $g_{ij} = \langle e_i, e_j\rangle_o$, and $w^\flat = w_i e^i$, we see that

$$\tilde{g}_{ij} = \frac{1}{(1+|w|_o^2)^{\frac{2}{3}}}\left(g_{ij}(1+|w|_o^2) - w_i w_j\right). \tag{3.13}$$

Proposition 3.2.3 *In local coordinates, the metric $\tilde{g}^{ij}$ on* 1*-forms is given by:*

$$\tilde{g}^{ij} = \frac{1}{(1+|w|_o^2)^{\frac{1}{3}}}(g^{ij} + w^i w^j).$$

Proof We compute:

$$\begin{aligned}
\tilde{g}_{ij}\tilde{g}^{jk} &= \frac{1}{(1+|w|_o^2)^{\frac{2}{3}}}\left(g_{ij}(1+|w|_o^2) - w_i w_j\right)\frac{1}{(1+|w|_o^2)^{\frac{1}{3}}}(g^{jk} + w^j w^k)\\
&= \frac{1}{(1+|w|_o^2)}\left((g_{ij}g^{jk} + g_{ij}w^j w^k)(1+|w|_o^2) - g^{jk}w_i w_j - w_i w_j w^j w^k\right)\\
&= \frac{1}{(1+|w|_o^2)}\left((\delta_i^k + w_i w^k)(1+|w|_o^2) - w_i w^k - |w|_o^2 w_i w^k\right)\\
&= \delta_i^k,
\end{aligned}$$

which completes the proof. ∎

Now with $\alpha = \alpha_i e^i$ and $\beta = \beta_j e^j$ two 1-forms, their new inner product is

$$\begin{aligned}
\langle \alpha, \beta\rangle_{\sim} = \alpha_i\beta_j\tilde{g}^{ij} &= \frac{1}{(1+|w|_o^2)^{\frac{1}{3}}}(\alpha_i\beta_j g^{ij} + \alpha_i w^i \beta_j w^j)\\
&= \frac{1}{(1+|w|_o^2)^{\frac{1}{3}}}\left(\langle \alpha, \beta\rangle_o + (w \lrcorner \alpha)(w \lrcorner \beta)\right).
\end{aligned} \tag{3.14}$$

From this expression we can derive a formula for the new metric $\langle\ ,\ \rangle_{\sim}$ on k-forms:

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Theorem 3.2.4 *Let α, β be k-forms. Then*

$$\langle \alpha, \beta \rangle_{\sim} = \frac{1}{(1 + |w|_o^2)^{\frac{k}{3}}} (\langle \alpha, \beta \rangle_o + \langle w \lrcorner \alpha, w \lrcorner \beta \rangle_o). \tag{3.15}$$

Proof We have already established it for the case $k = 1$ in (3.14), and the case $k = 0$ is trivial. For the general case, we will prove the statement on *decomposable* forms and it follows in general by linearity. Let $\alpha = e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k}$ and $\beta = e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_k}$. Then by the definition of the metric on k-forms,

$$\langle \alpha, \beta \rangle_{\sim} = \det \begin{pmatrix} \langle e^{i_1}, e^{j_1} \rangle_{\sim} & \langle e^{i_1}, e^{j_2} \rangle_{\sim} & \cdots & \langle e^{i_1}, e^{j_k} \rangle_{\sim} \\ \langle e^{i_2}, e^{j_1} \rangle_{\sim} & \langle e^{i_2}, e^{j_2} \rangle_{\sim} & \cdots & \langle e^{i_2}, e^{j_k} \rangle_{\sim} \\ \vdots & \vdots & \ddots & \vdots \\ \langle e^{i_k}, e^{j_1} \rangle_{\sim} & \langle e^{i_k}, e^{j_2} \rangle_{\sim} & \cdots & \langle e^{i_k}, e^{j_k} \rangle_{\sim} \end{pmatrix}.$$

Now from equation (3.14) each entry in the above matrix is of the form

$$\langle e^{i_a}, e^{j_b} \rangle_{\sim} = \frac{1}{(1 + |w|_o^2)^{\frac{1}{3}}} (g^{i_a j_b} + w^{i_a} w^{j_b})$$

and we have

$$\langle \alpha, \beta \rangle_{\sim} = \frac{1}{(1 + |w|_o^2)^{\frac{k}{3}}} \det \begin{pmatrix} g^{i_1 j_1} + w^{i_1} w^{j_1} & \cdots & g^{i_1 j_k} + w^{i_1} w^{j_k} \\ \vdots & \ddots & \vdots \\ g^{i_k j_1} + w^{i_k} w^{j_1} & \cdots & g^{i_k j_k} + w^{i_k} w^{j_k} \end{pmatrix}.$$

Now we apply Lemma A.3 to obtain

$$\langle \alpha, \beta \rangle_o + \sum_{l,m=1}^{k} (-1)^{l+m} w^{i_l} w^{j_m} \langle e^{i_1} \wedge \cdots \widehat{e^{i_l}} \cdots \wedge e^{i_k}, e^{j_1} \wedge \cdots \widehat{e^{j_m}} \cdots \wedge e^{j_k} \rangle_o$$

for the determinant above. Now with $w = w^i e_i$, we can take the interior product with both α and β:

$$w \lrcorner \alpha = \sum_{l=1}^{k} (-1)^{l-1} w^{i_l} e^{i_1} \wedge \cdots \widehat{e^{i_l}} \cdots \wedge e^{i_k},$$

$$w \lrcorner \beta = \sum_{m=1}^{k} (-1)^{m-1} w^{j_m} e^{j_1} \wedge \cdots \widehat{e^{j_m}} \cdots \wedge e^{j_k},$$

and hence the sum over l and m above is just $\langle w \lrcorner \alpha, w \lrcorner \beta \rangle_o$. Putting everything together, we arrive at (3.15):

$$\langle \alpha, \beta \rangle_{\sim} = \frac{1}{(1 + |w|_o^2)^{\frac{k}{3}}} (\langle \alpha, \beta \rangle_o + \langle w \lrcorner \alpha, w \lrcorner \beta \rangle_o). \qquad \blacksquare$$

To continue our analysis of the new G_2-structure $\tilde{\varphi}$, we now need to compute the new Hodge star $\tilde{*}$.

Theorem 3.2.5 *The Hodge star for the new metric on a k-form α is given by:*

$$\tilde{*}\alpha = (1+|w|_o^2)^{\frac{2-k}{3}}\big(*_o\alpha + (-1)^{k-1}w\lrcorner\big(*_o(w\lrcorner\alpha)\big)\big)$$
$$= (1+|w|_o^2)^{\frac{2-k}{3}}\big(*_o\alpha + w\lrcorner(w^\flat\wedge *_o\alpha)\big). \tag{3.16}$$

Proof The second form follows from the first from (A.2). Although it looks a little more cluttered, we prefer to use the first form for $\tilde{*}$. Notice that up to a scaling factor, the new star is given by 'twisting by w', taking the old star, then 'untwisting by w', and adding this to the old star. To establish this formula, let β be an arbitrary k-form and compute:

$$\begin{aligned}\beta\wedge\tilde{*}\alpha &= \langle\beta,\alpha\rangle_\sim \operatorname{vol}_\sim\\ &= \frac{1}{(1+|w|_o^2)^{\frac{k}{3}}}(\langle\alpha,\beta\rangle_o + \langle w\lrcorner\alpha, w\lrcorner\beta\rangle_o)(1+|w|_o^2)^{\frac{2}{3}}\operatorname{vol}_o\\ &= (1+|w|_o^2)^{\frac{2-k}{3}}\big(\beta\wedge *_o\alpha + (w\lrcorner\beta)\wedge *_o(w\lrcorner\alpha)\big).\end{aligned}$$

Now if we take the interior product with w of the 8-form

$$\beta\wedge *_o(w\lrcorner\alpha) = 0$$

we obtain

$$(w\lrcorner\beta)\wedge *_o(w\lrcorner\alpha) = (-1)^{k-1}\beta\wedge\big(w\lrcorner\big(*_o(w\lrcorner\alpha)\big)\big),$$

and this completes the proof, since β is arbitrary. ■

We now give a geometric description of the transformation $\varphi_o \mapsto \varphi_o + w\lrcorner *_o\varphi_o$. From (3.11) for the new metric $\tilde{g}$, with $v_1 = v$ and $v_2 = w$, we have

$$\begin{aligned}\langle v,w\rangle_\sim &= \frac{1}{(1+|w|_o^2)^{\frac{2}{3}}}(\langle v,w\rangle_o + \langle v,w\rangle_o|w|_o^2 - \langle v,w\rangle_o\langle w,w\rangle_o)\\ &= \frac{1}{(1+|w|_o^2)^{\frac{2}{3}}}\langle v,w\rangle_o.\end{aligned}$$

Hence we see that all the distances are *shrunk* by a factor of $(1+|w|_o^2)^{-\frac{2}{3}}$ in the direction of the vector field w. On the other hand, if either v_1 or v_2 is orthogonal to w in the old metric, then (3.11) gives

$$\begin{aligned}\langle v_1,v_2\rangle_\sim &= \frac{1}{(1+|w|_o^2)^{\frac{2}{3}}}(\langle v_1,v_2\rangle_o + \langle v_1,v_2\rangle_o|w|_o^2 - 0)\\ &= (1+|w|_o^2)^{\frac{1}{3}}\langle v_1,v_2\rangle_o.\end{aligned}$$

Thus in the directions perpendicular to the vector field w, the distances are *stretched* by a factor of $(1+|w|_o^2)^{1/3}$. Therefore this new metric is expanded in the 6 directions

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perpendicular to w and is compressed in the direction parallel to w. Of course, the situation is more complicated if neither v_1 nor v_2 is parallel or perpendicular to w. This produces a *tubular* manifold. For example in the case of $M = N \times S^1$, where N is a Calabi–Yau 3-fold and the metric on M is the product metric, if we take $w = \frac{\partial}{\partial \theta}$ where θ is a coordinate on S^1, then the Calabi–Yau manifold N is expanded and the circle factor S^1 is compressed under $\varphi_o \mapsto \varphi_o + w \lrcorner *_o \varphi_o$. By replacing w by tw and letting $t \to \infty$, we can make this "tube" as long and thin as we want. The total volume, however, always increases by $(1 + |w|_o^2)^{2/3}$ by Proposition 3.2.2.

In general, determining the class of G_2-structure that $\tilde{\varphi}$ belongs to for $\tilde{\varphi} = \varphi_o + w \lrcorner \varphi_o$ involves some very complicated differential equations on the vector field w. However, since $\tilde{\varphi}$ is *always* a positive 3-form for any w, it may be interesting to study some of these differential equations in the simplest cases to determine if one can choose w to produce a $\tilde{\varphi}$ in a strictly smaller subclass. From Theorem 3.2.5, we have

$$
\begin{aligned}
\tilde{*}\tilde{\varphi} &= (1 + |w|_o^2)^{-\frac{1}{3}} \big(*_o \tilde{\varphi} + w \lrcorner \big(*_o (w \lrcorner \tilde{\varphi}) \big) \big) \\
&= (1 + |w|_o^2)^{-\frac{1}{3}} \big(*_o \varphi_o + *_o (w \lrcorner *_o \varphi_o) + w \lrcorner *_o (w \lrcorner \varphi_o) \big) .
\end{aligned} \tag{3.17}
$$

For example this transformation will yield a manifold of holonomy G_2 if w satisfies the system

$$
\begin{aligned}
0 &= d(\varphi_o + w \lrcorner *_o \varphi_o) \\
0 &= d\Big((1 + |w|_o^2)^{-\frac{1}{3}} \big(*_o \varphi_o + *_o (w \lrcorner *_o \varphi_o) + w \lrcorner *_o (w \lrcorner \varphi_o) \big) \Big) .
\end{aligned}
$$

The ellipticity and other properties of this system under certain hypotheses is currently being investigated [24].

3.3 Infinitesmal Deformations in the $\bigwedge^3_7$ Direction

Since the decomposition of the space of forms corresponding to the G_2-structure φ *changes* when we add something in $\bigwedge^3_7$, we consider a one-parameter family φ_t of G_2-structures satisfying

$$
\frac{\partial}{\partial t} \varphi_t = w \lrcorner *_t \varphi_t \tag{3.18}
$$

for a fixed vector field w. That is, at each time t, we move in the direction $w \lrcorner *_t \varphi_t$ which is a 3-form in $\bigwedge^3_{7_t}$, the decomposition depending on t. Since the Hodge star $*_t$ is also changing in time, this is *a priori* a nonlinear equation. However, our first observation is that this is in fact not the case:

Proposition 3.3.1 *Under the flow described by equation* (3.18)*, the metric g does not change. Hence the volume form and Hodge star are also constant.*

Proof From (2.14) which gives the metric from the 3-form, we have:

$$
g_t(u, v) \operatorname{vol}_t = \frac{1}{6} (u \lrcorner \varphi_t) \wedge (v \lrcorner \varphi_t) \wedge \varphi_t .
$$

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Differentiating with respect to t, and using the differential equation (3.18),

$$6\frac{\partial}{\partial t}\big(g_t(u,v)\,\mathrm{vol}_t\big) = (u \lrcorner w \lrcorner *_t \varphi_t) \wedge (v \lrcorner \varphi_t) \wedge \varphi_t + (u \lrcorner \varphi_t) \wedge (v \lrcorner w \lrcorner *_t \varphi_t) \wedge \varphi_t$$
$$+ (u \lrcorner \varphi_t) \wedge (v \lrcorner \varphi_t) \wedge (w \lrcorner *_t \varphi_t).$$

Now from the proof of Lemma 3.2.1 (the linear term) we see that this expression is zero, by polarizing. From this it follows easily by taking determinants that vol_t is constant and thus so is g_t and $*_t$. ■

Therefore we can replace $*_t$ by $*_0 = *$ and equation (3.18) is actually *linear*. Moreover, the flow determined by this linear equation gives a one-parameter family of G_2-structures each yielding the *same* metric g. Our equation is now

$$\frac{\partial}{\partial t}\varphi_t = w \lrcorner * \varphi_t = A\varphi_t$$

where A is the linear operator $\alpha \mapsto A\alpha = w \lrcorner * \alpha$ on $\bigwedge^3$.

Proposition 3.3.2 *The operator A is skew-symmetric. Further, the eigenvalues of A are $\lambda = 0$ with multiplicity 21, and $\lambda = \pm i|w|$ each with multiplicity 7.*

Proof Let $e^1, e^2, \dots, e^{35}$ be a basis of $\bigwedge^3$. Then

$$A_{ij}\,\mathrm{vol} = \langle e^i, Ae^j\rangle\,\mathrm{vol} = e^i \wedge *(w \lrcorner * e^j)$$
$$= -e^i \wedge w^\flat \wedge e^j = w^\flat \wedge e^i \wedge e^j = -A_{ji}\,\mathrm{vol},$$

since 3-forms anti-commute. Therefore A is diagonalizable over $\mathbb{C}$. Suppose now that $\alpha \in \bigwedge^3$ is an eigenvector with eigenvalue $\lambda = 0$. Then

$$A\alpha = w \lrcorner * \alpha = - * (w^\flat \wedge \alpha) = 0,$$

so $w^\flat \wedge \alpha = 0$ and hence $\alpha = w^\flat \wedge \beta$ for some $\beta \in \bigwedge^2$. Therefore the multiplicity of $\lambda = 0$ is $\dim(\bigwedge^2) = 21$. If $A\alpha = \lambda\alpha$ for $\lambda \neq 0$, then $\alpha = \frac{1}{\lambda}(w \lrcorner * \alpha)$ and $w \lrcorner \alpha = 0$. Then we can write (A.10) as

$$|w|^2\alpha = -w \lrcorner * (w \lrcorner * \alpha) = -A^2\alpha = -\lambda^2\alpha,$$

and hence $\lambda = \pm i|w|$. Since the eigenvalues come in complex conjugate pairs and there are $35 - 21 = 14$ remaining, there must be 7 of each. ■

Now if α is an eigenvector for $\frac{\partial}{\partial t}\alpha_t = A\alpha_t = \lambda\alpha_t$, then $\alpha(t) = e^{\lambda t}\alpha(0)$. Let $u_1, u_2, \dots, u_{21}$ be a basis for the $\lambda = 0$ eigenspace, and $v_1, \dots, v_7$ and $\bar{v}_1, \dots, \bar{v}_7$

be bases of *complex* eigenvectors corresponding to the $\lambda = +i|w|$ and $\lambda = -i|w|$ eigenspaces, respectively. We can write

$$\phi_0 = \sum_{k=1}^{7} c_k v_k + \sum_{k=1}^{7} \bar{c}_k \bar{v}_k + \sum_{k=1}^{21} h_k u_k = \sum_{k=1}^{7} c_k v_k + \sum_{k=1}^{7} \bar{c}_k \bar{v}_k + \eta_0$$

where η_0 as defined by the above equation is the part of φ_0 in the kernel of A. Then the solution is given by

$$\begin{aligned} \varphi_t &= \sum_{k=1}^{7} c_k e^{i|w|t} v_k + \sum_{k=1}^{7} \bar{c}_k e^{-i|w|t} \bar{v}_k + \eta_0 \\ &= \cos(|w|t) \sum_{k=1}^{7} (c_k v_k + \bar{c}_k \bar{v}_k) + \sin(|w|t) \sum_{k=1}^{7} i(c_k v_k - \bar{c}_k \bar{v}_k) + \eta_0 \\ &= \cos(|w|t)\beta_0 + \sin(|w|t)\gamma_0 + \eta_0. \end{aligned} \tag{3.19}$$

All that remains is to determine β_0, γ_0, and η_0 in terms of the initial condition φ_0. Substituting $t = 0$ into (3.19), we have

$$\varphi_0 = \beta_0 + \eta_0.$$

Differentiating, we have

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_t &= -|w| \sin(|w|t)\beta_0 + |w| \cos(|w|t)\gamma_0, \\ A\varphi_t &= \cos(|w|t)A\beta_0 + \sin(|w|t)A\gamma_0 + A\eta_0. \end{aligned}$$

Comparing coefficients, we have

$$A\beta_0 = |w|\gamma_0, \quad A\gamma_0 = -|w|\beta_0, \quad A\eta_0 = 0.$$

From $\beta_0 = \varphi_0 - \eta_0$ and the equations above, we get $\gamma_0 = |w|^{-1} A\varphi_0$ and substituting this into the second equation, we obtain $\beta_0 = -|w|^{-2} A^2 \varphi_0$. Finally, we have:

Theorem 3.3.3 *The solution to the differential equation*

$$\frac{\partial}{\partial t}\varphi_t = w \lrcorner *_t \varphi_t$$

is given by

$$\varphi(t) = \varphi_0 + \frac{1 - \cos(|w|t)}{|w|^2} \big(w \lrcorner * (w \lrcorner * \varphi_0)\big) + \frac{\sin(|w|t)}{|w|} (w \lrcorner * \varphi_0). \tag{3.20}$$

The solution exists for all time and is closed curve in $\bigwedge^3$. Also, the path only depends on $\pm \frac{w}{|w|}$, and the norm $|w|$ only affects the speed of travel along this curve.

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Proof This is all immediate from the above discussion. ∎

Remark 3.3.4 In [4], it is shown that the set of G_2-structures on M which correspond to the same metric as that of a fixed G_2-structure φ_o is an $\mathbb{RP}^7$-bundle over the manifold M. The above theorem gives an explicit formula (3.20) for a path of G_2-structures all corresponding to the same metric g starting from an arbitrary vector field w on M.

Remark 3.3.5 This can also be compared to the Kähler case. Since the metric and the almost complex structure J are independent in this case, for a fixed metric g, the family of 2-forms $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ for varying J's are all Kähler forms corresponding to the same metric.

Remark 3.3.6 Even though the metric is unchanged under an infinitesmal deformation in the $\bigwedge^3_7$ direction, the class of G_2-structure *can* change. Therefore simply knowing that a metric on a 7-manifold arises from a G_2-structure and knowing the metric explicitly does not determine the class.

Remark 3.3.7 We can more generally consider the equation

$$\frac{\partial}{\partial t}\varphi_t = w_t \lrcorner *_t \varphi_t$$

where the vector field w_t now itself depends on the parameter t. Retracing the above steps, we find that the general solution in this case is of the form

$$\varphi(t) = \varphi_0 + \frac{1-\cos\big(f(t)\big)}{|w_t|^2}\big(w_t \lrcorner * (w_t \lrcorner * \varphi_0)\big) + \frac{\sin\big(f(t)\big)}{|w_t|}(w_t \lrcorner * \varphi_0)$$

where the function $f(t)$ is given by

$$f(t) = \int_0^t |w_s|\, ds.$$

We now apply this theorem to an example, where we reproduce known results.

Example 3.3.8 Let N be a Calabi–Yau threefold, with Kähler form ω and holomorphic $(3,0)$ form Ω. The complex coordinates will be denoted by $z^j = x^j + iy^j$. Then there is a natural G_2-structure φ on the product $N \times S^1$ given by

$$\varphi = \operatorname{Re}(\Omega) + d\theta \wedge \omega, \tag{3.21}$$

where θ is the coordinate on the circle S^1. This induces the product metric on $N \times S^1$, with the flat metric on S^1. With the orientation on $N \times S^1$ given by $(x^1, x^2, x^3, \theta, y^1, y^2, y^3)$, it is easy to check that

$$*\varphi = -d\theta \wedge \operatorname{Im}(\Omega) + \frac{\omega^2}{2}.$$

Now let $w = \frac{\partial}{\partial\theta}$ be a globally defined non-vanishing vector field on S^1 with $|w| = 1$. Then we have

$$w \lrcorner * \varphi = -\operatorname{Im}(\Omega),$$
$$*(w \lrcorner * \varphi) = -d\theta \wedge \operatorname{Re}(\Omega),$$
$$w \lrcorner * (w \lrcorner * \varphi) = -\operatorname{Re}(\Omega).$$

Thus for this choice of vector field w, the flow in (3.20) is given by

$$\varphi_t = \operatorname{Re}(\Omega) + d\theta \wedge \omega - \big(1 - \cos(t)\big) \operatorname{Re}(\Omega) - \sin(t) \operatorname{Im}(\Omega) = \operatorname{Re}(e^{it}\Omega) + d\theta \wedge \omega,$$

which is the canonical G_2 form on $N \times S^1$ with the Calabi–Yau structure on N given by $e^{it}\Omega$ and ω. It is well known that we can change the holomorphic volume form Ω by a phase and preserve the Ricci-flat metric. Here it arises naturally using the flow described by (3.20) and the canonical vector field $w = \frac{\partial}{\partial\theta}$.

4 Manifolds With a Spin(7)-Structure

4.1 Spin(7)-Structures

Let M be an oriented 8-manifold with a global 3-fold cross product structure. Such a structure will henceforth be called a Spin(7)*-structure*. Its existence is also given by topological conditions (see [12, 22, 25] for details). Similarly to the G_2 case, this cross product $X(\cdot,\cdot,\cdot)$ gives rise to an associated *Riemannian metric* g and an alternating 4-form Φ which are related by:

$$\Phi(a, b, c, d) = g\big(X(a, b, c), d\big). \tag{4.1}$$

As in the G_2 case, the metric and the cross product structure cannot be prescribed independently. We will see in Section 4.3 how the 4-form Φ determines the metric $g(\cdot\,,\cdot)$. For a Spin(7)-structure Φ, near a point $p \in M$ we can choose local coordinates $x^0, x^1, \dots, x^7$ so that *at the point* p, we have:

$$\begin{aligned}\Phi_p = {} & dx^{0123} - dx^{0167} - dx^{0527} - dx^{0563} + dx^{0415} + dx^{0426} + dx^{0437} \\ & + dx^{4567} - dx^{4523} - dx^{4163} - dx^{4127} + dx^{2637} + dx^{1537} + dx^{1526}\end{aligned} \tag{4.2}$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$. In these coordinates the metric at p is the standard Euclidean metric $g_p = \sum_{k=1}^{8} dx^k \otimes dx^k$ and $*\Phi = \Phi$, so Φ is *self-dual*.

Remark 4.1.1 As in the G_2 case, other conventions for (4.2) appear in the literature. With some conventions, the 4-form Φ is anti-self-dual.

The 4-forms that arise from a Spin(7)-structure are called *positive* or *non-degenerate*, and this set is denoted $\bigwedge^4_{\rm pos}$. The subgroup of SO(8) that preserves Φ_p is Spin(7). (see [3].) Hence at each point p, the set of Spin(7)-structures at p is isomorphic to $\mathrm{GL}(8,\mathbb{R})/\operatorname{Spin}(7)$, which is $64 - 21 = 43$ dimensional. This time, however, in contrast to the G_2 case, since $\bigwedge^4(\mathbb{R}^8)$ is 70 dimensional, the set $\bigwedge^4_{\rm pos}(p)$ of positive 4-forms at p is *not* an open subset of $\bigwedge^4_p$. One of the consequences of this is that the analogous non-infinitesimal deformation in the Spin(7) case will *not* work. This is discussed in Section 5.2.

4.2 Decomposition of $\bigwedge^*(M)$ Into Irreducible Spin(7)-Representations

There is an action of the group Spin(7) on $\mathbb{R}^8$, and hence on the spaces $\bigwedge^*$ of differential forms on M. We can decompose each space $\bigwedge^k$ into irreducible Spin(7)-representations [7, 22, 25]. The results of this decomposition are presented below. As before, the notation $\bigwedge^k_l$ refers to an l-dimensional irreducible Spin(7)-representation which is a subspace of $\bigwedge^k$, w is a vector field on M and vol is the volume form.

$$\bigwedge^0_1 = \{f \in C^\infty(M)\}, \qquad \bigwedge^1_8 = \left\{\alpha \in \Gamma\Big(\bigwedge^1(M)\Big)\right\},$$

$$\bigwedge^2 = \bigwedge^2_7 \oplus \bigwedge^2_{21}, \qquad \bigwedge^3 = \bigwedge^3_8 \oplus \bigwedge^3_{48},$$

$$\bigwedge^4 = \bigwedge^4_1 \oplus \bigwedge^4_7 \oplus \bigwedge^4_{27} \oplus \bigwedge^4_{35},$$

$$\bigwedge^5 = \bigwedge^5_8 \oplus \bigwedge^5_{48}, \qquad \bigwedge^6 = \bigwedge^6_7 \oplus \bigwedge^6_{21},$$

$$\bigwedge^7_8 = \{w \lrcorner \operatorname{vol}\}, \qquad \bigwedge^8_1 = \{f \operatorname{vol}; f \in C^\infty(M)\}.$$

This decomposition respects the *Hodge star* $*$ operator since $\operatorname{Spin}(7) \subset SO(8)$, so $*\bigwedge^k_l = \bigwedge^{8-k}_l$. Taking wedge product with Φ is either zero or an isomorphism onto its image on each irreducible summand.

Proposition 4.2.1 *If α is a* 1*-form, we have the following identities:*

$$*\big(\Phi \wedge *(\Phi \wedge \alpha)\big) = -7\alpha, \tag{4.3}$$

$$|\Phi \wedge \alpha|^2 = 7|\alpha|^2. \tag{4.4}$$

Proof This can be easily checked pointwise using local coordinates and (4.2). ■

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We now explicitly describe the decomposition of the space of forms for $k = 2, 3, 4$.

$$\textstyle\bigwedge^2_7 = \left\{ \beta \in \bigwedge^2; *(\Phi \wedge \beta) = 3\beta \right\}, \tag{4.5}$$

$$\textstyle\bigwedge^2_{21} = \left\{ \beta \in \bigwedge^2; *(\Phi \wedge \beta) = -\beta \right\}, \tag{4.6}$$

$$\textstyle\bigwedge^3_8 = \left\{ *(\Phi \wedge \alpha); \alpha \in \bigwedge^1_8 \right\} = \left\{ w \lrcorner \Phi; w \in \Gamma\big(T(M)\big) \right\}, \tag{4.7}$$

$$\textstyle\bigwedge^3_{48} = \left\{ \eta \in \bigwedge^3; \Phi \wedge \eta = 0 \right\}, \tag{4.8}$$

$$\textstyle\bigwedge^4_1 = \{ f\Phi; f \in C^\infty(M) \}, \tag{4.9}$$

$$\textstyle\bigwedge^4_7 = \left\{ \beta_i^j e^i \wedge (e_j \lrcorner \Phi) - \beta_j^i e^j \wedge (e_i \lrcorner \Phi); \beta_{ij} e^i \wedge e^j \in \bigwedge^2_7 \right\}, \tag{4.10}$$

$$\textstyle\bigwedge^4_{27} = \left\{ \sigma \in \bigwedge^4; *\sigma = \sigma, \sigma \wedge \Phi = 0, \sigma \wedge \tau = 0 \, \forall \tau \in \bigwedge^4_7 \right\}, \tag{4.11}$$

$$\textstyle\bigwedge^4_{35} = \left\{ \sigma \in \bigwedge^4; *\sigma = -\sigma \right\}. \tag{4.12}$$

4.3 The Metric of a Spin(7)-Structure

Here the situation differs significantly from the G_2 case. Because Φ is self-dual equation (4.3) gives us only one useful identity rather than the four identities in equations (2.4)–(2.6). In particular it was equation (2.6) which enabled us to prove Proposition 2.3.1 to obtain a formula for the metric from the 3-form φ in the G_2 case.

The prescription for obtaining the metric from the 4-form Φ in the Spin(7) case is much more complicated. Before we can do this, we need to collect some facts about various 2-forms which can be constructed from pairs of vector fields, as these facts will be used both to determine the metric and later to analyze how it changes under a $\bigwedge^4_7$ deformation in Section 5.2.

Proposition 4.3.1 *Let a, b, c, and d be vector fields. Define the 2-forms $\beta = a^\flat \wedge b^\flat = \beta_7 + \beta_{21}$ and $\mu = c^\flat \wedge d^\flat = \mu_7 + \mu_{21}$. Then we can construct other 2-forms $a \lrcorner b \lrcorner \Phi$ and $*\big((a \lrcorner \Phi) \wedge (b \lrcorner \Phi)\big)$ from a and b, and these are related to β by*

$$a \lrcorner b \lrcorner \Phi = -3\beta_7 + \beta_{21}, \tag{4.13}$$

$$*\big((a \lrcorner \Phi) \wedge (b \lrcorner \Phi)\big) = 2\beta_7 - 6\beta_{21}. \tag{4.14}$$

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Furthermore, if we define

$$A = \langle a \wedge b, c \wedge d\rangle = \langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle b, c\rangle, \tag{4.15}$$

$$B = \Phi(a, b, c, d) \tag{4.16}$$

then the following relations hold between these 2*-forms:*

$$(a \lrcorner b \lrcorner \Phi) \wedge (c^\flat \wedge d^\flat) \wedge \Phi = (-3A - 2B)\,\mathrm{vol}, \tag{4.17}$$

$$(a^\flat \wedge b^\flat) \wedge (c \lrcorner \Phi) \wedge (d \lrcorner \Phi) = (-4A + 2B)\,\mathrm{vol}, \tag{4.18}$$

$$(a \lrcorner b \lrcorner \Phi) \wedge (c \lrcorner d \lrcorner \Phi) \wedge \Phi = (6A + 7B)\,\mathrm{vol}\,. \tag{4.19}$$

Proof Let $\beta = a^\flat \wedge b^\flat = \beta_7 + \beta_{21}$ using the decompositions in (4.5) and (4.6). From Lemma A we can write

$$\begin{aligned} a \lrcorner b \lrcorner \Phi &= *\big(a^\flat \wedge *(b \lrcorner \Phi)\big) \\ &= - * (a^\flat \wedge b^\flat \wedge \Phi) = -3\beta_7 + \beta_{21}, \end{aligned}$$

where we have used the self-duality $*\Phi = \Phi$ and the characterizations of $\bigwedge^2_7$ and $\bigwedge^2_{21}$. Now since $\Phi \wedge \Phi = 14\,\mathrm{vol}$, we have

$$(w \lrcorner \Phi) \wedge \Phi = 7 w \lrcorner\, \mathrm{vol} = 7 * w^\flat,$$

where we have used (A.6). Taking the interior product on both sides with v,

$$\begin{aligned} (v \lrcorner w \lrcorner \Phi) \wedge \Phi - (w \lrcorner \Phi) \wedge (v \lrcorner \Phi) &= 7 v \lrcorner * w^\flat \\ &= -7 * (v^\flat \wedge w^\flat), \\ (-3\beta_7 + \beta_{21}) \wedge \Phi + (v \lrcorner \Phi) \wedge (w \lrcorner \Phi) &= -7 * \beta_7 - 7 * \beta_{21}, \\ -9 * \beta_7 - *\beta_{21} + (v \lrcorner \Phi) \wedge (w \lrcorner \Phi) &= -7 * \beta_7 - 7 * \beta_{21}, \end{aligned}$$

which can be rearranged to give (4.14). We also have

$$\begin{aligned} B\,\mathrm{vol} = \Phi(a, b, c, d)\,\mathrm{vol} &= a^\flat \wedge b^\flat \wedge c^\flat \wedge d^\flat \wedge \Phi \\ &= (\beta_7 + \beta_{21}) \wedge (3 * \mu_7 - *\mu_{21}) \\ &= (3\langle \beta_7, \mu_7\rangle - \langle \beta_{21}, \mu_{21}\rangle)\,\mathrm{vol}, \end{aligned}$$

and

$$A = \langle \beta, \mu\rangle = \langle \beta_7, \mu_7\rangle + \langle \beta_{21}, \mu_{21}\rangle,$$

which together give that

$$\langle \beta_7, \mu_7\rangle = \frac{A + B}{4}, \quad \langle \beta_{21}, \mu_{21}\rangle = \frac{3A - B}{4}.$$

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Hence, for example

$$\begin{aligned}(a\lrcorner b\lrcorner\Phi)\wedge(c\lrcorner d\lrcorner\Phi)\wedge\Phi &= (-3\beta_7+\beta_{21})\wedge(-9*\mu_7 - *\mu_{21})\\ &= 27\Big(\frac{A+B}{4}\Big)\,\mathrm{vol} - \Big(\frac{3A-B}{4}\Big)\,\mathrm{vol}\\ &= (6A+7B)\,\mathrm{vol},\end{aligned}$$

which is (4.19). The other two are obtained similarly. ∎

Proposition 4.3.1 immediately yields the following corollary, which is analogous to Proposition 2.3.1 in the G_2 case.

Corollary 4.3.2 *The following identity holds for v and w vector fields:*

$$(v\lrcorner w\lrcorner\Phi)\wedge(v\lrcorner w\lrcorner\Phi)\wedge\Phi = 6|v\wedge w|^2\,\mathrm{vol}\,. \tag{4.20}$$

Proof This follows from (4.19). ∎

If we polarize (4.20) in w, we obtain the useful equation:

$$\begin{aligned}(v\lrcorner w_1\lrcorner\Phi)\wedge(v\lrcorner w_2\lrcorner\Phi)\wedge\Phi &= 6\langle v\wedge w_1, v\wedge w_2\rangle\,\mathrm{vol}\\ &= 6(|v|^2\langle w_1,w_2\rangle - \langle v,w_1\rangle\langle v,w_2\rangle)\,\mathrm{vol}\,.\end{aligned} \tag{4.21}$$

We now derive the expression for the metric in terms of the 4-form Φ.

Theorem 4.3.3 *Let v be a non-zero tangent vector at a point p and let $e_0, e_1, \dots, e_7$ be any oriented basis for T_pM, so that* $\mathrm{vol}(e_0, e_1, \dots, e_7) > 0$. *Assume without loss of generality that $v^0 \neq 0$. Then the length $|v|$ of v is given by*

$$|v|^4 = \frac{(7)^3}{(6)^{\frac{7}{3}}}\frac{\Big(\det\Big(\big((e_i\lrcorner v\lrcorner\Phi)\wedge(e_j\lrcorner v\lrcorner\Phi)\wedge(v\lrcorner\Phi)\big)(e_1,e_2,\dots,e_7)\Big)\Big)^{\frac{1}{3}}}{\Big(((v\lrcorner\Phi)\wedge\Phi)(e_1,e_2,\dots,e_7)\Big)^3}.$$

Proof We work in local coordinates at the point p. In this notation $g_{ij} = \langle e_i, e_j\rangle$ with $0 \le i, j \le 7$. Let $\det_8(g)$ denote the 8×8 determinant of (g_{ij}) and let $\det_7(g)$ denote the 7×7 determinant of the submatrix where $1 \le i, j \le 7$. Using the fact that $\Phi^2 = 14\,\mathrm{vol} = 14\sqrt{\det_8(g)}e^0\wedge e^1\cdots\wedge e^7$, and writing $v = v^k e_k$, we compute

$$\begin{aligned}A(v) &= \big((v\lrcorner\Phi)\wedge\Phi\big)(e_1,e_2,\dots,e_7)\\ &= 7v^0\sqrt{\det_8(g)}.\end{aligned} \tag{4.22}$$

Now $\langle v, e_j\rangle = v^k g_{kj} = v_j$. We also have the 7×7 matrix (for $1 \le i, j \le 7$)

$$\begin{aligned}B_{ij}(v) &= \big((e_i\lrcorner v\lrcorner\Phi)\wedge(e_j\lrcorner v\lrcorner\Phi)\wedge(v\lrcorner\Phi)\big)(e_1,e_2,\dots,e_7)\\ &= 6(|v|^2 g_{ij} - v_i v_j)v^0\sqrt{\det_8(g)},\end{aligned} \tag{4.23}$$

where we have used Corollary 4.3.2. Now consider the 7×7 matrix $(|v|^2 g_{ij} - v_i v_j)$. By examining the proof of Lemma A.4, we see that its determinant is

$$|v|^{14} \det_7(g) - |v|^{12} |v^\flat \wedge e^0|^2 \det_8(g). \tag{4.24}$$

Now from Cramer's rule $\det_7(g) = g^{00} \det_8(g)$ and we also have $|v^\flat \wedge e^0|^2 = |v|^2 g^{00} - v^0 v^0$. Hence (4.24) becomes

$$|v|^{12} v^0 v^0 \det_8(g).$$

Returning to (4.23), we have now shown that

$$\begin{aligned} \det B_{ij}(v) &= 6^7 |v|^{12} (v^0)^2 \det_8(g) (v^0)^7 \big(\det_8(g)\big)^{\frac{7}{2}} \\ &= 6^7 |v|^{12} (v^0)^9 \big(\det_8(g)\big)^{\frac{9}{2}}, \end{aligned}$$

and hence

$$(\det B_{ij}(v))^{\frac{1}{3}} = 6^{\frac{7}{3}} |v|^4 (v^0)^3 \big(\det_8(g)\big)^{\frac{3}{2}}.$$

Finally, since from (4.22) we have

$$\big(A(v)\big)^3 = (7)^3 (v^0)^3 \big(\det_8(g)\big)^{\frac{3}{2}},$$

these two expressions can be combined to yield

$$|v|^4 = \frac{(7)^3}{(6)^{\frac{7}{3}}} \frac{(\det B_{ij}(v))^{\frac{1}{3}}}{\big(A(v)\big)^3}, \tag{4.25}$$

which completes the proof. ■

4.4 The Triple Cross Product of a Spin(7)-Structure

In this section we describe the triple cross product operation on a manifold with a Spin(7)-structure in terms of the 4-form Φ.

Definition 4.4.1 Let u, v, and w be vector fields on M. The *triple cross product*, denoted $X(u, v, w)$, is a vector field on M whose associated 1-form under the metric isomorphism satisfies:

$$\big(X(u, v, w)\big)^\flat = w \lrcorner v \lrcorner u \lrcorner \Phi. \tag{4.26}$$

This immediately yields the relation between X, Φ, and the metric g:

$$g\big(X(u, v, w), y\big) = \big(X(u, v, w)\big)^\flat (y) = y \lrcorner w \lrcorner v \lrcorner u \lrcorner \Phi = \Phi(u, v, w, y). \tag{4.27}$$

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Analogous to (2.18) we can write

$$\big(X(u,v,w)\big)^{\flat} = w \lrcorner v \lrcorner u \lrcorner \Phi = *(u^{\flat} \wedge v^{\flat} \wedge w^{\flat} \wedge \Phi). \tag{4.28}$$

As in the G_2 case, one can show [23] that

$$|X(u,v,w)|^2 = |u \wedge v \wedge w|^2. \tag{4.29}$$

More identities involving the 3-fold cross product can be found in [23]. In particular, one can show that the following lemma holds.

Lemma 4.4.2 *Let h, u_1, and u_2 be vector fields. Let $\sigma \in \bigwedge^4_7$ be given by*

$$\sigma = v^{\flat} \wedge (w \lrcorner \Phi) - w^{\flat} \wedge (v \lrcorner \Phi)$$

for two other vector fields v and w. Then

$$(h \lrcorner u_1 \lrcorner \Phi) \wedge (h \lrcorner u_2 \lrcorner \Phi) \wedge \sigma = 0.$$

Proof See [23] for a proof. ■

Remark 4.4.3 We can actually show the stronger result that in terms of the decompositions in (4.5) and (4.10), the wedge product map

$$\bigwedge^2_7 \times \bigwedge^2_7 \times \bigwedge^4_7 \to \bigwedge^8_1$$

is the zero map. This is a direct analogy with Theorem 2.4.4. However, we will not have occasion to use this fact.

4.5 The 4 Classes of Spin(7)-Structures

Similar to the classification of G_2-structures by Fernández and Gray in [9], Fernández studied Spin(7)-structures in [7]. In this case, the results are slightly different because a 4-form Φ which determines a Spin(7)-structure is self-dual. Such a manifold has holonomy a subgroup of Spin(7) if and only if $\nabla\Phi = 0$, which Fernández showed to be equivalent to

$$d\Phi = 0.$$

Again this equivalence was established by decomposing the space W that $\nabla\Phi$ belongs to into irreducible Spin(7)-representations, and comparing the invariant subspaces of W to the isomorphic spaces in $\bigwedge^*(M)$. In the Spin(7) case, this space W decomposes as

$$W = W_8 \oplus W_{48},$$

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where again the subscript k denotes the dimension of the irreducible representation W_k. Again in analogy with the G_2 case, we have a canonically defined 7-form ζ and 1-form θ, given by

$$\zeta = *d\Phi \wedge \Phi, \tag{4.30}$$

$$\theta = *\zeta = *(*d\Phi \wedge \Phi). \tag{4.31}$$

Note that $\theta = 0$ when the manifold has holonomy contained in Spin(7), and more generally θ vanishes if $\pi_8(d\Phi) = 0$. We will see below that in the case $\pi_{48}(d\Phi) = 0$ the form θ is closed.

This time we have only 4 classes of Spin(7)-structures: the classes $\{0\}$, W_8, W_{48}, and $W = W_8 \oplus W_{48}$. Table 4.1 describes the classes in terms of differential equations on the form Φ. Unlike the G_2 case, the inclusions between these classes are all strict, and this is discussed in [7].

Class	Defining Equations	Name	$d\theta$
$W_8 \oplus W_{48}$	no relation on $d\Phi$.		
W_8	$d\Phi + \frac{1}{7}\theta \wedge \Phi = 0$	LC Spin(7)	$d\theta = 0$
W_{48}	$\theta = 0$		$\theta = 0$
$\{0\}$	$d\Phi = 0$	Spin(7)	$\theta = 0$

Table 4.1: The 4 classes of Spin(7)-structures

Remark 4.5.1 Note that in the Spin(7) case, there is no analogue of an "integrable" structure, nor are there analogues of *almost* or *nearly* Spin(7)-structure as there are in the G_2 case. An almost Spin(7) manifold ($d\Phi = 0$) automatically has holonomy Spin(7). And $d\Phi$ does not have a one-dimensional component which would give us the analogue of a nearly G_2-structure.

We now prove the closedness of θ in the class W_8 as given in the final column of Table 4.1.

Lemma 4.5.2 *If Φ satisfies $d\Phi + \frac{1}{7}\theta \wedge \Phi = 0$, then $d\theta = 0$.*

Proof Suppose $d\Phi + \frac{1}{7}\theta \wedge \Phi = 0$. We differentiate this equation to obtain:

$$d\theta \wedge \Phi = \theta \wedge d\Phi = \theta \wedge \left(-\frac{1}{7}\theta \wedge \Phi\right) = 0.$$

But wedge product with Φ is an isomorphism from $\bigwedge^2$ to $\bigwedge^6$, so $d\theta = 0$. ∎

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5 Deformations of a Fixed Spin(7)-Structure

We begin with a fixed Spin(7)-structure on a manifold M in a certain class. We will deform the form Φ and see how this affects the class. This time there are only 4 classes, and only two intermediate classes. However, the ways we can deform Φ in the Spin(7) case are more complicated. Since $\Phi \in \bigwedge^4_1 \oplus \bigwedge^4_7 \oplus \bigwedge^4_{27} \oplus \bigwedge^4_{35}$, there are now four canonical ways to deform the 4-form Φ. Again, since $\bigwedge^4_1 = \{f\Phi\}$, adding to Φ an element of $\bigwedge^4_1$ amounts to conformally scaling Φ. This preserves the decomposition into irreducible representations. In all other cases, however, since the decomposition depends on Φ it will change for those deformations.

We will see that analogously to the G_2 case, flowing in the $\bigwedge^4_7$ direction gives us a path in the space of positive 4-forms, all corresponding to the same metric. However, this time simply deforming non-infinitesmally by an element of $\bigwedge^4_7$ will *not* yield a positive 4-form, in fact we can show that it *never* does. We will explain how much of the construction does carry over and give some reasons why it should not be a surprise that discovering an analogous construction in the Spin(7) case that works should be considerably more complicated.

5.1 Conformal Deformations of Spin(7)-Structures

Let f be a smooth, nowhere vanishing function on M. We conformally scale Φ by f^4, for notational convenience. Denote the new form by $\tilde{\Phi} = f^4\Phi_o$. We first compute the new metric $\tilde{g}$ and the new volume form $\mathrm{vol}_\sim$ in the following lemma.

Lemma 5.1.1 *The metric g_o on vector fields, the metric g_o^{-1} on one forms, and the volume form vol_o transform as follows:*

$$\tilde{g} = f^2 g_o, \quad \tilde{g}^{-1} = f^{-2} g_o^{-1}, \quad \mathrm{vol}_\sim = f^8\, \mathrm{vol}_o\,.$$

Proof We substitute $\tilde{\Phi} = f^4\Phi_o$ into equations (4.22) and (4.23) to obtain

$$\tilde{A}(v) = f^8 A_o(v),$$
$$\tilde{B}_{ij}(v) = f^{12}(B_o)_{ij}(v).$$

Substituting these expressions into (4.25) we compute

$$|v|^4_\sim = f^4 |v|^4_o,$$

from which we have $|v|^2_\sim = f^2|v|^2_o$ and the remaining conclusions now follow. ■

We now determine the new Hodge star $\tilde{*}$ in terms of the old $*_o$.

Lemma 5.1.2 *If α is a k-form, then $\tilde{*}\alpha = f^{8-2k} *_o \alpha$.*

Proof This is identical to Corollary 3.1.2. ■

From this we obtain the following:

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Lemma 5.1.3 *The exterior derivatives $d\tilde{\Phi}$ and $\tilde{*}d\tilde{\Phi}$ of the new 4-form are*

$$d\tilde{\Phi} = 4f^3 df \wedge \Phi_o + f^4 d\Phi_o$$
$$\tilde{*}d\tilde{\Phi} = 4f *_o (df \wedge \Phi_o) + f^2 *_o d\Phi_o.$$

Proof This is immediate from $\tilde{\Phi} = f^4\Phi_o$ and Lemma 5.1.2. ■

Using these results, we can determine which classes of Spin(7)-structures are conformally invariant. We can also determine what happens to the 7-form ζ and the associated 1-form $\theta = *\zeta$. This is all given in the following theorem:

Theorem 5.1.4 *Under the conformal deformation $\tilde{\Phi} = f^4\Phi_o$, we have:*

$$d\tilde{\Phi} + \frac{1}{7}\tilde{\theta} \wedge \tilde{\Phi} = f^4\Big(d\Phi_o + \frac{1}{7}\theta_o \wedge \Phi_o\Big), \tag{5.1}$$

$$\tilde{\zeta} = -28f^5 *_o df + f^6\zeta_o, \tag{5.2}$$

$$\tilde{\theta} = -28d\big(\log(f)\big) + \theta_o. \tag{5.3}$$

Hence, we see from Table 4.1 *and equation* (5.1) *that only the class W_8 is preserved under a conformal deformation of Φ. (This part was originally proved in* [7] *using a different method.) Also,* (5.3) *shows that θ changes by an exact form, so in the class W_8, where θ is closed, we have a well defined cohomology class* $[\theta]$ *which is unchanged under a conformal scaling.*

Proof We begin by using Lemma 5.1.3 and (4.30) to compute $\tilde{\zeta}$ and $\tilde{\theta}$:

$$\begin{aligned}\tilde{\zeta} &= \tilde{*}d\tilde{\Phi} \wedge \tilde{\Phi}\\ &= \big(4f *_o (df \wedge \Phi_o) + f^2 *_o d\Phi_o\big) \wedge f^4\Phi_o\\ &= 4f^5\Phi_o \wedge *_o(\Phi_o \wedge df) + f^6\zeta_o\\ &= -28f^5 *_o df + f^6\zeta_o,\end{aligned}$$

where we have used (4.3) in the last step. Now from Lemma 5.1.2, we get:

$$\tilde{\theta} = \tilde{*}\tilde{\zeta} = -28f^{-1}df + \theta_o = -28d\big(\log(f)\big) + \theta_o.$$

Now using the above expression for $\tilde{\theta}$, we have:

$$\begin{aligned}d\tilde{\Phi} + \frac{1}{7}\tilde{\theta} \wedge \tilde{\Phi} &= 4f^3 df \wedge \Phi_o + f^4 d\Phi_o + \frac{1}{7}(-28f^{-1}df + \theta_o) \wedge f^4\Phi_o\\ &= f^4\big(d\Phi_o + \frac{1}{7}\theta_o \wedge \Phi_o\big),\end{aligned}$$

which completes the proof. ■

The next result gives necessary and sufficient conditions for being able to achieve holonomy Spin(7) by conformally scaling.

Theorem 5.1.5 *Let Φ_o be a positive 4-form (associated to a* Spin(7)*-structure). Under the conformal deformation $\tilde{\Phi} = f^4\Phi_o$, the new 4-form $\tilde{\Phi}$ satisfies $d\tilde{\Phi} = 0$ if and only if Φ_o is already at least class W_8 and $28d\log(f) = \theta_o$. Hence in order to have $\tilde{\Phi}$ be closed (and hence correspond to holonomy* Spin(7)*), the original 1-form θ_o has to be exact. In particular if the manifold is simply-connected or more generally $H^1(M) = 0$ then this will always be the case if Φ_o is in the class W_8, since $d\theta_o = 0$.*

Proof From Lemma 5.1.3, for $d\tilde{\varphi} = 0$, we need

$$d\tilde{\Phi} = 4f^3 df \wedge \Phi_o + f^4 d\Phi_o = 0 \quad \Rightarrow \quad d\Phi_o = -4d\log(f) \wedge \Phi_o,$$

which says that $d\Phi_o \in \bigwedge^5_8$ by Proposition 4.2.1. Hence $\pi_{48}(d\Phi_o) = 0$ so Φ_o must be already of class W_8. Then to make $d\tilde{\Phi} = 0$, we need to eliminate the W_8 component, which requires $28d\log(f) = \theta_o$ by Theorem 5.1.4. ■

Remark 5.1.6 Note that if we start with a Spin(7)-structure Φ_o that is already holonomy Spin(7), then Theorem 5.1.4 shows that a conformal scaling by a non-constant f will always generate a non-zero W_8 component.

5.2 Deforming Φ By an Element of $\bigwedge^4_7$

We can continue our analogy with the G_2 case and now try to deform the Spin(7) 4-form Φ by an element of $\bigwedge^4_7$. Using (4.10), one can check that if we start with two vector fields v and w, we can construct a special kind of element $\sigma_7 \in \bigwedge^4_7$ by $\sigma_7 = v^\flat \wedge (w \lrcorner \Phi_o) - w^\flat \wedge (v \lrcorner \Phi_o)$. We will consider this type since at least locally every element in $\bigwedge^4_7$ is a linear combination of elements of this type. Now let $\tilde{\Phi} = \Phi_o + t\big(v^\flat \wedge (w \lrcorner \Phi_o) - w^\flat \wedge (v \lrcorner \Phi_o)\big)$, for $t \in \mathbb{R}$. Using the notation of Theorem 4.3.3, we have the following proposition.

Proposition 5.2.1 *Let $\sigma_7 = \big(v^\flat \wedge (w \lrcorner \Phi_o) - w^\flat \wedge (v \lrcorner \Phi_o)\big)$. Under the transformation $\tilde{\Phi} = \Phi_o + \sigma_7$, we have*

$$\tilde{\Phi}^2 = \left(1 + \frac{4}{7}|v \wedge w|^2_o\right)\Phi_o^2. \tag{5.4}$$

Proof This follows easily from $\Phi \wedge (w \lrcorner \Phi) = 7 * w^\flat$ and (4.18). ■

We continue the computation of the expressions needed to determine if $\tilde{\Phi}$ is indeed a Spin(7)-structure with the following lemma.

Lemma 5.2.2 *With $\tilde{\Phi} = \Phi_o + t\sigma$, in the expression*

$$(e_i \lrcorner u \lrcorner \tilde{\Phi}) \wedge (e_i \lrcorner u \lrcorner \tilde{\Phi}) \wedge \tilde{\Phi},$$

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which is a cubic polynomial in t, the linear and cubic terms both vanish, and the coefficient of the quadratic term is

$$\begin{aligned}&6\big(-\Phi_o(v,w,h,e_i)^2+|v\wedge w\wedge h\wedge e_i|^2-2\langle h\wedge e_i, v\wedge w\rangle\Phi_o(v,w,h,e_i)\big)\ \mathrm{vol}_o\\&\quad+6\big(\langle w\wedge e_i, w\wedge v\rangle\langle h\wedge e_i, h\wedge v\rangle+\langle e_i\wedge h, e_i\wedge v\rangle\langle w\wedge h, w\wedge v\rangle\big)\ \mathrm{vol}_o\\&\quad+6\big(\langle h\wedge e_i, h\wedge w\rangle\langle v\wedge e_i, v\wedge w\rangle+\langle e_i\wedge h, e_i\wedge w\rangle\langle v\wedge h, v\wedge w\rangle\big)\ \mathrm{vol}_o\\&\quad-12\langle h\wedge e_i, v\wedge w\rangle^2\,\mathrm{vol}_o\,.\end{aligned}$$

Proof See [23] for a proof. ■

If we now polarize the expression $(h\lrcorner e_i\lrcorner\tilde{\Phi})\wedge(h\lrcorner e_i\lrcorner\tilde{\Phi})\wedge\tilde{\Phi}$, take the interior product with h, and apply this to a basis extension $e_1, e_2, \dots, e_7$, as required by Theorem 4.3.3, one can check that

$$\begin{aligned}\frac{1}{6}\tilde{B}_{ij}&=|h|_o^2(1+|v\wedge w|_o^2)g_{ij}-\langle w,h\rangle^2 v_iv_j-\langle v,h\rangle^2 w_iw_j\\&\quad+\langle v,h\rangle\langle w,h\rangle(v_iw_j+w_iv_j)-(1+|v\wedge w|_o^2)h_ih_j-X_iX_j\\&\quad-\langle w,h\rangle(v_iX_j+X_iv_j)+\langle v,h\rangle(w_iX_j+X_iw_j),\end{aligned}$$

where X is the vector field $X(v,w,h)$. From this expression the determinant of $\tilde{B}_{ij}$ can be computed as

$$\det(\tilde{B}_{ij})=6^7|h|_o^{12}(1+|v\wedge w|_o^2)^6.$$

Now if this was indeed a Spin(7)-structure then Theorem 4.3.3 would imply that

$$|h|_\sim^4=\frac{(1+|v\wedge w|_o^2)^2}{(1+\frac{4}{7}|v\wedge w|_o^2)^3}|h|_o^4.$$

This would mean the metric changes conformally, but the conformal factor is *not* compatible with what would be the new volume form $\mathrm{vol}_\sim=\frac{1}{14}\tilde{\Phi}^2$ from Proposition 5.2.1. Hence this is *never* a Spin(7)-structure. Note that the construction very closely parallels the G_2 case. Even though the deformation does not yield a Spin(7)-structure, it is nevertheless true that $\det(\tilde{B}_{ij})$ turns out to be a positive definite quadratic form.

Recall now one major difference between the G_2 and Spin(7) cases: the space $\bigwedge^3_{\mathrm{pos}}$ of G_2-structures at a point is an *open subset* of the space $\bigwedge^3$ of 3-forms at that point. In contrast, the space $\bigwedge^4_{\mathrm{pos}}$ of Spin(7)-structures at a point is a 43-dimensional submanifold of the 70-dimensional vector space $\bigwedge^4$ of 4-forms at that point, and this submanifold is not linearly embedded. So we should not expect that moving linearly in $\bigwedge^4$ would keep us on this submanifold, in general (Bryant, personal communication.) In effect, the Spin(7) case is *more non-linear* than the G_2 case. There may still exist, however, some non-linear way of deforming Φ_o by an element of $\bigwedge^4_7$ to obtain a Spin(7)-structure. In Section 6 we present an argument as to why this may be.

5.3 Infinitesmal Deformations in the $\bigwedge^4_7$ Direction

Even though the non-infinitesmal $\bigwedge^4_7$ deformation did not produce a Spin(7)-structure, we will see that in analogy with the G_2 case, we can get a family of Spin(7)-structures all corresponding to the same metric by taking *infinitesmal deformations* in the $\bigwedge^4_7$ direction. Consider a one-parameter family Φ_t of Spin(7)-structures, satisfying

$$\frac{\partial}{\partial t}\Phi_t = w\lrcorner *_t (v\lrcorner\Phi_t) - v\lrcorner *_t (w\lrcorner\Phi_t) \tag{5.5}$$

for a pair of vector fields v and w. That is, at each time t, we move in the direction of a 4-form in $\bigwedge^4_{7_t}$, since the decomposition of $\bigwedge^4$ depends on Φ_t and hence is changing in time. Since the Hodge star $*_t$ is also changing in time, this is again *a priori* a nonlinear equation. However, just like in the G_2 case, it is actually linear:

Proposition 5.3.1 *Under the flow described by equation* (5.5)*, the metric g does not change. Hence the volume form and Hodge star are also constant.*

Proof From Theorem 4.3.3, Proposition 5.2.1, and Lemma 5.2.2 we see that if we expand the expression for $|h|^4$ for some vector field h, as a power series in t, there is no linear term and hence to first order the metric does not change. ∎

Therefore we can replace $*_t$ by $*_0 = *$ and equation (5.5) is actually *linear*. Moreover, the flow determined by this linear equation gives a one-parameter family of Spin(7)-structures each yielding the *same* metric g. Our equation is now

$$\frac{\partial}{\partial t}\Phi_t = w\lrcorner * (v\lrcorner\Phi_t) - v\lrcorner * (w\lrcorner\Phi_t) = B\Phi_t$$

where B is the linear operator $\alpha \mapsto B\alpha = w\lrcorner * (v\lrcorner\alpha) - v * (w\lrcorner\alpha)$ on $\bigwedge^4$.

Proposition 5.3.2 *The operator B is skew-symmetric. Furthermore, the eigenvalues λ of B are $\lambda = 0, \pm i|v \wedge w|$.*

Proof The proof is similar to the G_2 case and can be found in [23]. ∎

Now we proceed exactly as in the G_2 case. If we replace A by B and the non-zero eigenvalues by $\pm i|v \wedge w|$, then all the remaining calculations of Section 3.3 carry through. Therefore we have

$$\Phi_t = -\frac{1}{|v \wedge w|^2}\cos(|v \wedge w|t)B^2\Phi_0 + \frac{1}{|v \wedge w|}\sin(|v \wedge w|t)B\Phi_0 + \Phi_0 + \frac{1}{|v \wedge w|^2}B^2\Phi_0,$$

which we summarize as the following theorem.

Theorem 5.3.3 *The solution to the differential equation*

$$\frac{\partial}{\partial t}\Phi_t = w\lrcorner * (v\lrcorner\Phi_t) - v\lrcorner * (w\lrcorner\Phi_t)$$

is given by

$$\Phi(t) = \Phi_0 + \frac{1-\cos(|v\wedge w|t)}{|v\wedge w|^2}B^2\Phi_0 + \frac{\sin(|v\wedge w|t)}{|v\wedge w|}B\Phi_0, \tag{5.6}$$

where $B\alpha = v\lrcorner(w^\flat\wedge\alpha) - w\lrcorner(v^\flat\wedge\alpha)$. The solution exists for all time and is closed curve in $\bigwedge^4$.

Proof This follows from the above discussion. ■

Remark 5.3.4 In [4], it is shown that the set of Spin(7) s on M which correspond to the same metric as that of a fixed Spin(7)-structure Φ_o is an $O(8)/\operatorname{Spin}(7)$-bundle (which is rank 7) over the manifold M. The above theorem gives an explicit formula (5.6) for a path of Spin(7)-structures all corresponding to the same metric g starting from two vector fields v and w on M.

Remark 5.3.5 Again, even though the metric is unchanged under an infinitesmal deformation in the $\bigwedge^4_7$ direction, the class of Spin(7)-structure *can* change.

We apply this theorem to two examples, where we again reproduce known results.

Example 5.3.6 Let N be a Calabi–Yau fourfold, with Kähler form ω and holomorphic $(4,0)$ form Ω. The complex coordinates will be denoted by $z^j = x^j + iy^j$. Then N has a natural Spin(7)-structure Φ on it given by

$$\Phi = \operatorname{Re}(\Omega) + \frac{\omega^2}{2}. \tag{5.7}$$

It is easy to check in local coordinates that $\omega \in \bigwedge^2_7$ in the Spin(7) decomposition. Since we are computing pointwise, if we take two tangent vectors v and w for which $\pi_7(v^\flat\wedge w^\flat) = \omega$, then one can compute that

$$B\Phi = -\operatorname{Im}(\Omega) \quad \text{and} \quad B^2\Phi = -\operatorname{Re}(\Omega).$$

Thus for the element of $\bigwedge^4_7$ which corresponds to ω, the flow in (5.6) is given by

$$\Phi_t = \operatorname{Re}(\Omega) + \frac{\omega^2}{2} - \big(1-\cos(t)\big)\operatorname{Re}(\Omega) - \sin(t)\operatorname{Im}(\Omega) = \operatorname{Re}(e^{it}\Omega) + \frac{\omega^2}{2},$$

which is the canonical Spin(7) form on N where now the Calabi–Yau structure is given by $e^{it}\Omega$ and ω. Thus we arrive at the phase freedom for Calabi–Yau fourfolds.

Example 5.3.7 Consider a 7-manifold M with a G_2-structure φ. We can put a Spin(7)-structure Φ on the product $M \times S^1$ given by

$$\Phi = d\theta \wedge \varphi + *_7\varphi,$$

where $*_7\varphi$ is the 4-form dual to φ on M. This induces the product metric on $M \times S^1$, with the flat metric on S^1. Now let $v = \frac{\partial}{\partial\theta}$ be a globally defined non-vanishing vector field on S^1 with $|v| = 1$. Choose another vector field w on M. Then one computes

$$B\Phi = d\theta \wedge (w \lrcorner *_7 \varphi) + *_7(w \lrcorner *_7 \varphi),$$

$$B^2\Phi = d\theta \wedge \big(w \lrcorner *_7 (w \lrcorner *_7 \varphi)\big) + *_7\big(w \lrcorner *_7 (w \lrcorner *_7 \varphi)\big).$$

The flow in (5.6) gives

$$\Phi_t = d\theta \wedge \varphi_t + *_7\varphi_t,$$

where φ_t is the flow given by (3.20) for the vector field w. Thus, in the product case $M \times S^1$ we recover the results of Section 3.3.

6 Conclusion

In the construction of Calabi–Yau manifolds, we start from a Kähler manifold and we reduce the holonomy from $U(n)$ to $\mathrm{SU}(n)$, which is a drop of 1 in dimension. Hence it might be expected that it would involve the solution of an equation for one function. In going from SO(7) to G_2 we have a drop of 7 in dimension, so we might expect to need 7 conditions, which could involve an equation for a vector field (or equivalently an element of $\bigwedge^3_7$). Similarly the difference in dimension between SO(8) and Spin(7) is also 7, which could be related to an element $\bigwedge^4_7$.

Note that in the G_2 case, since elements of $\bigwedge^3_7$ are canonically identified with vector fields, they are intrinsic to the manifold without reference to a G_2-structure. For Spin(7)-structures, we need the 4-form Φ to define $\bigwedge^4_7$ and this introduces more non-linearity. To maintain the analogy with the G_2 case, there should be some (non-linear) way of transforming a Spin(7)-structure Φ using an element of $\bigwedge^4_7$ so that we get a new Spin(7)-structure whose new metric is related to the old one by

$$\langle u_1, u_2\rangle_{\sim} = f\big(\langle u_1, u_2\rangle_o + \langle X(v, w, u_1), X(v, w, u_2)\rangle_o\big),$$

where v and w are vector fields which determine the corresponding element of $\bigwedge^4_7$ and f is some positive function of $|v \wedge w|^2$.

A Some Linear Algebra

Here we collect together various identities involving the exterior and interior products and the Hodge star operator. Also we state some identities involving determinants. Proofs for all these identities can be found (for example) in [23]. Let M be a

Riemannian manifold of dimension n. Let $\langle\ ,\ \rangle$ denote the metric, as well as the induced metric on forms. In all that follows, α and γ are k-forms, β is a $(k-1)$-form, w is a vector field, and $w^\flat$ is the 1-form dual to w in the given metric. That is,

$$|w|^2 = \langle w, w\rangle = w^\flat(w) = \langle w^\flat, w^\flat\rangle.$$

Now $*$ takes k-forms to $(n-k)$-forms, and is defined by

$$\langle \alpha, \gamma\rangle \operatorname{vol} = \alpha \wedge *\gamma = \gamma \wedge *\alpha.$$

We also have

$$*^2 = (-1)^{k(n-k)} \tag{A.1}$$

on k-forms.

Lemma A.1 *We have the following four identities:*

$$*(w \lrcorner \alpha) = (-1)^{k+1}(w^\flat \wedge *\alpha), \tag{A.2}$$

$$(w \lrcorner \alpha) = (-1)^{nk+n} * (w^\flat \wedge *\alpha), \tag{A.3}$$

$$*(w \lrcorner * \alpha) = (-1)^{nk+n+1}(w^\flat \wedge \alpha), \tag{A.4}$$

$$(w \lrcorner * \alpha) = (-1)^{k} * (w^\flat \wedge \alpha), \tag{A.5}$$

and when $\alpha = \operatorname{vol}$, *the special case*

$$w \lrcorner \operatorname{vol} = *w^\flat. \tag{A.6}$$

We also have the useful relations

$$(X \lrcorner \alpha) \wedge *\beta = \alpha \wedge *(X^\flat \wedge \beta), \tag{A.7}$$

$$(X \lrcorner \alpha) \wedge \zeta = (-1)^{k+1} \alpha \wedge (X \lrcorner \zeta), \tag{A.8}$$

for any k-form α, $(k-1)$-form β, $(n+1-k)$-form ζ, and vector field X.

The next lemma gives further relations between a k-form α and a vector field w.

Lemma A.2 *With notation as above, we have the following three identities:*

$$|w|^2\alpha = w^\flat \wedge (w \lrcorner \alpha) + w \lrcorner (w^\flat \wedge \alpha), \tag{A.9}$$

$$|w|^2\alpha = (-1)^{nk+1} * \Big(w \lrcorner \big(*(w \lrcorner \alpha)\big)\Big) + (-1)^{nk+n+1}\big(w \lrcorner * (w \lrcorner * \alpha)\big), \tag{A.10}$$

$$|w|^2|\alpha|^2 = |w \lrcorner \alpha|^2 + |w \lrcorner * \alpha|^2. \tag{A.11}$$

The following lemma about determinants is used many times in the computation of the metrics and volume forms arising from G_2 and Spin(7)-structures.

Lemma A.3 *Let g_{ij} be an $n \times n$ matrix, v_i and w_j be two $n \times 1$ vectors, and C, K constants. Consider the matrix*

$$B_{ij} = C g_{ij} + K v_i w_j.$$

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Its determinant is given by

$$\det(B) = C^n \det(g) + \sum_{k,l=1}^{n} (-1)^{k+l} v_k w_l C^{n-1} K G_{kl}, \tag{A.12}$$

where G_{kl} is the (k, l)-th minor of the matrix g_{ij}. That is, it is the determinant of g_{ij} with the k-th row and l-th column removed.

We can obtain a special case of Lemma A.3 when the matrix g_{ij} is a metric. It is used several times in the text, most notably in the derivation of the metric from the 4-form Φ in the $\mathrm{Spin}(7)$ case in Theorem 4.3.3.

Lemma A.4 *Let $g_{ij} = \langle e_i, e_j \rangle$ be a Riemannian metric in local coordinates, and $v^\flat = v_i e^i$ and $w^\flat = w_j e^j$ be two one forms dual to the vector fields v and w. Then if we define*

$$B_{ij} = C g_{ij} + K v_i w_j,$$

we have

$$\det(B) = C^n \det(g) + C^{n-1} K \langle v, w \rangle \det(g).$$

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Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario
email: spiro@math.mcmaster.ca