ON THE CONES ASSOCIATED WITH BIORTHOGONAL SYSTEMS AND BASES IN BANACH SPACES

C. W. McARTHUR, IVAN SINGER, AND MARK LEVIN

1. Let E be a Banach space (by this we shall mean, for simplicity, a *real* Banach space) and (x_n, f_n) ($\{x_n\} \subset E$, $\{f_n\} \subset E^*$) a biorthogonal system, such that $\{f_n\}$ is total on E (i.e. the relations $x \in E, f_n(x) = 0, n = 1, 2, \ldots,$ imply $x = 0$). Then it is natural to consider the cone

$$K = K_{(x_n, f_n)} = \{x \in E | f_n(x) \geqq 0 \ (n = 1, 2, \ldots)\}, \tag{1}$$

which we shall call "the cone associated with the biorthogonal system (x_n, f_n)". In particular, if $\{x_n\}$ is a basis of E and $\{f_n\}$ the sequence of coefficient functionals associated with the basis $\{x_n\}$, this cone is nothing else but

$$K = K_{\{x_n\}} = \left\{ \sum_{i=1}^{\infty} \alpha_i x_i \in E | \ \alpha_n \geqq 0 \ (n = 1, 2, \ldots) \right\}, \tag{2}$$

and we shall call it "the cone associated with the basis $\{x_n\}$". Recently, Fullerton (**3**, Theorems 1, 2, and 3) and Gurevič (**6**, Theorems 1 and 4, Lemma 3) have given geometric conditions on the cone $K = K_{(x_n, f_n)}$ associated with a biorthogonal system (x_n, f_n), which are necessary and sufficient in order that $\{x_n\}$ be an unconditional basis of the space E, and Gurariĭ (**5**, p. 1239, Theorem 2) has given a condition on the cone $K = K_{(x_n, f_n)}$ which is sufficient in order that $\{x_n\}$ be a "basis of the cone K" (i.e. that for every $x \in K$ the series $\sum_{i=1}^{\infty} f_i(x) x_i$ be convergent to x). In § 2 of the present paper we shall further this study, giving conditions on K which are necessary and sufficient in order that $\{x_n\}$ be an unconditional basis of the cone K, and a sufficient condition in order that $\{x_n\}$ be an unconditional basis of K, which is also "boundedly complete on K".

Throughout this paper, by "cone" we shall understand "closed convex cone having the origin as extreme point", i.e. a closed set K such that $K + K \subset K$, $\lambda K \subset K$ ($\lambda \geqq 0$), and $K \cap (-K) = \{0\}$. (The assumption above, that $\{f_n\}$ is total on E, was made in order to ensure that this last condition is satisfied.) A subset B of a cone K is said to be a "base" of the cone K if B is closed and convex and if every $x \in K \sim \{0\}$ has a unique representation of the form $x = \lambda y$, with $\lambda > 0$, $y \in B$. Fullerton (**3**, Remark after Theorem 3′) has observed that the cone $K = K_{\{x_n\}}$ associated with a basis $\{x_n\}$ of a Banach

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space cannot have a base which is compact. (Actually, his argument in (**3**) shows that the cone may not have even a weakly compact base.) In § 3 we shall further this study, characterizing some types of bases $\{x_n\}$ of a Banach space E by geometric properties of the bases B of the associated cone $K = K_{\{x_n\}}$, or of the bases $B^{\{\epsilon_n\}}$ of the cones $K_{\{\epsilon_n x_n\}}$ associated with the bases $\{\epsilon_n x_n\}$ of E, where $\epsilon_n = \pm 1$ ($n = 1, 2, \ldots$). The question of finding geometric properties of B corresponding to certain properties of $\{x_n\}$ and the converse question, to find properties of $\{x_n\}$ which correspond to certain geometric properties of B, may deserve further interest.

2. We recall that a cone K induces a natural partial order relation on E, namely, $x \geqq y$ if and only if $x - y \in K$. (In particular, $x \geqq 0$ if and only if $x \in K$.) Let us also recall that the cone K is said to be (a) generating, if $E = K - K$; (b) minihedral, if for every $x, y \in K$ there exists $z_0 = \sup(x, y)$ (i.e. the element $z_0 \geqq x, y$ with the property $z \geqq x, y \Rightarrow z \geqq z_0$); (c) normal, if there exists a constant $\delta > 0$ such that

$$(3) \qquad ||x + y|| \geqq \delta \qquad (x, y \in K, ||x|| = ||y|| = 1).$$

Consequently, any cone K which is contained in a normal cone is normal. It is well known (see, e.g., **8**, Chapter 1, § 1.2.2) that K is normal if and only if the norm on E is "semi-monotone", i.e. there exists a constant $L > 0$ such that

$$(4) \qquad 0 \leqq x \leqq y \Rightarrow ||x|| \leqq L||y||.$$

The cone $K = K_{\{x_n\}}$ associated with a basis $\{x_n\}$ is normal if and only if it is regular, i.e. the relations $y_1 \leqq y_2 \leqq \ldots \leqq y_n \leqq \ldots \leqq z$ imply the norm-convergence of the sequence $\{y_n\}$ (see **8**, Chapter 1, § 1.2.2; **6**).

THEOREM 1. *Let E be a Banach space and let (x_n, f_n) ($\{x_n\} \subset E$, $\{f_n\} \subset E^*$) be a biorthogonal system such that $\{f_n\}$ is total and that $\{x_n\}$ is a basis of the associated cone $K = K_{(x_n, f_n)}$. The following statements are equivalent:*

(1°) *For every $x \in K$ the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is unconditionally convergent (i.e. $\{x_n\}$ is an unconditional basis of the cone K);*

(2°) *For every $x \in K$ the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is weakly unconditionally Cauchy;*

(3°) *K is normal;*

(4°) *For every $x \in K$, the set $P_x = K \cap (x - K) = \{y \in E | \, 0 \leqq y \leqq x\}$ is bounded;*

(5°) *For every $x \in K$, the set P_x above is linearly homeomorphic to a finite cube or a cube of Hilbert.*

Moreover, if we have (1°), *then K is minihedral.*

Proof. The implication (1°) $\Rightarrow$ (2°) is trivial. Assume now that we have (2°).

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Let $x, y \in K$ be such that $y \leqq x$, $0 \leqq f_i(y) \leqq f_i(x)$ $(i = 1, 2, \ldots)$, and let $f \in E^*$ be arbitrary. Then since $\{x_n\}$ is a basis of K and by (2°),

$$|f(y)| = \left|\sum_{i=1}^{\infty} f_i(y) f(x_i)\right| \leqq \sum_{i=1}^{\infty} f_i(y)|f(x_i)| \leqq \sum_{i=1}^{\infty} f_i(x)|f(x_i)| = M_{f,x} < \infty$$

(M a positive constant),

which shows that for every $x \in K$, the set

$$P_x = \{y \in E \mid 0 \leqq y \leqq x\}$$

is weakly bounded, whence also strongly bounded. Thus (2°) $\Rightarrow$ (4°). The equivalence (3°) $\Leftrightarrow$ (4°) is well known (see, e.g., **1**, p. 1165, Lemma 2). Assume now that we have (3°). Let $x \in K$ and $\epsilon > 0$ be arbitrary. Since $\{x_n\}$ is a basis of K, there exists a positive integer N such that

$$\left\|\sum_{i=N}^{\infty} f_i(x) x_i\right\| < \epsilon/L, \tag{5}$$

where L is the constant occurring in (4). Now let $\sum_{i=1}^{\infty} f_{n_i}(x) x_{n_i}$ be an arbitrary subseries of $\sum_{i=1}^{\infty} f_i(x) x_i$. Choose i_0 such that $n_i \geqq N$ whenever $i \geqq i_0$. We have then, for any $p, q \geqq i_0$,

$$0 \leqq \sum_{i=p}^{q} f_{n_i}(x) x_{n_i} \leqq \sum_{i=N}^{\infty} f_i(x) x_i,$$

whence by (4) and (5),

$$\left\|\sum_{i=p}^{q} f_{n_i}(x) x_{n_i}\right\| \leqq L\left\|\sum_{i=N}^{\infty} f_i(x) x_i\right\| < L(\epsilon/L) = \epsilon,$$

which proves (since E is complete) that $\sum_{i=1}^{\infty} f_i(x) x_i$ is unconditionally convergent. Thus (3°) $\Rightarrow$ (1°). Furthermore, the implication (5°) $\Rightarrow$ (4°) is trivial (since the cubes in (5°) are compact), and the implication (1°) $\Rightarrow$ (5°) follows, observing that in the proof by Fullerton (**3**, Theorem 2) of the similar statement for $\{x_i\}$ an unconditional basis of the whole space E, only expansions of elements of K are used. (Let us also mention that one can show directly, with the standard ϵ-net method, that for each $x \in K$ the set P_x is compact, and then apply a result of Klee (**7**, p. 31, Corollary 1.3), to conclude that P_x is homeomorphic to the fundamental cube of Hilbert whenever $f_n(x) > 0$ $(n = 1, 2, \ldots)$.) Thus (1°) $\Leftrightarrow \ldots \Leftrightarrow$ (5°). Assume, finally, that we have (1°) and let $x, y \in K$ be arbitrary. Then the series $\sum_{i=1}^{\infty} [f_i(x) + f_i(y)] x_i$ is unconditionally convergent, whence so is the series $\sum_{i=1}^{\infty} [\sup(f_i(x), f_i(y))] x_i$, and the sum of this latter series is obviously $\sup(x, y)$. Thus (1°) $\Rightarrow K$ is minihedral, which completes the proof of the theorem.

Remark 1. A basis $\{x_n\}$ of the whole space E can have property (1°) without

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being an unconditional basis of E, as shown, for example, by the basis

(6) $$x_n = \sum_{i=1}^{n} e_i, \qquad n = 1, 2, \ldots \text{ (where } e_i = \{\delta_{ij}\}_{j=1}^{\infty}),$$

of the space $E = c_0$.

Remark 2. The converse of the last assertion of Theorem 1 is not valid, as shown by the following example: Let E be the closed hyperplane

$$\left\{x = \{\xi_n\} \in l^1 \,\middle|\, \sum_{i=1}^{\infty} \xi_i = 0\right\}$$

in the space l^1, and let

(7) $$x_n = e_n - e_{n-1} \qquad (n = 1, 2, \ldots).$$

Then $\{x_n\}$ is a basis of E (see, e.g., **11**, p. 364), with the associated sequence of coefficient functionals

(8) $$f_n(x) = \sum_{i=1}^{n} \xi_i \qquad (x = \{\xi_n\} \in E),$$

whence

(9) $$K = \left\{x = \{\xi_n\} \in E \,\middle|\, \sum_{i=1}^{n} \xi_i \geqq 0 \quad (n = 1, 2, \ldots)\right\}.$$

The cone K is minihedral and generating. In fact, if $\sum_{i=1}^{\infty} \alpha_i x_i \in E$, we have

$$\sum_{i=1}^{\infty} \alpha_i x_i = \sum_{i=1}^{\infty} \alpha_i (e_i - e_{i+1}) = \alpha_1 e_1 + \sum_{i=2}^{\infty} (\alpha_i - \alpha_{i-1}) e_i$$

(where $\{e_n\}$ denotes the unit vector basis of l^1), i.e.

$$|\alpha_1| + \sum_{i=2}^{\infty} |\alpha_i - \alpha_{i-1}| < \infty,$$

and conversely. Since $|\,|\alpha_i| - |\alpha_{i-1}|\,| \leqq |\alpha_i - \alpha_{i-1}|$ $(i = 2, 3, \ldots)$, it follows that $\sum_{i=1}^{\infty} |\alpha_i| x_i$ converges whenever $\sum_{i=1}^{\infty} \alpha_i x_i$ converges, and thus for each $x \in E$ there exists the element $|x| \in E$, whence also the elements $x_+ = \sup(x, 0)$, $x_- = \sup(-x, 0)$, whence K is minihedral and generating. However, K is not normal, since for the sequences $\{x_n\}$, $\{z_n\} \subset E$ defined by

$$y_n = (1/n)[e_1 + e_3 + \ldots + e_{2n-1}] - (1/n)[e_2 + e_4 + \ldots + e_{2n}] \qquad (n = 1, 2, \ldots),$$

$$z_n = (1/n)[e_1 - e_{2n}],$$

we have $0 \leqq y_n \leqq z_n$, $||y_n|| = 2$, $||z_n|| = 2/n$ $(n = 1, 2, \ldots)$. (One can also observe that if K would be normal, then since it is generating, $\{x_n\}$ would be an unconditional basis of E (**6**, Theorems 1 and 4, Lemma 3), which is not the case (**11**, p. 364).)

Remark 3. It is essential in Theorem 1 to assume that $\{x_n\}$ is a basis of K, as shown by the example of the unit vectors x_n in the space $E = m$, for which we have (3°) and (4°) but not (1°), (2°) or (5°). Let us mention that if $\{f_n\}$ is total on E and K is "acute angled" in the sense that

$$(10) \qquad ||x + y|| - 1 \geqq \delta(t) > 0 \qquad (x, y \in K, ||x|| \geqq 1, ||y|| \geqq t),$$

then $\{x_n\}$ is a basis of K. (**5**, Theorem 2); obviously, every acute angled cone is normal, but the converse is not true, as shown by simple two-dimensional examples. Furthermore, the natural positive cone of the space $E = m$, mentioned in this remark is also normal but not acute angled. Let us also observe that for the usual Schauder basis $\{x_n\}$ of the space $E = C([0, 1])$ the associated cone $K = K_{\{x_n\}}$ is contained in the natural positive cone of the space E, which is obviously normal. Therefore K is normal, whence, by Theorem 1, for every $x \in K$ the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is unconditionally convergent (although $\{x_n\}$ is not an unconditional basis of E). However, K is not acute angled. (It is easy to give two consecutive elements, x_k and x_{k+1}, of the Schauder basis $\{x_n\}$ such that $||x_k|| = ||x_{k+1}|| = ||x_k + x_{k+1}|| = 1$.)

In the case when K is also sequentially weakly complete, we have the following result.

THEOREM 2. *Let E be a Banach space and let (x_n, f_n) ($\{x_n\} \subset E$, $\{f_n\} \subset E^*$) be a biorthogonal system such that the associated cone $K = K_{(x_n, f_n)}$ is normal and sequentially weakly complete. Then $\{x_n\}$ is an unconditional basis of K, which is also boundedly complete on K (i.e., the relations $a_i \geqq 0$ ($i = 1, 2, \ldots$), $\sup_n ||\sum_{i=1}^{n} a_i x_i|| < \infty$ imply that $\sum_{i=1}^{\infty} a_i x_i$ converges).*

Proof. Let us first prove that $\{x_n\}$ is boundedly complete on K. Let $\{a_n\}$ be a sequence of scalars such that $a_n \geqq 0$ ($n = 1, 2, \ldots$), $\sup_n ||\sum_{i=1}^{n} a_i x_i|| = M < \infty$, and let J be an arbitrary finite set of positive integers. Choose n such that $J \subset [1, n]$. Then

$$0 \leqq \sum_{i \in J} a_i x_i \leqq \sum_{i=1}^{n} a_i x_i,$$

whence, since K is normal,

$$\left\| \sum_{i \in J} a_i x_i \right\| \leqq L \left\| \sum_{i=1}^{n} a_i x_i \right\| \leqq LM.$$

Consequently, by a theorem of Gel′fand (**4**, p. 242, Proposition 2), the series $\sum_{i=1}^{\infty} a_i x_i$ is weakly unconditionally Cauchy, i.e.

$$\sum_{i=1}^{\infty} a_i |f(x_i)| = \sum_{i=1}^{\infty} |f(a_i x_i)| < \infty \qquad (f \in E^*),$$

whence, since K is sequentially weakly complete, every subseries $\sum_{i=1}^{\infty} a_{n_i} x_{n_i}$ converges weakly to an element of K. Therefore, by the Orlicz-Pettis theorem

(**2**, p. 318, Theorem 1), the series $\sum_{i=1}^{\infty} a_i x_i$ converges strongly. Thus $\{x_n\}$ is boundedly complete on K. Now, since (x_n, f_n) is a biorthogonal system, for every $x \in K$ we have

$$0 \leqq \sum_{i=1}^{n} f_i(x)x_i \leqq x \qquad (n = 1, 2, \ldots),$$

whence, since K is normal,

$$\sup_n \left\| \sum_{i=1}^{n} f_i(x)x_i \right\| \leqq L\|x\| < \infty,$$

and therefore, by the above, the series $\sum_{i=1}^{\infty} f_i(x)x_i$ converges strongly to an element $y \in K$. Since $\{f_n\}$ is total on E, we must have $y = x$, and thus $\{x_n\}$ is a basis of K, whence, by the normality of K and by Theorem 1, it is also an unconditional basis of K. This completes the proof.

Remark 4. The converse of Theorem 2 is not valid, as shown by the following example. Let E and $\{x_n\}$ be as in Remark 1. Then $\{x_n\} \subset K$ is a weak Cauchy sequence which is not weakly convergent to any element of E, and thus K is not sequentially weakly complete. However, as observed in Remark 1, $\{x_n\}$ is an unconditional basis of K and it is also boundedly complete on K since the relations $a_n \geqq 0$ $(n = 1, 2, \ldots)$,

$$\sup_n \left\| \sum_{i=1}^{n} a_i x_i \right\| = \sup_n \sup_{1 \leqq j \leqq n} \left| \sum_{i=j}^{n} a_i \right| = \sup_n \left| \sum_{i=1}^{n} a_i \right| < \infty$$

imply that $\sum_{i=1}^{\infty} a_i < \infty$ and that the series $\sum_{i=1}^{\infty} a_i x_i$ converges to $\sum_{j=1}^{\infty} (\sum_{i=j}^{\infty} a_i)e_j \in K$ (where $\{e_n\}$ is the unit vector basis of $E = c_0$).

3. Let us now turn to a base B of the cone $K = K_{(x_n, f_n)}$ associated with a biorthogonal system (x_n, f_n). The problem of the existence of such a base has an affirmative answer, namely, we have the following result.

PROPOSITION 1. *Let E be a Banach space and let (x_n, f_n) ($\{x_n\} \subset E$, $\{f_n\} \subset E^*$) be a biorthogonal system such that $\{f_n\}$ is total. Then the associated cone $K = K_{(x_n, f_n)}$ has an unbounded base.*

Proof. Define $f \in E^*$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i \|f_i\|} f_i(x) \qquad (x \in E). \tag{11}$$

It is true that the set

$$B = \{y \in K | f(y) = 1\} \tag{12}$$

is an unbounded base of K. In fact, B is convex and closed and for every $x \in K \sim \{0\}$ we have $x = \lambda y$, where $\lambda = f(x) > 0$ and $y = (1/f(x))x \in B$. This representation is unique since the relations $x = \lambda_1 y_1 = \lambda_2 y_2$, $\lambda_1, \lambda_2 > 0$,

$y_1, y_2 \in B$ imply by (12) that $f(x) = \lambda_1 = \lambda_2$, whence also $y_1 = y_2$; therefore B is a base of K. Furthermore, we have

$$f(2^n||f_n||x_n) = \sum_{i=1}^{\infty} \frac{1}{2^i||f_i||} f_i(2^n||f_n||x_n) = 1 \qquad (n = 1, 2, \ldots),$$

i.e. $2^n||f_n||x_n \in B$ $(n = 1, 2, \ldots)$ and this sequence is unbounded, since

$$||2^n||f_n||x_n|| = 2^n||f_n|| \, ||x_n|| \geqq 2^n|f_n(x_n)| = 2^n \qquad (n = 1, 2, \ldots).$$

This completes the proof of Proposition 1.

All the results stated in the remainder of the paper for normalized bases $\{x_n\}$, i.e., bases satisfying $||x_n|| = 1$ $(n = 1, 2, \ldots)$, remain valid, obviously, for "bounded" bases (**12**, p. 546, Theorem 1.6), i.e. bases satisfying

$$0 < \inf_n ||x_n|| \leqq \sup_n ||x_n|| < \infty;$$

we state them here only for normalized bases in order to avoid confusion with boundedness of a base B of K. We recall that a normalized basis $\{x_n\}$ of a Banach space E is said to be *of type* l_+ (**11**, p. 353) if there exists a constant $\eta > 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geqq \eta \sum_{i=1}^{n} \alpha_i \tag{13}$$

for any finite sequence $\alpha_1, \ldots, \alpha_n \geqq 0$. As shown in (**11**, Proposition 1), this happens if and only if there exists a functional $f \in E^*$ such that

$$f(x_n) \geqq 1 \qquad (n = 1, 2, \ldots). \tag{14}$$

THEOREM 3. *A normalized basis $\{x_n\}$ of a Banach space E is of type l_+ if and only if the associated cone $K = K_{\{x_n\}}$ has a bounded base.*

Proof. Assume that $\{x_n\}$ is a normalized basis of type l_+. Put

$$B = \{y \in K | f(y) = 1\}, \tag{15}$$

where $f \in E^*$ is any functional satisfying (14). Then B is a base of K and for every $y = \sum_{i=1}^{\infty} \alpha_i x_i \in B$ we have, taking into account $||x_n|| = 1$ and $\alpha_n \geqq 0$ $(n = 1, 2, \ldots)$,

$$||y|| \leqq \sum_{i=1}^{\infty} \alpha_i \leqq \sum_{i=1}^{\infty} \alpha_i f(x_i) = f\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = 1,$$

i.e. B is bounded. Conversely, assume that the cone K associated with the normalized basis $\{x_n\}$ has a bounded base B. Then $0 \notin B$ (since otherwise by the convexity of B, for any $y \in B$ one would have $\frac{1}{2}y \in B$, whence the element $y \in K$ would have two representations $y = 1 \cdot y = 2 \cdot \frac{1}{2}y$, i.e. B would not be a base), whence, since B is closed and convex, there exists a functional $f \in E^*$ such that

$$\inf_{y \in B} f(y) = \delta > 0. \tag{16}$$

Since $x_n \in K \sim \{0\}$, there exists a unique representation $x_n = \lambda_n y_n$, with $\lambda_n > 0$, $y_n \in B$, whence $x_n/\lambda_n \in B$, whence $1/\lambda_n = ||x_n/\lambda_n|| \leqq \sup_{y \in B}||y|| = C < \infty$ and thus $\lambda_n \geqq 1/C$ $(n = 1, 2, \ldots)$. Therefore, taking also into account (16), we obtain

$$f(x_n) = \lambda_n f(x_n/\lambda_n) \geqq \delta/C \qquad (n = 1, 2, \ldots),$$

which proves that $\{x_n\}$ is of type l_+. This completes the proof of Theorem 3.

We recall that a normalized basis $\{x_n\}$ of a Banach space E is said to be (a) shrinking, if $||f\,|[x_n, x_{n+1}, x_{n+2}, \ldots]|| \to 0$ as $n \to \infty$, for all $f \in E^*$ (where $[x_n, x_{n+1}, x_{n+2}, \ldots]$ denotes the closed linear subspace spanned by $\{x_j\}_{j=n}^{\infty}$); (b) of type P (**11**, p. 354) if $\sup_n ||\sum_{i=1}^{n} x_i|| < \infty$.

COROLLARY 1. *If $\{x_n\}$ is a normalized shrinking basis or a normalized basis of type* P *of a Banach space E, then every base B of the associated cone $K = K_{\{x_n\}}$ is unbounded.*

In fact, every shrinking basis and every basis of type P is not of type l_+ (**11**, Theorem 1).

In connection with the above results, the following proposition on general cones (not necessarily associated with biorthogonal systems) will be useful.

PROPOSITION 2. *If a cone K in a Banach space E has a bounded base, then K is normal.*

Proof. Let B be a bounded base of K. Then $\sup_{y \in B} ||y|| = M < \infty$, and since B is closed and $0 \notin B$, we also have $\inf_{y \in B} ||y|| = m > 0$. Let $0 \leqq x \leqq z$ be arbitrary with $0 \neq x \neq z$. Then x, z, and $z - x$ have unique representations $x = \lambda_1 y_1$, $z = \lambda_2 y_2$, $z - x = \lambda_3 y_3$, with $\lambda_i > 0$, $y_i \in B$ $(i = 1, 2, 3)$. Hence

$$\lambda_2 y_2 = z = (z - x) + x = \lambda_3 y_3 + \lambda_1 y_1 = (\lambda_3 + \lambda_1)\left[\frac{\lambda_3}{\lambda_3 + \lambda_1} y_3 + \frac{\lambda_1}{\lambda_3 + \lambda_1} y_1\right].$$

Since B is convex, we have

$$\frac{\lambda_3}{\lambda_3 + \lambda_1} y_3 + \frac{\lambda_1}{\lambda_3 + \lambda_1} y_1 \in B,$$

whence by the unique representation property occurring in the definition of a base of a cone,

$$\lambda_3 + \lambda_1 = \lambda_2 \quad \text{and} \quad \frac{\lambda_3}{\lambda_3 + \lambda_1} y_3 + \frac{\lambda_1}{\lambda_3 + \lambda_1} y_1 = y_2.$$

We observe that the second equality is also a consequence of the first equality, since it amounts to

$$\frac{z - x}{\lambda_1 + \lambda_3} + \frac{x}{\lambda_1 + \lambda_3} = \frac{z}{\lambda_2}.$$

Since $\lambda_3 > 0$, from the first of these equalities we obtain $\lambda_1 < \lambda_2$, whence

$$||x|| = \lambda_1||y_1|| \leqq \lambda_1 M < \lambda_2 M = \lambda_2 m \frac{M}{m} \leqq \lambda_2||y_2|| \frac{M}{m} = ||z|| \frac{M}{m},$$

which completes the proof of Proposition 2.

Obviously, the converse of Proposition 2 is not valid, since, e.g., the positive cone K associated with the unit vector basis $\{x_n\}$ of the space $E = c_0$ is normal, but has no bounded base (since $\{x_n\}$ is not of type l_+). From Propositions 1 and 2 also follows the following result.

COROLLARY 1. *If $\{x_n\}$ is a normalized basis of type l_+ of a Banach space E, then the associated cone $K = K_{\{x_n\}}$ is normal.*

One can also prove this result directly, observing that if $\{x_n\}$ is a normalized basis of type l_+ and $0 \leqq \alpha_i \leqq \beta_i$ $(i = 1, 2, \ldots, n)$, then

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| \leqq \sum_{i=1}^{n} \alpha_i \leqq \sum_{i=1}^{n} \beta_i \leqq \frac{1}{\eta}\left\|\sum_{i=1}^{n} \beta_i x_i\right\|$$

(where $\eta > 0$ is the constant occurring in (13)), whence the same also holds for convergent infinite series $\sum_{i=1}^{\infty} \alpha_i x_i$, $\sum_{i=1}^{\infty} \beta_i x_i$ with $0 \leqq \alpha_i \leqq \beta_i$ $(i = 1, 2, \ldots)$.

Taking also into account Theorem 1, we obtain the following result.

COROLLARY 2. *If $\{x_n\}$ is a normalized basis of type l_+ of a Banach space E, and $K = K_{\{x_n\}}$ is the associated cone, then for every $x \in K$ the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is unconditionally convergent.*

Actually, for bases of type l_+, one can prove more, namely the result which follows.

PROPOSITION 3. *A normalized basis $\{x_n\}$ of a Banach space E is of type l_+ if and only if there exists a constant $M > 0$ such that for every $x \in K$ the series $\sum_{i=1}^{\infty} f_i(x)x_i$ is absolutely convergent (i.e. $\sum_{i=1}^{\infty} ||f_i(x)x_i|| < \infty$) and*

$$\sum_{i=1}^{\infty} ||f_i(x)x_i|| \leqq M||x|| \qquad (x \in K). \tag{17}$$

Proof. If $\{x_n\}$ is a normalized basis of type l_+, then for every $x \in K$ and $n = 1, 2, \ldots,$ we have

$$\sum_{i=1}^{n} ||f_i(x)x_i|| = \sum_{i=1}^{n} f_i(x) \leqq \frac{1}{\eta}\left\|\sum_{i=1}^{n} f_i(x)x_i\right\|,$$

whence, taking $n \to \infty$, we obtain (17) with $M = 1/\eta$. Conversely, if $\{x_n\}$ is a normalized basis satisfying (17), then for any $\alpha_1, \ldots, \alpha_n \geqq 0$ we have, setting $x = \sum_{i=1}^{n} \alpha_i x_i$ in (17),

$$\frac{1}{M}\sum_{i=1}^{n} \alpha_i = \frac{1}{M}\sum_{i=1}^{n} ||\alpha_i x_i|| \leqq \left\|\sum_{i=1}^{n} \alpha_i x_i\right\|,$$

i.e., $\{x_n\}$ is of type l_+, which completes the proof.

In particular, the bases equivalent to the unit vector basis of the space l^1 can be characterized as follows.

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THEOREM 4. *A normalized basis $\{x_n\}$ of a Banach space E is equivalent to the unit vector basis of l^1 if and only if the associated cone $K = K_{\{x_n\}}$ is generating and has a bounded base.*

Proof. The cone $K_{\{e_n\}}$ associated with the unit vector basis $\{e_n\}$ of l^1 is generating and by Theorem 3 it has a bounded base. Therefore, the cone $K_{\{x_n\}}$ associated with any basis $\{x_n\}$ equivalent to $\{e_n\}$ has the same properties. Conversely, assume that $\{x_n\}$ is a normalized basis such that $K = K_{\{x_n\}}$ is generating and has a bounded base B. Then by Proposition 2, K is normal, whence, since it is also generating, $\{x_n\}$ is an unconditional basis (by Theorem 1, or by (**6**)). On the other hand, by Theorem 3, $\{x_n\}$ is of type l_+. Consequently (**11**, p. 353, Remark 1), $\{x_n\}$ is equivalent to the unit vector basis of l^1, which completes the proof.

The sufficiency part of Theorem 4 can also be proved using Proposition 3, as follows. By Theorem 3, $\{x_i\}$ is of type l_+. Let $x \in E$ be arbitrary. Then, since K is generating, $x = y - z$, with $y, z \in K$, whence, by Proposition 3,

$$\sum_{i=1}^{\infty} |f_i(x)| \leqq \sum_{i=1}^{\infty} |f_i(y)| + \sum_{i=1}^{\infty} |f_i(z)| = \sum_{i=1}^{\infty} ||f_i(y)x_i|| + \sum_{i=1}^{\infty} ||f_i(z)x_i|| < \infty .$$

The converse implication ($\sum_{i=1}^{\infty} |\alpha_i| < \infty \Rightarrow \sum_{i=1}^{\infty} \alpha_i x_i$ converges) being obvious (since $\{x_n\}$ is normalized and E is complete), $\{x_n\}$ is equivalent to the unit vector basis of l^1, which completes the proof.

We shall call a subset B of a cone K in a Banach space E a *hyperbase* of K if there exists a strictly positive functional $f \in E^*$ (i.e. $f(x) > 0$ for all $x \in K \sim \{0\}$) such that $B = \{y \in K | f(y) = 1\}$. It is immediate that every hyperbase is a base, but the converse is not true, even for compact bases, as shown by the following example. Consider in the space $E = l^2$ the compact convex set

$$Q = \{x = \{\xi_n\} \in l^2 | \, |\xi_j| \leqq 1/j \ (j = 1, 2, \ldots)\}.$$

Then the linear subspace $G = \bigcup_{n=1}^{\infty} nQ$ spanned by Q is dense in $E = l^2$ (since it contains all almost zero sequences), but does not coincide with E (since otherwise by the theorem of Baire (**2**, p. 20, Theorem 9) some $n_0 Q$ would have an interior point, in contradiction with $\dim E = \infty$). Take an arbitrary $x \in E \sim G$ and put

$$K = \{\lambda(y - x) | \, y \in Q, \lambda \geqq 0\}.$$

Then one can show that K is a cone and $B = Q - x = \{y - x | \, y \in Q\}$ is a compact base of K, but not a hyperbase of K.

If (x_n, f_n) ($\{x_n\} \subset E$, $\{f_n\} \subset E^*$) is a biorthogonal system (respectively, if $\{x_n\}$ is a normalized basis of E) then for every sequence $\{\epsilon_n\}$ with $\epsilon_n = \pm 1$ ($n = 1, 2, \ldots$), the sequence $(\epsilon_n x_n, \epsilon_n f_n)$ is also a biorthogonal system (respectively, $\{\epsilon_n x_n\}$ is also a normalized basis of E). One can therefore consider the associated cone

$$K^{\{\epsilon_n\}} = K_{(\epsilon_n x_n, \epsilon_n f_n)} = \{x \in E | \, \epsilon_n f_n(x) \geqq 0 \ (n = 1, 2, \ldots)\}$$

(respectively, $K^{\{\epsilon_n\}} = K_{\{\epsilon_n x_n\}}$), and a hyperbase $B^{\{\epsilon_n\}}$ of $K^{\{\epsilon_n\}}$. This will permit us to characterize geometrically some other classes of bases in Banach spaces. We shall use the notation

$$B_n^{\{\epsilon_j\}} = B^{\{\epsilon_j\}} \cap [x_n, x_{n+1}, x_{n+2}, \ldots] \qquad (n = 1, 2, \ldots).$$

We recall (**11**, p. 354) that a normalized basis $\{x_n\}$ of a Banach space E is said to be (a) of type P*, if $\sup_n ||\sum_{i=1}^n f_i|| < \infty$, where $\{f_n\}$ is the associated sequence of coefficient functionals; (b) of type al_+, if there exists a sequence $\{\epsilon_n\}$, where $\epsilon_n = \pm 1$ $(n = 1, 2, \ldots)$, such that $\{\epsilon_n x_n\}$ is of type l_+; (c) of type wc_0, if

$$x_n \xrightarrow{w} 0$$

(i.e. $f(x_n) \to 0$ for all $f \in E^*$).

PROPOSITION 4. *A normalized basis* $\{x_n\}$ *of a Banach space E is*

(a) *of type* P*, *if and only if there exists a hyperbase B of K containing all* x_n $(n = 1, 2, \ldots)$;

(b) *not of type* al_+, *if and only if for every* $\{\epsilon_n\}$, $\epsilon_n = \pm 1$ *and for every hyperbase* $B^{\{\epsilon_n\}}$ *of the cone* $K^{\{\epsilon_n\}}$ *the (unique) numbers* $\lambda_n > 0$ *for which* $B^{\{\epsilon_n\}} \supset \{\lambda_n \epsilon_n x_n\}$ *satisfy* $\sup_n \lambda_n = \infty$;

(c) *of type* wc$_0$, *if and only if for every* $\{\epsilon_n\}$, $\epsilon_n = \pm 1$, *and every hyperbase* $B^{\{\epsilon_n\}}$ *of the cone* $K^{\{\epsilon_n\}}$, *the (unique) numbers* $\lambda_n > 0$ *for which* $B^{\{\epsilon_n\}} \supset \{\lambda_n \epsilon_n x_n\}$ *satisfy* $\lim_{n\to\infty} \lambda_n = \infty$;

(d) *shrinking, only if for every* $\{\epsilon_n\}$, $\epsilon_n = \pm 1$, *and every hyperbase* $B^{\{\epsilon_n\}}$ *of the cone* $K^{\{\epsilon_n\}}$ *we have* $\operatorname{dist}(0, B_n^{\{\epsilon_j\}}) \to \infty$ *as* $n \to \infty$.

Proof. (a) If $\{x_n\}$ is of type P*, then by (**11**, Proposition 3), there exists an $f \in E^*$ such that $f(x_n) = 1$ $(n = 1, 2, \ldots)$. Put $B = \{y \in K \mid f(y) = 1\}$. Then B is a hyperbase of K containing all x_n $(n = 1, 2, \ldots)$. Conversely, if B is a hyperbase of K such that $x_n \in B$ $(n = 1, 2, \ldots)$, then there exists an $f \in E^*$ such that $B = \{y \in K \mid f(y) = 1\}$. Then $f(x_n) = 1$ $(n = 1, 2, \ldots)$ and therefore, by (**11**, Proposition 3), $\{x_n\}$ is of type P*.

(b) If $\{x_n\}$ is not of type al_+, then by (**11**, Proposition 1, we have $\inf_n |f(x_n)| = 0$ $(f \in E^*)$. Let $\epsilon_n = \pm 1$ $(n = 1, 2, \ldots)$ and let $B^{\{\epsilon_n\}}$ be an arbitrary hyperbase of the cone $K^{\{\epsilon_n\}}$. Then there exists $f \in E^*$ such that $B^{\{\epsilon_n\}} = \{y \in K^{\{\epsilon_n\}} \mid f(y) = 1\}$, whence $(1/f(\epsilon_n x_n))\epsilon_n x_n \in B^{\{\epsilon_n\}}$ and thus

$$\lambda_n = 1/f(\epsilon_n x_n) \qquad (n = 1, 2, \ldots),$$

whence $\sup_n \lambda_n = \infty$. Conversely, if $\{x_n\}$ is of type al_+, then by (**11**, Proposition 1), there exists an $f \in E^*$ such that $|f(x_n)| \geqq 1$ $(n = 1, 2, \ldots)$. Put $\epsilon_n = \operatorname{sign} f(x_n)$. Then $f(\epsilon_n x_n) \geqq 1$ $(n = 1, 2, \ldots)$, whence the set $B^{\{\epsilon_n\}} = \{y \in K^{\{\epsilon_n\}} \mid f(y) = 1\}$ is a hyperbase of the cone $K^{\{\epsilon_n\}}$ and

$$(1/f(\epsilon_n x_n))\epsilon_n x_n \in B^{\{\epsilon_n\}} \qquad (n = 1, 2, \ldots),$$

whence $\lambda_n = 1/f(\epsilon_n x_n) \leqq 1$ $(n = 1, 2, \ldots)$.

(c) The proof is similar to that of (b), with slightly more computation in the converse part.

(d) If $\{x_n\}$ is shrinking, let $\epsilon_n = \pm 1$ $(n = 1, 2, \ldots)$ and let $B^{\{\epsilon_n\}}$ be an arbitrary hyperbase of the cone $K^{\{\epsilon_n\}}$. Then there exists an $f \in E^*$ such that $B^{\{\epsilon_n\}} = \{y \in K^{\{\epsilon_n\}} | f(y) = 1\}$, whence for any $y \in B_n^{\{\epsilon_j\}}$,

$$y = \sum_{i=n}^{\infty} \alpha_i x_i, \qquad \alpha_i \geqq 0 \ (i = n, n+1, \ldots)$$

we have

$$\frac{1}{||y||} = f\left(\frac{y}{||y||}\right) < \epsilon \quad \text{for } n > N_{(\epsilon)}$$

(since $\{x_n\}$ is shrinking). Therefore $||y|| > 1/\epsilon$ for all $y \in B_n^{\{\epsilon_j\}}$ whenever $n > N(\epsilon)$, which completes the proof.

Remark 5. For a biorthogonal system $(x_i, f_i\}$ with $\{f_i\}$ total, if E is the closed linear span of $\{x_i\}$ and W is the closure of the set

$$\left\{ \sum_{i=1}^{n} a_i x_i : a_i \geqq 0, \, i = 1, 2, \ldots ; n = 1, 2, \ldots \right\},$$

Schaeffer (**10**, p. 139; **9**, p. 251) has shown that $\{x_i, f_i\}$ is an unconditional basis for E if and only if W is a normal b-cone.

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The Romanian Academy of Sciences,
Bucharest, Rumania;
The Florida State University,
Tallahassee, Florida

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