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DUALITY AND LAGRANGE MULTIPLIERS FOR NONSMOOTH MULTIOBJECTIVE PROGRAMMING

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Without any constraint qualification, the necessary and sufficient optimality conditions are established in this paper for nonsmooth multiobjective programming involving generalised convex functions. With these optimality conditions, a mixed dual model is constructed which unifies two dual models. Several theorems on mixed duality and Lagrange multipliers are established in this paper.

1. INTRODUCTION

Consider the non-smooth multiobjective programming problem (MP):

$$\text{(MP)} \qquad \begin{aligned} \min \quad & \big(f_1(x), f_2(x), \dots, f_q(x)\big) \\ \text{such that} \quad & g_j(x) \leqslant 0, j = 1, \dots, m, \\ & x \in X \subset R^n. \end{aligned}$$

where $f_i, g_j, i = 1, \dots, q, j = 1, \dots, m$ are locally Lipschitz functions on X. In the following, we denote

$$f(x) = \big(f_1(x), \dots, f_q(x)\big), g(x) = \big((g_1(x), \dots, g_m(x)\big).$$

Ben-Israel, Ben-Tal and Zlobec [2] gave a necessary and sufficient condition for a vector to be an optimal solution of a convex programming problem without using a constraint qualification. With this necessary and sufficient condition, Mond and Zlobec [4], Egudo, Weir and Mond [3] defined a dual programming to the (single-objective or multi-objective) convex programming and established duality theory without any constraint qualification.

The convexity conditions were weakened by Weir and Mond [5], and the optimality necessary and sufficient condition for a generalised convex programming problem without the need of any constraint qualification is established in [5]. Also, some duality results

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involving generalised convex were given in [5]. However, all functions in [5] must be smooth.

In [6, 7], we established the necessary and sufficient optimality conditions and mixed duality for the minimax programming and minimax fractional programming without a constraint qualification.

In 1992, Egudo, Weir and Mond[3] established the dual problems for convex and generalised convex multiobjective programs without requiring a constraint qualification.

Recently, Zhou, Zhang and Wang in [8] establish the first-order necessary and sufficient optimality conditions for non-smooth generalised convex programming, which unified the results of Ben-Israel, Ben-Tal and Zlobec [2] and Weir and Mond [5].

The purpose of this paper is to extend the first-order results in [8] to the non-smooth generalised convex multiobjective programming problem where no constraint qualification is needed. Based on these necessary and sufficient optimality conditions, we define a mixed type dual programming of multiobjective programming problem, which unifies the Mond-Weir type dual programming and Wolfe type dual programming. Some theorems of duality and existence of Lagrangian saddle point are established without any constraint qualification.

2. Preliminary Concepts and Results

Throughout this paper, let R^n be n-dimensional Euclidean space, and R^n_+ be its nonnegative orthant.

Let x and y be in R^n, we denote

$$x < y \Leftrightarrow x_i < y_i \text{ for } i = 1, 2, \dots, n.$$
$$x \leqq y \Leftrightarrow x_i \leqq y_i \text{ for } i = 1, 2, \dots, n.$$
$$x \leqslant y \Leftrightarrow x_i \leqq y_i \text{ for } i = 1, 2, \dots, n, \text{ but } x \neq y.$$

For $J = \{1, 2, \dots, m\}$, we denote

$$S = \{x \in R^n \mid g_j(x) \leqslant 0, j = 1, 2, \dots, m\}, \quad Q = \{1, 2, \dots, q\}.$$

A feasible point x^* for a multiobjective programming problem is said to be an efficient solution if there exists no feasible x for the multiobjective programming problem such that $f(x) \leqslant f(x^*)$. A feasible point x^* for a multiobjective programming problem is said to be a weak sufficient solution if there exists no feasible x for the multiobjective programming problem such that $f(x) < f(x^*)$. A feasible point x^* for a multiobjective programming problem is said to be a proper efficient solution if there exists a scalar $M > 0$ such that for each i, we have

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leqq M \tag{2.1}$$

for some j such that $f_j(x) > f_j(x^*)$ whenever x is feasible for the multiobjective programming problem and $f_i(x) < f_i(x^*)$.

For a fixed $r \in Q, x^* \in R^n$, we denote

$$\begin{aligned} Q^r &= Q \backslash \{r\} = \{k \in Q \mid k \neq r\}; \\ F^r(x^*) &= \{x \mid f_i(x) \leqslant f_i(x^*), i \in Q^r\}; \\ Q^{r=}(x^*) &= \{i \in Q^r \mid f_i(x) = f_i(x^*), \forall x \in F^r(x^*)\}; \\ Q^{=}(x^*) &= \bigcup_{i \in Q} Q^{r=}(x^*) = \{i \in Q \mid x \in F^r(x^*) \Rightarrow f_i(x) = f_i(x^*) \text{ for some } r \in Q \text{ holds}\}. \end{aligned}$$

Also we denote

$$\begin{aligned} J &= \{1, 2, \ldots, m\}, \quad J(x) = \{j \in J \mid g_j(x) = 0\}, \\ J^{=} &= \{j \in J\, id g_j(x) = 0, \forall\ x \in S\}, \ S^{=} = \{x \in R^n \mid g_j(x) = 0, j \in J^{=}\}, \\ J^{<}(x) &= J(x) \setminus J^{=} = \{i \in J(x) \mid \exists x_i \in S \text{ such that } g_i(x_i) < 0\}, \\ J^{<} &= J \setminus J^{=} = \bigcup_{x \in S} J^{<}(x) = \{i \in J \mid \exists x_i \in S \text{ such that } g_i(x_i) < 0\}. \end{aligned}$$

First, we give the definitions of quasiconvex, pseudoconvex and regular function.

DEFINITION 2.1: A function $f : S \subseteq R^n \to R$ is said to be quasiconvex if

$$f(\lambda x_1 + (1 - \lambda) x_2) \leqslant \max\{f(x_1), f(x_2)\}, \forall x_1, x_2 \in S, \lambda \in [0, 1]. \tag{2.2}$$

If the strict inequality "<" holds, then the function f is said to be strictly quasiconvex.

DEFINITION 2.2: A function $f : S \subseteq R^n \to R$ is pseudoconvex if for any $x_1, x_2 \in S, 0 \leqslant \lambda \leqslant 1$,

$$f(x_2) < f(x_1) \text{ implies } f(\lambda x_2 + (1 - \lambda) x_1) \leqslant f(x_1) - \lambda \beta(x_1, x_2), \tag{2.3}$$

where $\beta(x_1, x_2) > 0, \beta(x_1, x_2) = o(\|x_1 - x_2\|)$.

If x_1 is fixed, we say that f is pseudoconvex at x_1.

Obviously, if $f : R^n \to R$ is pseudoconvex and regular at x^*, then we have

$$f(x) < f(x^*) \Rightarrow f^0(x^*, x - x^*) < 0,$$

where $f^0(\cdot, \cdot)$ is the Clarke directional derivative defined in Definition 2.3.

DEFINITION 2.3: Let f be locally Lipschitzian around x, the directional derivative $f'(x, d)$ is defined by

$$f'(x, d) = \lim_{t \to 0} \frac{f(x + td) - f(x)}{t};$$

the Clarke directional derivative $f^0(x, d)$ is defined by

$$f^0(x, d) = \lim_{x' \to x,\ t \downarrow 0} \sup \frac{f(x' + td) - f(x')}{t}.$$

If $f^0(x,d) = f'(x,d)$, then f is said to be regular at x with respect to vector $d \in X$. When this holds for all $d \in R^n$, one says that f is regular at x.

Now, we give some notations. Let S be a convex set in R^n, a vector $d \in R^n$ is said to be a feasible direction for S at x^* if there is $T > 0$ such that

$$x^* + td \in S \qquad \text{for all } 0 < t \leqslant T .$$

The set of all feasible directions for S at x^* is denoted by $F(S, x^*)$.

For f: $R^n \to R$ and $x^* \in \operatorname{dom}(f)$, we denote

$$\begin{aligned}
D_f^{<}(x^*) &= \{d \in R^n \mid \exists T > 0 \text{ such that } f(x^* + td) < f(x^*), \forall t \in (0,T)\},\\
D_f^{=}(x^*) &= \{d \in R^n \mid \exists T > 0 \text{ such that } f(x^* + td) = f(x^*), \forall t \in (0,T)\},\\
D_f^{\leqslant}(x^*) &= \{d \in R^n \mid \exists T > 0 \text{ such that } f(x^* + td) \leqslant f(x^*), \forall t \in (0,T)\}.
\end{aligned}$$

Let $\{g_i : i \in J\}$ be a set of functions indexed by a set J, we denote

$$D_i^{<}(x^*) = D_{g_i}^{<}(x^*), D_J^{<}(x^*) = \bigcap_{i \in J} D_i^{<}(x^*), \tag{2.4}$$

$$D_i^{=}(x^*) = D_{g_i}^{=}(x^*), D_J^{=}(x^*) = \bigcap_{i \in J} D_i^{=}(x^*). \tag{2.5}$$

$$D_i^{\leqslant}(x^*) = D_{g_i}^{\leqslant}(x^*), D_J^{\leqslant}(x^*) = \bigcap_{i \in J} D_i^{\leqslant}(x^*). \tag{2.6}$$

Lemma 2.4. ([8]) *If f is pseudoconvex and regular at x^*, then*

(a) $D_f^{<}(x^*) = \{d \mid f^0(x^*, d) < 0\} = \{d \mid \xi^T d < 0, \forall \xi \in \partial f(x^*)\}$.

(b) $D_f^{<}(x^*)$ is a convex cone.

(c) *If $D_f^{<}(x^*) \neq \emptyset$, then*

$$\left(D_f^{<}(x^*)\right)^* = \{\mu\xi \mid \mu \leqslant 0, \xi \in \partial f(x^*)\} = -\overline{\operatorname{cone}}(\partial f(x^*)).$$

For the following single-objective nonlinear programming (P):

$$\begin{aligned}
\min \quad & f(x)\\
\text{such that} \quad & g_i(x) \leqslant 0, i = 1, \dots, m,\\
& x \in R^n,
\end{aligned}$$

where $f, g_i : R^n \to R, i = 1, \dots, m$, are locally Lipschitz functions, Ben-Israel and Mond have given the following lemmas.

Lemma 2.5. ([1]) *If $f, g_i, i \in J$, are continuous functions, then*

$$F(S, x^*) = D_{J(x^*)}^{\leqslant}(x^*).$$

Lemma 2.6. ([1]) *If $f, g_i, i \in J$, are all pseudoconvex, then*

(a) *The set $S^= = \{x \in R^n \mid g_{J^=}(x) = 0\}$ is convex.*

(b) *If x is feasible for (P) and $u \in S^=$, then*

$$(x-u)^T y \geqslant 0 \quad \text{for all} \quad y \in \left[D^{=}_{\overline{J}^=}(u)\right]^*.$$

In the following, we give a new conclusion for the multiobjective programming problem.

Lemma 2.7. *If $f_i, g_j, i \in Q, j \in J$, are all regular pseudoconvex, and x is a feasible solution of the multiobjective programming problem, $u \in S^=$ and $f_{Q^=(u)}(x) = f_{Q^=(u)}(u)$, then*

$$(x-u)^T y \geqslant 0 \quad \forall\, y \in \left[D^{=}_{\overline{Q}^=(u)\cup J^=}(u)\right]^*,$$

where the cone $\left[D^{=}_{\overline{Q}^=(u)\cup J^=}(u)\right]^*$ is the polar cone of $D^{=}_{\overline{Q}^=(u)\cup J^=}(u)$.

Proof: For a fixed u, we denote $h_{Q^=(u)}(x) = f_{Q^=(u)}(x) - f_{Q^=(u)}(u)$. From the pseudoconvexity of f_i, we know that h_i is also pseudoconvex for all $i \in Q^=(u)$. By Lemma 2.6, the set $\{x \in R^n : g_{J^=}(x) = 0\}$ and $\{x \in R^n : f_{Q^=(u)}(x) = f_{Q^=(u)}(u)\}$ are convex sets, so the set

$$\{x \in R^n : g_{J^=}(x) = 0, f_{Q^=(u)}(x) = f_{Q^=(u)}(u)\}$$

is also a convex set.

From the definition of the cone $\left[D^{=}_{\overline{Q}^=(u)\cup J^=}(u)\right]^*$, to prove that $x - u \in D^{=}_{\overline{Q}^=(u)\cup J^=}(u)$, it is enough to show that

$$g_j\big(u + \lambda(x-u)\big) = g_j(u) = 0, \qquad \forall\, \lambda \in (0,1), \forall\, j \in J^=,$$

$$f_i\big(u + \lambda(x-u)\big) = f_i(u) = 0, \qquad \forall\, \lambda \in (0,1), \forall\, i \in Q^=(u).$$

If there exist indexes $j_0 \in J^=$, $i_0 \in Q^=(u)$ and a positive scalar $\lambda_1 \in (0,1)$, such that

$$g_{j_0}\big(u + \lambda_1(x-u)\big) < 0$$

or

$$f_{i_0}\big(u + \lambda_1(x-u)\big) < 0,$$

this contradicts the convexity of the set

$$\{x \in R^n : g_{J^=}(x) = 0, f_{Q^=(u)}(x) = f_{Q^=(u)}(u)\}.$$

By the definition of polar cone, our conclusion holds. ∎

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3. Optimality Necessary Condition for multiobjective programming problem

In this section, we shall establish the necessary and sufficient optimality conditions for the nondifferential multiobjective generalised convex programming problem where no constraint qualification is needed.

For the single-objective nonlinear programming (P), we have established the following first-order necessary and sufficient optimality conditions in [8].

Theorem 3.1. ([8]) *Let x^* be a feasible solution of (P) and*

$$x^* \in \bigcap_{J^<(x^*)} \text{int dom}\,(g_i) \bigcap \text{int dom}\,(f).$$

If $f, g_i, i \in J$, are all pseudoconvex and regular at x^*, then x^* is optimal for (P) if and only if there exist nonnegative scalars $\lambda_i \geqslant 0, i \in J^<(x^*)$ such that

$$0 \in \partial f(x^*) + \sum_{J^<(x^*)} \lambda_i \partial g_i(x^*) - \left[D^{=}_{\overline{J^=}}(x^*)\right]^*.$$

The following result is given in [3].

Lemma 3.2. ([3]) *A feasible point x^* for the multiobjective programming problem is an efficient solution if and only if x^* is optimal for each of the following single-objective programs $(P^r(x^*))$:*

$$\begin{array}{lrl} (P^r(x^*)) & \min & f_r(x) \\ & \text{such that} & f_i(x) \leqslant f_i(x^*), i \in Q^r, \\ & & g_j(x) \leqslant 0, j = 1, \ldots, m, \\ & & x \in X \subset R^n, \end{array}$$

where $r = 1, 2, \ldots, q$.

Now, we give the first-order necessary and sufficient optimality conditions for multiobjective programming problem.

Theorem 3.3. *If $f_i, g_j, i \in Q, j \in J$, are all pseudoconvex, and a feasible point x^* for the multiobjective programming problem is an efficient solution, then there exist vectors $\lambda_i^* > 0, i \in Q, \sum_{i \in Q} \lambda_i^* = 1, \mu_j^* \geqslant 0, j \in J^<(x^*)$, such that*

$$0 \in \sum_{i \in Q} \lambda_i^* \partial f_i(x^*) + \sum_{j \in J^<(x^*)} \mu_j^* \partial g_j(x^*) - \left[D^{=}_{\overline{Q^=(x^*) \cup J^=}}(x^*)\right]^*. \tag{3.1}$$

Conversely, for the feasible point x^* of multiobjective programming problem, if there exist $\lambda_i^* > 0, i \in Q, \sum_{i \in Q} \lambda_i^* = 1, \mu_j^* \geqslant 0, j \in J^<(x^*)$, such that (3.1) holds, the function $\sum_{i \in Q} \lambda_i^* f_i(x)$ is regular pseudoconvex at x^*, and $\sum_{j \in J^<(x^*)} \mu_j^* g_j(x)$ is strictly regular pseudoconvex at x^*, then x^* is an efficient solution of multiobjective programming problem.

PROOF: Suppose that x^* is an efficient solution for the multiobjective programming problem. By Lemma 3.2, x^* is optimal for each of the following single-objective programs $\big(P^r(x^*)\big)$. For each $i \in Q$, applying Theorem 3.1, there exist $\lambda_i^r > 0, \mu_j^r \geqslant 0, j \in J^<(x^*)$, such that

$$0 \in \partial f_r(x^*) + \sum_{i \in Q^r \setminus Q^{r=}(x^*)} \lambda_i^r \partial f_i(x^*) + \sum_{j \in J^<(x^*)} \mu_j^r \partial g_j(x^*) - \big[D_{\overline{Q}^{r=}(x^*) \cup J^=}(x^*)\big]^*.$$

$$r = 1, 2, \ldots, q.$$

Summing the above over $r \in Q$, scaling appropriately, there exist

$$\lambda_i^* > 0, i \in Q, \sum_{i \in Q} \lambda_i^* = 1, \mu_j^* \geqslant 0, j \in J^<(x^*)$$

satisfying (3.1).

Suppose that there exist

$$\lambda_i^* > 0, i \in Q, \sum_{i \in Q} \lambda_i^* = 1, \mu_j^* \geqslant 0, j \in J^<(x^*)$$

such that (3.1) holds, then there exist

$$\xi_i \in \partial f_i(x^*), i \in Q, \zeta_j \in \partial g_j(x^*), j \in J^<(x^*), d \in \big[D_{\overline{Q}^{=}(x^*) \cup J^=}(x^*)\big]^*$$

such that

$$\sum_{i \in Q} \lambda_i^* \xi_i + \sum_{j \in J^<(x^*)} \mu_j^* \zeta_j = d. \tag{3.2}$$

Suppose to the contrary that x^* is an efficient solution for the multiobjective programming problem. By the definition of efficient solution, there exists $u \in S$, such that $f_{i_0}(u) < f_{i_0}(x^*)$ for some $i_0 \in Q$, and for all $i \in Q, i \neq i_0$, we have $f_i(u) \leqslant f_i(x^*)$. So, we have

$$\sum_{i \in Q} \lambda_i^* f_i(u) < \sum_{i \in Q} \lambda_i^* f_i(x^*). \tag{3.3}$$

By Lemma 2.7, we have $(u - x^*)^T d \geqslant 0$. Now, from (3.2), we have

$$(u - x^*)^T \sum_{i \in Q} \lambda_i^* \xi_i \geqslant -(u - x^*)^T \sum_{j \in J^<(x^*)} \mu_j^* \zeta_j.$$

Based on the pseudoconvexity of $\sum_{i \in Q} \lambda_i^* f_i(x)$ and (3.3), we can conclude that

$$(u - x^*)^T \sum_{i \in Q} \lambda_i^* \xi_i < 0.$$

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Hence

$$(u - x^*)^T \sum_{j \in J^<(x^*)} \mu_j^* \zeta_j > 0.$$

Again, by the strict pseudoconvexity of $\sum_{j \in J^<(x^*)} \mu_j^* g_j(x)$, we get

$$\sum_{j \in J^<(x^*)} \mu_j^* g_j(u) > \sum_{j \in J^<(x^*)} \mu_j^* g_j(x^*).$$

However, by the definition of the cone $J^<(x^*)$, we have

$$\sum_{j \in J^<(x^*)} \mu_j^* g_j(u) > 0.$$

This contradicts the feasibility of the point u. Then, x^* is an efficient solution of multiobjective programming problem. ∎

Theorems 3.4 and 3.5 in [**3**] can be deduced from the above result.

4. Mixed-Type Dual Model for multiobjective programming problem

In this section, we shall introduce the mixed type duality for multiobjective programming problem, and prove the weak dual theorem and strong dual theorem without any constraint qualification.

The following dual problem is said to be a mixed-type dual problem:

(XDMP) max $\left((f_1(y) + \mu_{J_1}^T g_{J_1}(y), \dots, f_q(y) + \mu_{J_1}^T g_{J_1}(y)\right)$

(4.1) such that $0 \in \sum_{i \in Q} \lambda_i \partial f_i(y) + \sum_{j=1}^{m} \mu_j \partial g_j(y) - \left[D^{=}_{\overline{Q}^=(y) \cup J^=}(x^*)\right]^*,$

(4.2) $\mu_j g_j(y) \geqslant 0, j \in J_2,$

(4.3) $\lambda_i \geqslant 0, i \in Q, \sum_{i \in Q} \lambda_i = 1.$

(4.4) $\mu_j \geqslant 0, j = 1, 2, \dots, m,$

(4.5) $g_{S^=}(y) = 0,$

where the index set J_1 is a subset of J, $J_2 = J \backslash J_1$.

In the following, let $f_i, i = 1, 2, \dots, q, g_j, j = 1, 2, \dots, m$ be all pseudoconvex, and D the feasible set of (XDMP).

Theorem 4.1. (Weak duality) *Let $x \in S, (y, \lambda, \mu) \in D$. If $f_i + \sum_{j \in J_1} \mu_j g_j, i \in Q$ is strictly pseudoconvex, then the following inequalities*

$$\begin{cases} f_{i_0}(x) < f_{i_0}(y) + \mu_{J_1}^T g_{J_1}(y), & \exists\, i_0 \in Q, \\ f_i(x) \leqslant f_i(y) + \mu_{J_1}^T g_{J_1}(y), & \forall i \in Q, i \neq i_0, \end{cases}$$

do not hold.

PROOF: Since $(y, \lambda, \mu) \in D$, we have

$$0 \in \sum_{i \in Q} \lambda_i \partial f_i(y) + \sum_{j=1}^{m} \mu_j \partial g_j(y) - \left[D^{=}_{Q^{=}(y) \cup J^{=}}(x^*)\right]^*,$$

that is, there exist $\xi_i \in \partial f_i(y), i \in Q, \zeta_j \in \partial g_j(y), j = 1, 2, \ldots, m, d \in \left[D^{=}_{Q^{=}(y) \cup J^{=}}(y)\right]^*$ satisfying

$$\sum_{i \in Q} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j = d. \tag{4.6}$$

By Lemma 2.7, we have

$$(x - y)^T \left[\sum_{i \in Q} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j\right] \geqslant 0. \tag{4.7}$$

Suppose that there exist $x \in S$ and$(y, \lambda, \mu) \in D$ satisfying above inequalities. By the strict pseudoconvexity of $f_i + \sum_{j \in J_1} \mu_j g_j, i = 1, 2, \ldots, q$ and regularity of each function, we have

$$(x - y)^T \left[\xi_{i_0} + \sum_{j \in J_1} \mu_j \zeta_j\right] < 0,$$

$$(x - y)^T \left[\xi_i + \sum_{j \in J_1} \mu_j \zeta_j\right] \leqslant 0, i = 1, 2, \ldots, q, i \neq i_0.$$

Multiplying each of these inequalities by λ_i and summing over $i \in Q$, we have

$$(x - y)^T \left[\sum_{i \in Q} \lambda_i \xi_i + \sum_{j \in J_1} \mu_j \zeta_j\right] < 0. \tag{4.8}$$

Since $x \in S, (y, \lambda,$ and $\mu) \in D$, we have

$$\mu_j g_j(x) - \mu_j g_j(y) \leqslant 0, \quad j \in J_2.$$

Since $\mu_j g_j$ is pseudoconvex, hence quasiconvex, then

$$(x - y)^T \mu_j \zeta_j \leqslant 0, \quad j \in J_2.$$

It follows that

$$(x - y)^T \sum_{j \in J_2} \mu_j \zeta_j \leqslant 0. \tag{4.9}$$

Summing (4.8) and (4.9) gives

$$(x - y)^T \left[\sum_{i \in Q} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j\right] < 0.$$

This is a contradiction to (4.7), therefore, x^* is an efficient solution of multiobjective programming problem. ∎

Theorem 4.2. (Strong Duality) *If x^* is an efficient solution of the multiobjective programming problem, and for $\forall \lambda_i > 0, i = 1, 2, \ldots, q, \mu_j \geqslant 0, j = 1, 2, \ldots, m, i \in Q$, each $f_i + \sum_{j \in J_1} \mu_j g_j, i \in Q$ is strictly pseudoconvex, then there exist $\lambda_i^* > 0, i \in Q, \mu_j^* \geqslant 0, j = 1, 2, \ldots, m$ such that (x^*, λ^*, μ^*) is efficient solution of (XDMP), and the objective values of the multiobjective programming problem at x^* and (XDMP) at (x^*, λ^*, μ^*) are equal.*

Proof: Since x^* is an efficient solution of multiobjective programming problem, then by Theorem3.3, there exist $\lambda_i^* > 0, i \in Q, \sum_{i \in Q} \lambda_i^* = 1$, and $\mu_j^* \geqslant 0, j \in J^{<}(x^*)$, such that

$$0 \in \sum_{i \in Q} \lambda_i^* \partial f_i(x^*) + \sum_{j \in J^{<}(x^*)} \mu_j^* \partial g_j(x^*) - \left[D_{\overline{Q}^{=}(x^*) \cup J^{=}}(x^*)\right]^*, \tag{4.10}$$

$$\mu_j^* g_j(x^*) = 0, \forall\ j \in J^{<}(x^*). \tag{4.11}$$

Setting $\mu_j^* = 0$ for $j \notin J^{<}(x^*)$, we know $(x^*, \lambda_i^*, i = 1, 2, \ldots, q, \mu_j^*, j = 1, 2, \ldots, m)$ is feasible for (XDMP).

If (x^*, λ^*, μ^*) is not efficient for (XDMP), the definition of efficient solution implies that there exists an efficient solution (y, λ, μ) of (XDMP) such that

$$\begin{cases} f_{i_0}(y) + \sum_{j \in J_1} \mu_j g(y) > f_{i_0}(x^*) + \sum_{j \in J_1} \mu_j^* g(x^*), & \exists\ i_0 \in Q, \\ f_i(y) + \sum_{j \in J_1} \mu_j g(y) \geqslant f_i(x^*) + \sum_{j \in J_1} \mu_j^* g(x^*), & \forall i \in Q, i \neq i_0. \end{cases}$$

From (4.11)we have $\sum_{j \in J_1} \mu_j^* g(x^*) = 0$, it follows that

$$\begin{cases} f_{i_0}(y) + \sum_{j \in J_1} \mu_j g(y) > f_{i_0}(x^*), & \exists\ i_0 \in Q, \\ f_i(y) + \sum_{j \in J_1} \mu_j g(y) \geqslant f_i(x^*), & \forall i \in Q, i \neq i_0. \end{cases}$$

This is a contradiction to the weak dual theorem. Clearly, the objective values of multiobjective programming problem at x^* and (XDMP) at (x^*, λ^*, μ^*) are equal. ∎

5. The Lagrangian Saddle Point of The Incomplete Lagrange Vector Value Functions

In this section, we shall establish the Lagrange saddle point existence theorem without the need of any constraint qualification.

Definition 5.1: The incomplete vector value Lagrange function $L(x, \mu_{J_1}) : X \times R_+^{|J_1|} \to R^q$ is defined

$$L(x, \mu_{J_1}) = \left(L_1(x, \mu_{J_1}), \ldots, L_q(x, \mu_{J_1})\right),$$

where $L_i(x, \mu_{J_1}) = f_i(x) + \mu_{J_1}^T g_{J_1}(x), i = 1, \ldots, q$.

DEFINITION 5.2: A point $(\overline{x}, \overline{\mu}_{J_1}) \in X \times R_+^{|J_1|}$ is said to be a saddle point of the incomplete Lagrange function $L(x, \mu_{J_1})$, if the following conditions hold:

$$L(x, \overline{\mu}_{J_1}) \nleq L(\overline{x}, \overline{\mu}_{J_1}), \qquad \forall x \in X, \tag{5.1}$$

$$L(\overline{x}, \overline{\mu}_{J_1}) \nleq L(\overline{x}, \mu_{J_1}), \qquad \forall \mu_{J_1} \in R_+^{|J_1|}. \tag{5.2}$$

If (x, μ_{J_1}) satisfying the following conditions:

$$L(x, \overline{\mu}_{J_1}) \nless L(\overline{x}, \overline{\mu}_{J_1}), \qquad \forall x \in X, \tag{5.3}$$

$$L(\overline{x}, \overline{\mu}_{J_1}) \nless L(\overline{x}, \mu_{J_1}), \qquad \forall \mu_{J_1} \in R_+^{|J_1|}, \tag{5.4}$$

then the point $(\overline{x}, \overline{\mu}_{J_1})$ is said to be a weak saddle point of the incomplete Lagrange function $L(x, \mu_{J_1})$.

Clearly, if $J_2 = \emptyset$, then $J_1 = J$, the incomplete Lagrange function $L(x, \mu_{J_1})$ just is the Lagrange function.

THEOREM 5.3. *If $(\overline{x}, \overline{\mu}_{J_1})$ is a saddle point of the incomplete Lagrange function $L(x, \mu_{J_1})$, and for the multiobjective programming problem, $\overline{x}$ satisfyies*

$$g_{J_2}(x) \leqq 0, \tag{5.5}$$

then $\overline{x}$ is a proper efficient solution of the multiobjective programming problem.

PROOF: Since $(\overline{x}, \overline{\mu}_{J_1})$ is a saddle point of the incomplete Lagrange function $L(x, \mu_{J_1})$, (5.2) implies that for $\forall \mu_{J_1} \in R_+^{|J_1|}$, there exists a i, $i = 1, 2, \ldots, q$, satisfying $L_i(\overline{x}, \mu_{J_1}) \leqq L_i(\overline{x}, \overline{\mu}_{J_1})$. Now there exists a i, $i = 1, 2, \ldots, q$, satisfying

$$f_i(\overline{x}) + \mu_{J_1}^T g_{J_1}(\overline{x}) \leqq f_i(\overline{x}) + \overline{\mu}_{J_1}^T g_{J_1}(\overline{x}).$$

It follows that

$$(\mu_{J_1} - \overline{\mu}_{J_1})^T g_{J_1}(\overline{x}) \leqq 0, \qquad \forall \mu_{J_1} \in R_+^{|J_1|}. \tag{5.6}$$

For $j = j_0 \in J_1$, letting

$$\mu_k = \overline{\mu}_k, \quad \forall\, k \in J_1 \setminus \{j_0\},$$
$$\mu_{j_0} = \overline{\mu}_{j_0} + 1,$$

along with (5.6), we have

$$g_{j_0}(\overline{x}) \leqq 0.$$

Repeating this course for all $j \in J_1$, we obtain

$$g_{J_1}(\overline{x}) \leqq 0. \tag{5.7}$$

Along with (5.9), $\overline{x}$ is feasible for the multiobjective programming problem.

Again, since $\overline{\mu}_{J_1} \in R_+^{|J_1|}$ and $g_{J_1}(\overline{x}) \leqq 0$, hence $\overline{\mu}_{J_1} g_{J_1}(\overline{x}) \leqq 0$. Because $(\mu_{J_1} - \overline{\mu}_{J_1})^T g_{J_1}(\overline{x}) \leqq 0$ holds for all $\mu_{J_1} \in R_+^{|J_1|}$, letting $\mu_{J_1} = 0$, we have $\overline{\mu}_{J_1}^T g_{J_1}(\overline{x}) \leqq 0$. It follows that

$$\overline{\mu}_{J_1}^T g_{J_1}(\overline{x}) = 0. \tag{5.8}$$

Now, if $\overline{x}$ was not an efficient solution of multiobjective programming problem, then there exists a feasible x for the multiobjective programming problem, satisfying $g(x) \leqq 0$ and $f(x) \leqslant f(\overline{x})$. So we have

$$f(x) + \overline{\mu}_{J_1}^T g_{J_1}(x) e \leqslant f(\overline{x}) = f(\overline{x}) + \overline{\mu}_{J_1}^T g_{J_1}(\overline{x}) e,$$

where $e = (1, 1, \ldots, 1)$.

It follows that

$$L(x, \overline{\mu}_{J_1}) \leqslant L(\overline{x}, \overline{\mu}_{J_1}).$$

This is a contradiction to (5.1).Then $\overline{x}$ is efficient for multiobjective programming problem.

If $\overline{x}$ were not a proper efficient solution of multiobjective programming problem, by the definition of the proper efficient solution, we can prove that there exists some feasible $\widetilde{x}$, satisfying

$$\mu^T\big(f(\widetilde{x}) - f(\overline{x})\big) < 0, \qquad \forall \mu > 0, \mu \in R_+^m.$$

By the artrabiariness of μ, there exists least one $i_0 \in \{1, 2, \ldots, q\}$, satisfying $f_{i_0}(\widetilde{x}) < f_{i_0}(\overline{x})$, but for all $i \in \{1, 2, \ldots, q\}, i \neq i_0$, we have $f_i(\widetilde{x}) \leqq f_i(\overline{x})$. Hence $f(\widetilde{x}) \leqslant f(\overline{x})$.It follows that

$$f(\widetilde{x}) + \overline{\mu}_{J_1}^T g_{J_1}(\widetilde{x}) e \leqslant f(\overline{x}) = f(\overline{x}) + \overline{\mu}_{J_1}^T g_{J_1}(\overline{x}) e,$$

that is, $L(\widetilde{x}, \overline{\mu}_{J_1}) \leqslant L(\overline{x}, \overline{\mu}_{J_1})$. This is a contradiction to (5.2). Thus $\overline{x}$ is a proper efficient solution of multiobjective programming problem. ∎

We have the following conclusion about weak efficient solutions.

Theorem 5.4. *If $(\overline{x}, \overline{\mu}_{J_1})$ is a weak saddle point of the incomplete Lagrange function $L(x, \mu_{J_1})$, and for the multiobjective programming problem, $\overline{x}$ satisfyies*

$$g_{J_2}(x) \leqq 0, \tag{5.9}$$

then $\overline{x}$ is a weak efficient solution of multiobjective programming problem.

Proof: The proof is similar to the Theorem 5.3. ∎

Then we give a necessary condition for a vector to be an efficient for multiobjective programming problem without using a constraint qualification.

THEOREM 5.5. *Let $\bar{x}$ be efficient for the multiobjective programming problem. Suppose for all $x \in S, \forall \lambda \in R^q$, with $\lambda_i > 0$ and $\sum_{i=1}^{q} \lambda_i = 1, \mu \in R^m$, and $\mu_j \geqslant 0$,*

$$g_{J_2}(x) \leqq 0, \forall x \in X. \tag{5.10}$$

If $f_i + \sum_{j \in J_1} \mu_j g_j, i \in Q$ is strictly pseudoconvex, then there exist $\bar{\mu} \in R^m, \bar{\mu}_j \geqslant 0$, such that $(\bar{x}, \bar{\mu}_{J_1})$ is a saddle point of the incomplete Lagrange function $L(x, \mu_{J_1})$.

PROOF: Since $\bar{x}$ is efficient for multiobjective programming problem, by Theorem 3.3, there exist $\bar{\lambda}_i > 0, i \in Q, \sum_{i \in Q} \bar{\lambda}_i = 1, \bar{\mu}_j \geqslant 0, j \in J^{<}(\bar{x})$, such that

$$0 \in \sum_{i \in Q} \bar{\lambda}_i \partial f_i(\bar{x}) + \sum_{j \in J^{<}(\bar{x})} \bar{\mu}_j \partial g_j(\bar{x}) - \left[D^{=}_{\bar{Q}^{=}(\bar{x}) \cup J^{=}}(\bar{x})\right]^*, \tag{5.11}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \forall\, j \in J^{<}(\bar{x}). \tag{5.12}$$

For $j \notin J^{<}(\bar{x})$, letting $\bar{\mu}_j = 0$, then

$$0 \in \sum_{i \in Q} \bar{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\bar{x}) - \left[D^{=}_{\bar{Q}^{=}(\bar{x}) \cup J^{=}}(\bar{x})\right]^*, \tag{5.13}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \forall\, j = 1, 2, \ldots, m. \tag{5.14}$$

From the proof of the weak theorem, there exist $\xi_i \in \partial f_i(y), i \in Q, \zeta_j \in \partial g_j(y), j = 1, 2, \ldots, m, d \in \left[D^{=}_{\bar{Q}^{=}(y) \cup J^{=}}(y)\right]^*$ such that

$$(x - y)^T \left[\sum_{i \in Q} \bar{\lambda}_i \xi_i + \sum_{j=1}^{m} \bar{\mu}_j \zeta_j\right] \geqslant 0. \tag{5.15}$$

If there exists a $x_0 \in X$ such that

$$L(x_0, \bar{\mu}_{J_1}) \leqslant L(\bar{x}, \bar{\mu}_{J_1}),$$

that is,

$$\begin{cases} f_{i_0}(x_0) + \bar{\mu}_{J_1}^T g_{J_1}(x_0) < f_{i_0}(\bar{x}) + \mu_{J_1}^T g_{J_1}(\bar{x}), & \exists\, i_0 \in Q, \\ f_i(x_0) + \bar{\mu}_{J_1}^T g_{J_1}(x_0) \leqslant f_i(\bar{x}) + \bar{\mu}_{J_1}^T g_{J_1}(\bar{x}), & \forall i \in Q, i \neq i_0. \end{cases}$$

Since $f_i + \sum_{j \in J_1} \bar{\mu}_j g_j, i = 1, 2, \ldots, q$ is strictly pseudoconvex, hence strictly quasiconvex, along with the regularity of each function, we have

$$(x - y)^T \left[\xi_{i_0} + \sum_{j \in J_1} \bar{\mu}_j \zeta_j\right] < 0,$$

$$(x - y)^T \left[\xi_i + \sum_{j \in J_1} \bar{\mu}_j \zeta_j\right] \leqslant 0, i = 1, 2, \ldots, q, i \neq i_0.$$

Multiplying each of these inequalities by $\overline{\lambda}_i$ and summing over $i \in Q$, noting that $\sum_{i=1}^{m} \overline{\lambda}_i = 1$, we have

(5.16) $$(x-y)^T\left[\sum_{i\in Q}\overline{\lambda}_i\xi_i + \sum_{j\in J_1}\overline{\mu}_j\zeta_j\right] < 0.$$

On the other hand, by the condition (5.10),we have

$$g_j(x_0) \leqslant g_j(\overline{x}), \quad \forall j \in J_2.$$

Again by the pseudoconvexity of each of g_j, we have

$$(x-y)^T\overline{\mu}_j\zeta_j \leqslant 0, \quad j \in J_2.$$

It follows that

(5.17) $$(x-y)^T\sum_{j\in J_2}\overline{\mu}_j\zeta_j \leqslant 0.$$

Summing (5.16) and (5.17) gives

$$(x-y)^T\left[\sum_{i\in Q}\overline{\lambda}_i\xi_i + \sum_{j=1}^{m}\overline{\mu}_j\zeta_j\right] < 0.$$

This is a contradiction to (5.15). Then we have

(5.18) $$L(x,\overline{\mu}_{J_1}) \not\leqslant L(\overline{x},\overline{\mu}_{J_1}), \qquad \forall x \in X.$$

Now suppose that there exists a scalar $\mu_{J_1} \in R_+^{|J_1|}$ such that

$$L(\overline{x},\overline{\mu}_{J_1}) \leqslant L(\overline{x},\mu_{J_1}),$$

that is,

$$f(\overline{x}) + \overline{\mu}_{J_1}^T g_{J_1}(\overline{x}) \leqslant f(\overline{x}) + \mu_{J_1}^T g_{J_1}(\overline{x}).$$

It follows that

$$0 = \overline{\mu}_{J_1}^T g_{J_1}(\overline{x}) < \mu_{J_1}^T g_{J_1}(\overline{x}).$$

This is a contradiction to the feasibility of $\overline{x}$. Hence

(5.19) $$L(\overline{x},\overline{\mu}_{J_1}) \not\leqslant L(\overline{x},\mu_{J_1}), \qquad \forall \mu_{J_1} \in R_+^{|J_1|}.$$

Along with (5.18)-(5.19), $(\overline{x},\overline{\mu}_{J_1})$ is a saddle point of the incomplete Lagrange function $L(x,\mu_{J_1})$. ∎

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6. Conclusion

In this paper, we have established a mixed dual problem of multiobjective programming problem, which unifies the Mond-Weir type dual and Wolfe type dual. On the other hand, applications to typical cases including the convex(not necessarily differentiable) and differentiable generalised convex multiobjective programs without requiring a constraint qualification are provided. Therefore, our work is a natural extension to the work of Egudo, Weir and Mond [3] to nonsmooth multiobjective programming.

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