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ON SEMIPERFECT MODULES

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ABSTRACT. Sandomierski (Proc. A.M.S. 21 (1969), 205–207) has proved that a ring is semiperfect if and only if every simple module has a projective cover. This is generalized to semiperfect modules as follows: If P is a projective module then P is semiperfect if and only if every simple homomorphic image of P has a projective cover and every proper submodule of P is contained in a maximal submodule.

Let R be a ring (with identity) and let M be a left R-module. A submodule $N \subseteq M$ is said to be *small* in M if $N+K=M$ where K is a submodule of M implies $K=M$. The sum $J(M)$ of all the small submodules of M is called the *radical* of M and it is easily verified that $J(M)$ is the intersection of all the maximal submodules of M. An epimorphism $P \xrightarrow{\pi} M \rightarrow 0$ is called a *projective cover* of M if P is projective and $ker(\pi)$ is small in P. The semiperfect rings of Bass can be described as those rings each of whose cyclic modules has a projective cover. Mares [4] generalized this notion to modules by calling a projective module P *semiperfect* if each homomorphic image of P has a projective cover. She then gave the following characterization of these semiperfect modules: ([4] Theorem 5.1).

THEOREM. (MARES) *A projective module P is semiperfect if and only if it satisfies the following three conditions:*

(1) *$J(P)$ is small in P.*
(2) *$P/J(P)$ is semisimple.*
(3) *Every idempotent of* $\mathrm{Hom}_R[P/J(P), P/J(P)]$ *is induced by an idempotent of* $\mathrm{Hom}_R[P, P]$.

This generalizes a result of Bass [1] that a ring R is semiperfect if and only if $R/J(R)$ is semisimple and idempotents can be lifted modulo $J(R)$.

The following result is an immediate consequence of Proposition 2.7 of [1].

LEMMA 1. *If P is a projective module then $J(P)=J(R)P$. Moreover, if $P=N\oplus M$ with $N\subseteq J(P)$ then $N=0$.*

We shall need the following:

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LEMMA 2. *Let M be an $R/J(R)$-module which has a projective cover as an R-module. Then M is projective as an $R/J(R)$-module.*

Proof. Let $0 \to K \to P \to M \to 0$ be a projective cover of M viewed as an R-module. Then the induced sequence

$$R/J \otimes_R K \to R/J \otimes_R P \to R/J \otimes_R M \to 0$$

is also exact. We have that $M \simeq R/J \otimes M$ as R/J-modules and that $R/J \otimes P$ is a projective R/J-module. Furthermore $K \subseteq J(P) = J(R)P$ so $R/J \otimes K \to R/J \otimes P$ is the zero map. The result follows. □

Sandomierski [5] has shown that a ring is semiperfect if and only if every simple module has a projective cover. We generalize this to projective modules as follows:

THEOREM. *Let P be a projective module. P is a semiperfect module if and only if every proper submodule is contained in a maximal submodule and P/M has a projective cover for every maximal submodule M of P.*

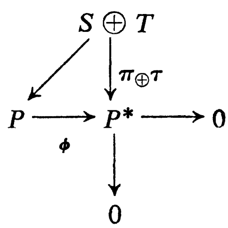
Proof. Let P be semiperfect. Since $J(P)$ is small in P and $P/J(P)$ is semisimple, it follows that each proper submodule of P is contained in a maximal submodule. The necessity of the other condition is clear.

For the converse, we verify the three conditions in Mares' Theorem. First of all $J(P)$ is small in P. For if $J(P)+K=P$ where $K \neq P$, we can include K in a maximal submodule M and so obtain $P=J(P)+K \subseteq M$.

Now we show that $P^*=P/J(P)$ is semisimple. If $M^*=M/J(P)$ is a maximal submodule of P^* then $P^*/M^* \simeq P/M$ has a projective cover. Since P^*/M^* is an $R/J(R)$-module, the lemma implies that M^* is a direct summand of P^*. Now suppose $\text{soc}(P^*) \neq P^*$. Then we can include $\text{soc}(P^*)$ in a maximal submodule of P^* which is a direct summand. This contradiction implies that P^* is semisimple.

Finally we must show that idempotents in $\text{Hom}_R[P/J(P), P/J(P)]$ are induced by idempotents in $\text{Hom}_R[P, P]$. Let $\phi: P \to P^*$ denote the natural map. It suffices to show that if $P^*=A^* \oplus B^*$ then we can write $P=M \oplus N$ where $\phi(M)=A^*$ and $\phi(N)=B^*$.

Now if $P^*=A^* \oplus B^*$, write $A^* = \oplus_{i \in J} S_i$ and $B^* = \oplus_{i \in J} T_j$ where the S_i and T_j are simple. Since each S_i and T_j is a homomorphic image of P, they have projective covers by hypothesis, say $P_i \xrightarrow{\pi_i} S_i \to 0$ and $Q_j \xrightarrow{\tau_j} T_j \to 0$. If $S = \oplus P_i$ and $T = \oplus Q_j$ then $S \oplus T$ is projective and so we have the following diagram



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where $\pi=\oplus\pi_i$, and $\tau=\oplus\tau_j$. Since $ker(\phi)=J(P)$ is small in P, $P\xrightarrow{\phi}P^*\to 0$ is a projective cover of P^*. Hence, by the uniqueness of projective covers, $S\oplus T=N\oplus P'$ where $N\subseteq ker(\pi\oplus\tau)$ and $f_{|P'}$ is an isomorphism. But $ker(\pi\oplus\tau)$ is the sum of the kernels of all the π_i and τ_j and so is contained in $J(S\oplus T)$. It follows that $N\subseteq J(S\oplus T)$ and so $N=0$ by Lemma 1. But then f is an isomorphism so $P=f(S)\oplus f(T)$. Since $\phi[f(S)]=A^*$ and $\phi[f(T)]=B^*$, we have lifted the decomposition $P^*=A^*\oplus B^*$. Hence P is semiperfect by Mares' theorem. □

COROLLARY 1. (*Sandomierski*). *A ring R is semiperfect if and only if each simple left R module has a projective cover.*

COROLLARY 2. *A finitely generated projective module P is semiperfect if and only if P/N has a projective cover for each maximal submodule N.*

If M is an R-module and N is a submodule a *supplement* of N in M is a submodule K such that $N+K=M$ and $N+V\neq M$ for all submodules $V\subset K$. If $N+K=M$, it is easy to verify that K is a supplement of N if and only if $N\cap K$ is small in K. Kasch and Mares have shown in [3] that a projective module is semiperfect if and only if every submodule has a supplement. In order to obtain a stronger result, we need the following result which appears as Proposition 3.1 of [2].

LEMMA 3. *Let P be a projective module and let $P=N+K$ where N and K are submodules each of which is a supplement of the other. Then $P=N\oplus K$.*

COROLLARY 3. *Let P be a projective module. P is semiperfect if and only if it satisfies the following two conditions:*

(1) *Every maximal submodule and every cyclic submodule has a supplement.*
(2) *Every proper submodule is contained in a maximal submodule.*

Proof. If P is semiperfect, condition (1) follows easily from the uniqueness of projective covers (Lemma 2.3 of [1].) For the converse, let M be a maximal submodule of P and let K be a supplement of M. Then, if $x\in K\backslash M$ we have $Rx\subseteq K$ and $M+Rx=P$ so $K=Rx$. Hence let N be a supplement of K. We claim that K is a supplement of N, that is $K\cap N$ is small in K. Indeed, if $(K\cap N)+V=K$ then $P=(K\cap N)+V+M$. Since $K\cap N$ is small in N and hence in P, this implies $P=V+M$ with $V\subseteq K$. It follows that $V=K$.

But then Lemma 3 implies that $P=N\oplus K$ and it follows that K is projective. Hence the exact sequence

$$0\to M\cap K\to P\to P/M\to 0$$

is a projective cover of P/M. □

We remark that the proof of Corollary 3 can be readily adapted to give a proof of the result of Kasch and Mares.

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References

1. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

2. J. S. Golan, *Quasiperfect Modules*, Quart. J. Math. Oxford (2), **22** (1971), 173–82.

3. F. Kasch and E. A. Mares, *Eine Kennzeichnung Semi-perfecter Moduln*, Nagoya Math. J. **27** (1966), 525–529.

4. E. A. Mares, *Semiperfect modules*, Math. Zeitschr. **82** (1963), 347–360.

5. F. L. Sandomierski, *On semiperfect and perfect rings*, Proc. Amer. Math. Soc. **21** (1969), 205–207.

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