Proceedings of the Edinburgh Mathematical Society (2025) **68**, 1–15
doi:10.1017/S0013091524000683

SIMPLICITY OF NON-ASSOCIATIVE SKEW LAURENT POLYNOMIAL RINGS

PER BÄCK[1] AND JOHAN RICHTER[2]

[1] *Division of Mathematics and Physics, Mälardalen University, Västerås, Sweden*
[2] *Department of Mathematics and Natural Sciences, Blekinge Institute of Technology, Karlskrona, Sweden*

Corresponding author: Per Bäck, email: per.back@mdu.se

(Received 5 April 2024)

Abstract We introduce non-associative skew Laurent polynomial rings and characterize when they are simple. Thereby, we generalize results by Jordan, Voskoglou, and Nystedt and Öinert.

Keywords: non-associative skew Laurent polynomial rings; simplicity

2020 Mathematics subject classification: Primary 17A99; 17D99
Secondary 16S35

1. Introduction

In 1903, Hilbert [4] introduced a ring of formal Laurent series with a *skewed* or *twisted* multiplication to show the existence of a non-commutative ordered division ring. Nowadays, the rings and their corresponding multiplication are thus referred to as *skew* or *twisted Laurent series rings* and *Hilbert's twist* [6], respectively. Thirty years later, Ore [12] initiated the study of what he called 'non-commutative polynomial rings', today more commonly known as *Ore extensions*. Since their introductions, skew Laurent series rings, Ore extensions and the closely related *skew Laurent polynomial rings* have been studied quite extensively (see e.g. [3, 6, 7] for comprehensive introductions). Moreover, some years ago, Nystedt, Öinert and Richter [10] introduced a non-associative generalization of Ore extensions.

In this article, we introduce a non-associative generalization of skew Laurent polynomial rings and characterize when such rings are simple. Thereby, we extend results on simplicity of skew Laurent polynomial rings by Jordan [5] (see Theorem 2.3) and Voskoglou [14] (see Theorem 2.4) to the non-associative setting (Theorem 4.1 and

Theorem 4.2, respectively). Moreover, our construction of *non-associative skew Laurent polynomial rings* is a generalization of that by Nystedt and Öinert [9], and we obtain a generalization (Theorem 4.2) of a simplicity result by them (see Theorem 2.2).

The article is organized as follows:

In § 2, we provide conventions and preliminaries from non-associative ring theory (§ 2.1). We also recall some results about graded non-associative rings (§ 2.2) and remind what skew Laurent polynomial rings are (§ 2.3).

In § 3, we introduce non-associative skew Laurent polynomial rings and examples thereof.

In § 4, we characterize when non-associative skew Laurent polynomial rings are simple. We then apply our results to the examples introduced in § 3.

2. Preliminaries

2.1. Non-associative ring theory

We denote by $\mathbb{N}$ the natural numbers, including zero. By a *non-associative ring*, we mean a unital ring which is not necessarily associative. If R is a non-associative ring, by a *left R-module*, we mean an additive group M equipped with a biadditive map $R \times M \to M$, $(r, m) \mapsto rm$ for any $r \in R$ and $m \in M$. A subset B of M is a basis if for any $m \in M$, there are unique $r_b \in R$ for $b \in B$, such that $r_b = 0$ for all but finitely many $b \in B$, and $m = \sum_{b \in B} r_b b$. A left R-module that has a basis is called *free*.

For a non-associative ring R, the *commutator* is the function $[\cdot,\cdot]\colon R \times R \to R$ defined by $[r, s] = rs - sr$ for any $r, s \in R$. The *commuter* of R, denoted by $C(R)$, is the additive subgroup $\{r \in R\colon [r, s] = 0 \text{ for all } s \in R\}$ of R. The *associator* is the function $(\cdot,\cdot,\cdot)\colon R \times R \times R \to R$ defined by $(r, s, t) = (rs)t - r(st)$ for all $r, s, t \in R$. Using the associator, we define three sets: the *left nucleus* of R, $N_l(R) := \{r \in R\colon (r, s, t) = 0 \text{ for all } s, t \in R\}$, the *middle nucleus* of R, $N_m(R) := \{s \in R\colon (r, s, t) = 0 \text{ for all } r, t \in R\}$, and the *right nucleus* of R, $N_r(R) := \{t \in R\colon (r, s, t) = 0 \text{ for all } r, s \in R\}$. From the so-called associator identity

$$u(r, s, t) + (u, r, s)t + (u, rs, t) = (ur, s, t) + (u, r, st)$$

which holds for all $r, s, t, u \in R$, it follows that $N_l(R)$, $N_m(R)$ and $N_r(R)$ are all associative subrings of R. We also define the *nucleus* of R, $N(R) := N_l(R) \cap N_m(R) \cap N_r(R)$, and the *centre* of R, $Z(R) := C(R) \cap N(R)$.

The next two propositions are standard results in non-associative ring theory (see, e.g., the proofs of [10, Proposition 2.1 and Proposition 2.3]).

Proposition 2.1. *Let R be a non-associative ring. Then the following equalities hold:*

(i) $Z(R) = C(R) \cap N_l(R) \cap N_m(R)$;
(ii) $Z(R) = C(R) \cap N_l(R) \cap N_r(R)$;
(iii) $Z(R) = C(R) \cap N_m(R) \cap N_r(R)$.

Proposition 2.2. *If R is a simple non-associative ring, then $Z(R)$ is a field.*

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

Let R be a non-associative ring. Take $u \in R$. Recall that u is said to be left (right) invertible if there is $v \in R$ ($w \in R$) such that $vu = 1$ ($uw = 1$); in that case v (or w) is called a left (or right) inverse of u. We let $R^\times$ denote the set of elements of R that are both left and right invertible.

Remark 2.1. Suppose $u \in N_m(R) \cap R^\times$. It is easy to show that u has a unique left inverse v, u has a unique right inverse w and $v = w$. We let u^{-1} denote the element $v = w$.

The following small result should be known. However, we have not been able to find a reference, and so we provide a proof of it.

Lemma 2.1. *Let R be a non-associative ring and let $u \in N_m(R) \cap R^\times$. Then the following assertions hold:*

(i) *If $u \in N_l(R)$, then $u^{-1} \in N_l(R)$;*
(ii) *If $u \in N_r(R)$, then $u^{-1} \in N_r(R)$;*
(iii) *If $u \in N_r(R)$ and $\left(u, u^{-1}, R\right) = \{0\}$, then $u^{-1} \in N_m(R) \cap N_r(R)$.*

Proof. Let $r, s \in R$ and $u \in N_m(R) \cap R^\times$. By Remark 2.1, u has a unique two-sided inverse u^{-1}, so the statement makes sense. Now we have the following:

(i) Let $u \in N_l(R)$. Then

$$\begin{aligned} u^{-1}(rs) &= u^{-1}\left(\left(\left(uu^{-1}\right)r\right)s\right) = u^{-1}\left(\left(u\left(u^{-1}r\right)\right)s\right) = u^{-1}\left(u\left(\left(u^{-1}r\right)s\right)\right) \\ &= \left(u^{-1}u\right)\left(\left(u^{-1}r\right)s\right) = \left(u^{-1}r\right)s. \end{aligned}$$

(ii) Let $u \in N_r(R)$. Then

$$\begin{aligned} r\left(su^{-1}\right) &= \left(r\left(su^{-1}\right)\right)\left(uu^{-1}\right) = \left(\left(r\left(su^{-1}\right)\right)u\right)u^{-1} = \left(r\left(\left(su^{-1}\right)u\right)\right)u^{-1} \\ &= \left(r\left(s\left(u^{-1}u\right)\right)\right)u^{-1} = (rs)u^{-1}. \end{aligned}$$

(iii) Let $u \in N_r(R)$ and $(u, u^{-1}, R) = \{0\}$. By (ii), $u^{-1} \in N_r(R)$, and

$$\begin{aligned} r\left(u^{-1}s\right) &= \left(r\left(u^{-1}u\right)\right)\left(u^{-1}s\right) = \left(\left(ru^{-1}\right)u\right)\left(u^{-1}s\right) = \left(ru^{-1}\right)\left(u\left(u^{-1}s\right)\right) \\ &= \left(ru^{-1}\right)\left(\left(uu^{-1}\right)s\right) = \left(ru^{-1}\right)s. \end{aligned}$$

□

For a general introduction to non-associative algebra, see, e.g., Schafer's book [13].

2.2. Graded non-associative rings

In [9], Nystedt and Öinert study group-graded non-associative rings. Recall that a non-associative ring R is said to be *graded by a group* G, or G*-graded*, if there is a collection of additive subgroups $\{R_g\}_{g\in G}$ of R, called *homogenous components*, such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ hold for all $g, h \in G$. An ideal I of R is called *graded* if

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

$I = \bigoplus_{g\in G} I \cap R_g$. The ring R is said to be *graded simple* if the only graded ideals of R are $\{0\}$ and R. The group G is called *hypercentral* if every non-trivial factor group of G has a non-trivial centre. In particular, all abelian groups are hypercentral.

With the terminology introduced above, the authors then prove the following theorem:

Theorem 2.1 ([9, Theorem 4]). *If a non-associative ring is graded by a hypercentral group, then the ring is simple if and only if it is graded simple and the centre of the ring is a field.*

If R is a G-graded non-associative ring, then we define $\mathrm{Supp}(R) := \{g \in G : R_g \neq \{0\}\}$. The ring R is said to be *faithfully G-graded* if for any $g, h \in \mathrm{Supp}(R)$ and non-zero $r \in R_g$, we have $rR_h \neq \{0\} \neq R_h r$. Denoting the identity element of G by e, the authors then use Theorem 2.1 to prove the following result:

Corollary 2.1 ([9, Corollary 32]). *If R is a faithfully G-graded ring with $\mathrm{Supp}(R) = G$, where G is a torsion-free hypercentral group, then R is simple if and only if R is graded simple and $Z(R) \subseteq R_e$ holds.*

Let $\sigma_1, \ldots, \sigma_n$ be automorphisms of R. We say that an ideal I of R is a $(\sigma_1, \ldots, \sigma_n)$-*ideal* if $\sigma_i(I) = I$ holds for any $i \in \{1, \ldots, n\}$. Moreover, R is said to be $(\sigma_1, \ldots, \sigma_n)$-*simple* if $\{0\}$ and I are the only $(\sigma_1, \ldots, \sigma_n)$-ideals of R. Note that the definition of $(\sigma_1, \ldots, \sigma_n)$-simplicity in [9] contains a mistake and should be the same as the one just given, which has also been confirmed by its authors. In loc. cit., non-associative skew Laurent polynomial rings are then introduced as a class of *non-associative skew group rings*. With the corrected definition of $(\sigma_1, \ldots, \sigma_n)$-simplicity above, the authors then prove the following theorem:

Theorem 2.2 ([9, Theorem 52]). *Let R be a non-associative ring with pairwise commuting automorphisms $\sigma_1, \ldots, \sigma_n$. Then $R[X_1^{\pm}, \ldots, X_n^{\pm}; \sigma_1, \ldots, \sigma_n]$ is simple if and only if R is $(\sigma_1, \ldots, \sigma_n)$-simple and there do not exist $u \in N(R)^{\times}$ and a non-zero $(m_1, \ldots, m_n) \in \mathbb{Z}^n$, such that for all $r \in R$ and $i \in \{1, \ldots, n\}$, the following equalities hold:*

(i) $(\sigma_1^{m_1} \circ \cdots \circ \sigma_n^{m_n})(r) = u^{-1} r u$;
(ii) $\sigma_i(u) = u$.

In § 3, we will generalize the above construction of non-associative skew Laurent polynomial rings, and in § 4, prove that a generalization of Theorem 2.2 holds for them.

2.3. Skew Laurent polynomial rings

Let us recall the definition of (associative) skew Laurent polynomial rings.

Definition 2.1 (Skew Laurent polynomial ring). *Let S be a ring, R a subring of S containing the multiplicative identity element 1 and $x \in S^{\times}$. Then S is called a skew Laurent polynomial ring of R if the following axioms hold:*

($S1$) S is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$;

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

(S2) $xR = Rx$;
(S3) S *is associative.*

To construct skew Laurent polynomial rings, one considers *generalized Laurent polynomial rings* $R[X^{\pm};\sigma]$ where R is an associative ring and $\sigma\colon R \to R$ is an automorphism. As an additive group, $R[X^{\pm};\sigma]$ equals the ordinary Laurent polynomial ring $R[X^{\pm}]$. The multiplication in $R[X^{\pm};\sigma]$ is then defined by the biadditive extension of the relations

$$(rX^m)\,(sX^n) = (r\sigma^m(s))X^{m+n} \tag{2.1}$$

for any $r, s \in R$ and $m, n \in \mathbb{Z}$. In particular, the product (2.1) makes $R[X^{\pm};\sigma]$ a $\mathbb{Z}$-graded ring and a Laurent polynomial ring of R with $x = X$ (see, e.g., the proof of Proposition 3.2). Moreover, every skew Laurent polynomial ring of R is isomorphic to a generalized Laurent polynomial ring $R[X^{\pm};\sigma]$ (see, e.g., the proof of Proposition 3.3).

The next theorem, due to Jordan [5], characterizes when generalized Laurent polynomial rings are simple:

Theorem 2.3 ([5, Theorem O]). *Let R be an associative ring with a ring automorphism σ. Then $R[X^{\pm};\sigma]$ is simple if and only if R is σ-simple and there do not exist $u \in R^{\times}$ and a non-zero $n \in \mathbb{Z}$, such that for all $r \in R$, the following equalities hold:*

(i) $\sigma^n(r) = u^{-1}ru$;
(ii) $\sigma(u) = u$.

If $\sigma_1, \ldots, \sigma_n$ are pairwise commuting automorphisms, then we may construct an iterated generalized Laurent polynomial ring of R as follows (see also Exercise 1W in [3]). First, we set $S_1 := R[X_1^{\pm};\sigma_1]$. Then σ_2 extends to a ring automorphism $\widehat{\sigma}_2$ on S_1, defined by $\widehat{\sigma}_2(rX_1^m) = \sigma_2(r)X_1^m$ for any $m \in \mathbb{Z}$. Next, we set $S_2 := S_1[X_2^{\pm};\widehat{\sigma}_2]$. Once S_i has been constructed for some $i < n$, we define $S_{i+1} := S_i[X_{i+1}^{\pm};\widehat{\sigma}_{i+1}]$ where $\widehat{\sigma}_{i+1}$ is the ring automorphism on S_i defined by $\widehat{\sigma}_{i+1}(rX_1^{m_1}\cdots X_n^{m_n}) = \sigma_{i+1}(r)X_1^{m_1}\cdots X_n^{m_n}$. We may now construct an iterated generalized Laurent polynomial ring $R[X_1^{\pm};\sigma_1]\cdots[X_n^{\pm};\widehat{\sigma}_n]$, which we denote by $R[X_1^{\pm},\ldots,X_n^{\pm};\sigma_1,\ldots,\sigma_n]$.

Voskoglou [14] has generalized Theorem 2.3 and characterized when iterated generalized Laurent polynomial rings are simple:

Theorem 2.4 ([14, Corollary 3.7]). *Let R be an associative ring with pairwise commuting automorphisms $\sigma_1, \ldots, \sigma_n$. Then $R[X_1^{\pm},\ldots,X_n^{\pm};\sigma_1,\ldots,\sigma_n]$ is simple if and only if R is $(\sigma_1,\ldots,\sigma_n)$-simple and there do not exist $u \in R^{\times}$ and a non-zero $(m_1,\ldots,m_n) \in \mathbb{Z}^n$, such that for all $r \in R$ and $i \in \{1,\ldots,n\}$, the following equalities hold:*

(i) $(\sigma_1^{m_1} \circ \cdots \circ \sigma_n^{m_n})(r) = u^{-1}ru$;
(ii) $\sigma_i(u) = u$.

We note that Theorem 2.4 is the special case of Theorem 2.2 when R is associative.

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

3. Non-associative skew Laurent polynomial rings

We wish to define non-associative skew Laurent polynomial rings in an analogous fashion to how non-associative Ore extensions in [10] are defined; hence, we follow the same line of reasoning as in loc. cit. First, we note that the product (2.1) equips the additive group $R[X^{\pm};\sigma]$ of generalized Laurent polynomials over any non-associative ring R with a non-associative ring structure for any additive bijection σ on R respecting 1. In order to define non-associative skew Laurent polynomial rings, we, therefore, wish to adopt the axioms (S1), (S2) and (S3) so that these rings still correspond to the above generalized Laurent polynomial rings. We suggest the following definition:

Definition 3.1 (Non-associative skew Laurent polynomial ring). *Let S be a non-associative ring, R a subring of S containing the multiplicative identity element* 1 *and $x \in S^{\times}$. Then S is called a non-associative skew Laurent polynomial ring of R if the following axioms hold:*

(N1) S is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$;
(N2) $xR = Rx$;
(N3) $(S,S,x) = (S,x,S) = \{0\}$.

Note that x^{-1} in the above definition does indeed exist by (N3) and Remark 2.1. Moreover, (N3) together with (ii) in Lemma 2.1 ensure that the elements x and x^{-1} are power associative, so that x^m is well-defined for any $m \in \mathbb{Z}$.

Let R be a non-associative ring. We denote by $R[X^{\pm};]$ the set of formal sums $\sum_{i\in\mathbb{Z}} r_i X^i$ where $r_i \in R$ is zero for all but finitely many $i \in \mathbb{Z}$, equipped with pointwise addition. Now, let σ be an additive bijection on R respecting 1. The *generalized Laurent polynomial ring* $R[X^{\pm};\sigma]$ over R is defined as the additive group $R[X^{\pm}]$ with multiplication defined by (2.1). One readily verifies that this makes $R[X^{\pm};\sigma]$ a $\mathbb{Z}$-graded non-associative ring.

Remark 3.1. If R is a non-associative ring with an additive bijection σ that respects 1, then $R[X^{\pm};\sigma]$ is a so-called *Ore monoid ring* $R[G;\pi]$ as introduced in [11]. Here $G = \mathbb{Z}$ and $\pi = \{\pi_b^a\}_{a,b\in G}$ where $\pi_b^a = \sigma^a$ if $a = b$ and $\pi_b^a = 0$ otherwise.

Proposition 3.1. *Let R be a non-associative ring with an additive bijection σ that respects 1. If $S = R[X^{\pm};\sigma]$, then $X^n \in N_m(S) \cap N_r(S)$ for any $n \in \mathbb{Z}$.*

Proof. This is a special case of [11, Proposition 3]. □

Proposition 3.2. *Let R be a non-associative ring with an additive bijection σ that respects 1. Then $R[X^{\pm};\sigma]$ is a non-associative skew Laurent polynomial ring of R with $x = X$.*

Proof. $R[X^{\pm};\sigma]$ is clearly a non-associative ring, and R can be identified with a subring of $R[X^{\pm};\sigma]$ that contains the multiplicative identity element. X has a two-sided inverse, and so we only need to verify that the axioms (N1), (N2) and (N3) hold. For the proof that (N1) holds, we refer the reader to the proof of [10, Proposition 3.2] which is nearly identical. For (N2), we have $XR = (1X)\left(RX^0\right) = \sigma(R)X \subseteq RX$ and

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

$RX = \sigma\left(\sigma^{-1}(R)\right)X = (1X)\left(\sigma^{-1}(R)X^0\right) = X\sigma^{-1}(R) \subseteq XR$. That (N3) holds follows from Proposition 3.1. □

Proposition 3.3. *Every non-associative skew Laurent polynomial ring of R is isomorphic to a generalized Laurent polynomial ring $R[X^{\pm};\sigma]$.*

Proof. The proof is similar to the proof of [10, Proposition 3.3]. However, computations are a bit more involved, and hence we provide it here for the convenience of the reader.

Let R be a non-associative ring, S a skew Laurent polynomial ring of R defined by x, and $r, s \in R$. Then, from (N1) and (N2), $xr = \sigma(r)x$ for some unique coefficient $\sigma(r) \in R$. Moreover, from (N2), $rx = x\overline{\sigma}(r)$ for some $\overline{\sigma}(r) \in R$, and since $x \in N_m(S)$ from (N3), $x^{-1}(rx) = x^{-1}(x\overline{\sigma}(r)) = (x^{-1}x)\overline{\sigma}(r) = 1\overline{\sigma}(r) = \overline{\sigma}(r)1$. Using (N1), $\overline{\sigma}(r)$ is then unique, and hence σ and $\overline{\sigma}$ define functions $\sigma\colon R \to R$ and $\overline{\sigma}\colon R \to R$. We have $rx = x\overline{\sigma}(r) = \sigma(\overline{\sigma}(r))x$, and so by (N1), $\sigma \circ \overline{\sigma} = \mathrm{id}_R$. We also have that $xr = \sigma(r)x = x\overline{\sigma}(\sigma(r))$. Using that $x \in N_m(S)$, $r = (x^{-1}x)r = x^{-1}(xr) = x^{-1}(x\overline{\sigma}(\sigma(r))) = (x^{-1}x)\overline{\sigma}(\sigma(r)) = \overline{\sigma}(\sigma(r))$. Hence $\overline{\sigma} \circ \sigma = \mathrm{id}_R$, and so we can conclude that σ is bijective with $\sigma^{-1} = \overline{\sigma}$. Since $rx = x\sigma^{-1}(r)$, we have $(x^{-1}r)x = x^{-1}(rx) = x^{-1}(x\sigma^{-1}(r)) = (x^{-1}x)\sigma^{-1}(r) = \sigma^{-1}(r)$ and therefore $x^{-1}r = (x^{-1}r)(xx^{-1}) = ((x^{-1}r)x)x^{-1} = \sigma^{-1}(r)x^{-1}$. Now, on the one hand $x(r+s) = \sigma(r+s)x$, and on the other hand, $x(r+s) = xr + xs = \sigma(r)x + \sigma(s)x$ by distributivity. Hence, by (N1), σ is additive. Since the multiplicative identity element in R is the multiplicative identity element 1 in S by assumption, $1x = x1 = \sigma(1)x$, and so by (N1), $\sigma(1) = 1$.

We claim that $(rx^m)(sx^n) = (r\sigma^m(s))x^{m+n}$ for any $m, n \in \mathbb{Z}$. To prove this, as an intermediate step, let us first show that $x(x^{-1}u) = u$ for any $u \in S$. Indeed, by (N1), we may set $u = \sum_{i\in\mathbb{Z}} u_i x^i$ for some $u_i \in R$. Then $x(x^{-1}u) = x\left(x^{-1}\left(\sum_{i\in\mathbb{Z}} u_i x^i\right)\right) = x\left(\sum_{i\in\mathbb{Z}} x^{-1}(u_i x^i)\right) = x\left(\sum_{i\in\mathbb{Z}}(x^{-1}u_i)x^i\right) = x\left(\sum_{i\in\mathbb{Z}}(\sigma^{-1}(u_i)x^{-1})x^i\right) = x\left(\sum_{i\in\mathbb{Z}}\sigma^{-1}(u_i)(x^{-1}x^i)\right) = x\left(\sum_{i\in\mathbb{Z}}\sigma^{-1}(u_i)x^{i-1}\right) = \sum_{i\in\mathbb{Z}} x(\sigma^{-1}(u_i)x^{i-1}) = \sum_{i\in\mathbb{Z}}(x\sigma^{-1}(u_i))x^{i-1} = \sum_{i\in\mathbb{Z}}(u_i x)x^{i-1} = \sum_{i\in\mathbb{Z}} u_i x^i = u$. Since $x \in N_m(S) \cap N_r(S)$ by assumption, from (iii) in Lemma 2.1, it now follows that $x^{-1} \in N_m(S) \cap N_r(S)$. Recall from § 2.1 that $N_m(S)$ and $N_r(S)$ are rings, and hence they are closed under multiplication. Therefore $x^n \in N_m(S) \cap N_r(S)$ for any $n \in \mathbb{Z}$.

Now, let us return to the proof of the equality $(rx^m)(sx^n) = (r\sigma^m(s))x^{m+n}$ for any $m, n \in \mathbb{Z}$. First, we prove by induction on m that $x^m s = \sigma^m(s)x^m$. The base case $m = 0$ is immediate. We now split the induction step into two cases. First, assume that $m \leq 0$ and set $p = -m$. We have $x^{-(p+1)}s = x^{m-1}s = (x^m x^{-1})s = x^m(x^{-1}s) = x^m(\sigma^{-1}(s)x^{-1}) = (x^m\sigma^{-1}(s))x^{-1} = (\sigma^m(\sigma^{-1}(s))x^m)x^{-1} = \sigma^{m-1}(s)x^{m-1} = \sigma^{-(p+1)}(s)x^{-(p+1)}$, which completes the negative part of the induction step. Now assume that $m \geq 0$. Then $x^{m+1}s = (x^m x)s = x^m(xs) = x^m(\sigma(s)x) = (x^m\sigma(s))x = (\sigma^m(\sigma(s))x^m)x = \sigma^{m+1}(s)x^{m+1}$, which completes the positive part of the induction step. Since $x^m, x^n \in N_m(S) \cap N_r(S)$, we get $(rx^m)(sx^n) = r(x^m(sx^n)) = r((x^m s)x^n) = r((\sigma^m(s)x^m)x^n) = r(\sigma^m(s)(x^m x^n)) = r(\sigma^m(s)x^{m+n}) = (r\sigma^m(s))x^{m+n}$.

Last, define a function $f\colon S \to R[X^{\pm};\sigma]$ by the additive extension of the relations $f(rx^m) = rX^m$ for any $r \in R$ and $m \in \mathbb{Z}$. Then f is an isomorphism of additive groups, and moreover, for any $r, s \in R$ and $m, n \in \mathbb{Z}$, $f\left((rx^m)(sx^n)\right) = f\left((r\sigma^m(s))x^{m+n}\right) = (r\sigma^m(s))X^{m+n} = (rX^m)(sX^n) = f(rx^m)f(sx^n)$. □

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

Proposition 3.4. *Let R be a non-associative ring with an additive bijection σ that respects 1. If $S = R[X^{\pm};\sigma]$, then*

(i) *$R \subseteq N_l(S)$ if and only if R is associative;*
(ii) *$X \in N_l(S)$ if and only if σ is an automorphism;*
(iii) *S is associative if and only if R is associative and σ is an automorphism.*

Proof.

(i) This follows from [11, Proposition 7].
(ii) By [11, Proposition 7], we know that $(X, S, S) \subseteq (X, R, R)S$. So it is enough to prove that $(X, R, R) = \{0\}$ if and only if σ is an automorphism. However, if $r, s \in R$, then the condition $X(rs) = (Xr)s$ is clearly equivalent to $\sigma(rs) = \sigma(r)\sigma(s)$.
(iii) The conditions are clearly necessary. Conversely, if they are satisfied then also $X^{-1} \in N(S)$, so S is generated by elements that belong to $N(S)$ and must be associative.

□

Example 3.1. On the ring $\mathbb{C}$ we can define $\sigma_q(a + bi) = a + qbi$ for any $a, b \in \mathbb{R}$ and $q \in \mathbb{R}^{\times}$. Then σ_q is an additive bijection that respects 1, and we can accordingly define $\mathbb{C}[X^{\pm};\sigma_q]$. Moreover, σ_q is a ring automorphism if and only if $q = \pm 1$, and so by (iii) in Proposition 3.4, $\mathbb{C}[X^{\pm};\sigma_q]$ is associative if and only if $q = \pm 1$.

Example 3.2. Let T be a non-associative ring and let $q \in Z(T)^{\times}$. Define a ring automorphism $\sigma_q \colon T[Y^{\pm}] \to T[Y^{\pm}]$ by the T-algebra extension of the relation $\sigma_q(Y) = qY$. The *non-associative quantum torus* over T is the generalized Laurent polynomial ring $T[Y^{\pm}][X^{\pm};\sigma_q]$. By (iii) in Proposition 3.6, $T[Y^{\pm}][X^{\pm};\sigma_q]$ is associative if and only if T is associative.

Example 3.3. Let $f \colon \mathbb{N} \to \mathbb{N}$ be any bijection with $f(0) = 0$. Suppose K is a field and put $R = K[[Y]]$. Let σ be the additive bijection on R defined by the K-linear and continuous (in the usual topology of formal power series rings) extension of the relations $\sigma(Y^n) = Y^{f(n)}$, if $n \neq 1$, and $\sigma(Y) = Y^{f(1)} + 1$. Note that σ respects 1 and is well-defined since $\lim_{n\to\infty} f(n) = \infty$. However, σ is not a ring homomorphism since $\sigma(Y^2) = Y^{f(2)}$ but $\sigma(Y)^2 = (Y^{f(1)} + 1)^2 = Y^{2f(1)} + 2Y^{f(1)} + 1 \neq Y^{f(2)}$. By (iii) in Proposition 3.6, $R[X^{\pm};\sigma]$ is not associative. Later (see (iii) in Corollary 4.2) we will use a specific choice of bijection denoted by g and defined by

$$g(n) = \begin{cases} 0 \text{ if } n = 0, \\ 2^k \text{ if } n = 2^{k+1} \text{ for some } k \in \mathbb{N}, \\ \min\{j \in \mathbb{N} \colon j > n \text{ and } j \text{ is not a power of } 2\} \text{ otherwise.} \end{cases}$$

Hence $g(1) = 3$, $g(2) = 1$, $g(3) = 5$ and so on. We denote the corresponding additive bijection by σ_g.

Recall that a *ring anti-automorphism* σ on a non-associative ring R is an additive bijection such that for any $r, s \in R$, $\sigma(rs) = \sigma(s)\sigma(r)$. In particular, $\sigma(1)\sigma(r) = \sigma(r1) =$

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

$\sigma(r) = \sigma(1r) = \sigma(r)\sigma(1)$, so by the uniqueness of 1, $\sigma(1) = 1$. Hence any ring anti-automorphism σ on R naturally gives rise to a generalized Laurent polynomial ring $R[X^{\pm};\sigma]$.

Lemma 3.1. *Let R be a non-associative ring with a ring anti-automorphism σ. Then $R[X^{\pm};\sigma]$ is associative if and only if R is associative and commutative.*

Proof. By (iii) in Proposition 3.6, $R[X^{\pm};\sigma]$ is associative if and only if R is associative and σ is a ring automorphism. We claim that σ is a ring automorphism if and only if R is commutative. It is clear that σ is a ring automorphism if R is commutative. To prove the converse, assume that σ is a ring automorphism. For any $r, s \in R$, $r = \sigma(r^prime)$ and $s = \sigma(s^prime)$ for some $r^prime, s^prime \in R$. Since σ is both a ring automorphism and a ring anti-automorphism, $rs = \sigma(r^prime)\sigma(s^prime) = \sigma(r^prime s^prime) = \sigma(s^prime)\sigma(r^prime) = sr$. □

Example 3.4. Let R be a non-associative ring and let σ be any of the $n!(n^2-n)!$ maps on the non-associative matrix ring $M_n(R)$ defined by permuting diagonal and non-diagonal elements separately. Then σ is an additive bijection that respects 1, and so we can define $M_n(R)[X^{\pm};\sigma]$. As a concrete example, one could, e.g., take σ to be matrix transpose, σ_T. Since σ_T is an anti-automorphism, by Lemma 3.1, $M_n(R)[X^{\pm};\sigma_T]$ is associative if and only if $n=1$ and R is commutative, or if R is the zero ring.

Let K be an associative and commutative ring, and let A be a non-associative K-algebra. Recall that an *involution* of A is a K-linear map $*\colon A \to A$, also written $a \mapsto a^*$, such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$ hold for any $a, b \in A$. In particular, $*$ is an anti-automorphism. A non-associative algebra with an involution $*$ is referred to as a non-associative $*$-*algebra* (for an introduction to $*$-algebras, see e.g. [8, Section 2.2]). If A is a non-associative $*$-algebra, then A naturally gives rise to a generalized Laurent polynomial ring $A[X^{\pm};*]$, which by Lemma 3.10 is associative if and only if A is associative and commutative.

Example 3.5. $M_n(\mathbb{C})$ with $*$ given by conjugate transpose is an associative $*$-algebra over $\mathbb{R}$ which is commutative if and only if $n=1$. Hence, by Lemma 3.10, $M_n(\mathbb{C})[X^{\pm};*]$ is associative if and only if $n=1$ (in which case we get the ring $\mathbb{C}[X^{\pm};\sigma_q]$ with $q=-1$ in Example 3.1).

Starting from any non-associative $*$-algebra, the so-called Cayley–Dickson construction gives a new non-associative $*$-algebra. In particular, by starting from the real numbers viewed as a real $*$-algebra with $* = \mathrm{id}_{\mathbb{R}}$ and then repeatedly applying the Cayley–Dickson construction, we get the following real, non-associative $*$-algebras where $*$ is given by conjugation: the complex numbers, the quaternions ($\mathbb{H}$), the octonions ($\mathbb{O}$) and so on. For more details on this construction, see e.g. [1].

Example 3.6. Let A be any of the real, non-associative $*$-algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \ldots$ with $*$ given by conjugation. Then A is commutative if and only if $A = \mathbb{R}$ or $\mathbb{C}$. By Lemma 3.10, $A[X^{\pm};*]$ is associative if and only if $A = \mathbb{R}$ or $\mathbb{C}$.

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

4. Simplicity

In this section, we examine when generalized Laurent polynomial rings are simple. If R is a non-associative ring with additive bijections $\sigma_1, \ldots, \sigma_n$ that respect 1, then, just as in the case of automorphisms in § 2.2, an ideal I of R is said to be a $(\sigma_1, \ldots, \sigma_n)$-*ideal* if $\sigma_i(I) = I$ holds for all $i \in \{1, \ldots, n\}$. The ring R is called σ-*simple* if $\{0\}$ and I are the only $(\sigma_1, \ldots, \sigma_n)$-ideals of R. Also recall from § 2.2 that an ideal I of a G-graded ring $R = \bigoplus_{g\in G} R_g$ is called graded if $I = \bigoplus_{g\in G} I \cap R_g$, and that R is said to be graded simple if the only graded ideals of R are $\{0\}$ and R.

With $R[X^{\pm}; \sigma]$ viewed as a $\mathbb{Z}$-graded ring, we have the following result:

Proposition 4.1. *Let R be a non-associative ring with an additive bijection σ that respects 1. Then $R[X^{\pm}; \sigma]$ is graded simple if and only if R is σ-simple.*

Proof. If I is a proper non-zero σ-ideal of R, then the elements in $S := R[X^{\pm}; \sigma]$ with coefficients from I form a proper, non-zero graded ideal of S.

Conversely, suppose that R is σ-simple. If I is a non-zero graded ideal of S, then it has non-zero intersection with some homogeneous component. The coefficients of that intersection is a non-zero σ-ideal of R, so I contains X^n for some $n \in \mathbb{Z}$. However, then I contains 1, and we have $I = S$. □

Proposition 4.2. *Let R be a non-associative ring with an additive bijection σ that respects 1. Then $Z(R[X^{\pm}; \sigma])$ equals the set of elements of the form $\sum_{i\in\mathbb{Z}} r_i X^i$, $r_i \in N_m(R) \cap N_r(R)$, such that for all $r \in R$ and $i \in \mathbb{Z}$, the following equalities hold:*

(i) $r_i \sigma^i(r) = r r_i$;
(ii) $\sigma(r r_i) = \sigma(r) r_i$.

Proof. Let $S = R[X^{\pm}; \sigma]$ and $p = \sum_{i\in\mathbb{Z}} r_i X^i \in Z(S)$. For any $r \in R$,

$$\begin{aligned} 0 &= \left[\textstyle\sum_{i\in\mathbb{Z}} r_i X^i, r\right] = \textstyle\sum_{i\in\mathbb{Z}} \left[r_i X^i, r\right] = \textstyle\sum_{i\in\mathbb{Z}} \left(r_i X^i\right) r - r\left(r_i X^i\right) \\ &= \textstyle\sum_{i\in\mathbb{Z}} \left(r_i \sigma^i(r) - r r_i\right) X^i, \end{aligned}$$

so by comparing coefficients,

$$r_i \sigma^i(r) = r r_i, \quad \text{for any } i \in \mathbb{Z}. \tag{4.1}$$

Moreover,

$$\begin{aligned} 0 &= \left[\textstyle\sum_{i\in\mathbb{Z}} r_i X^i, X\right] = \textstyle\sum_{i\in\mathbb{Z}} \left[r_i X^i, X\right] = \textstyle\sum_{i\in\mathbb{Z}} \left(r_i X^{i+1} - \sigma(r_i) X^{i+1}\right) \\ &= \textstyle\sum_{i\in\mathbb{Z}} (r_i - \sigma(r_i)) X^{i+1}, \end{aligned}$$

so $\sigma(r_i) = r_i$ for any $i \in \mathbb{Z}$. Also,

$$\begin{aligned} 0 &= \left(X, r, \textstyle\sum_{i\in\mathbb{Z}} r_i X^i\right) = \textstyle\sum_{i\in\mathbb{Z}} \left(X, r, r_i X^i\right) = \textstyle\sum_{i\in\mathbb{Z}} (Xr)\left(r_i X^i\right) - X\left(r\left(r_i X^i\right)\right) \\ &= \textstyle\sum_{i\in\mathbb{Z}} (\sigma(r)X)\left(r_i X^i\right) - \sigma(r r_i) X^{i+1} = \textstyle\sum_{i\in\mathbb{Z}} (\sigma(r)\sigma(r_i) - \sigma(r r_i)) X^{i+1}, \end{aligned}$$

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

so $\sigma(rr_i) = \sigma(r)\sigma(r_i)$ for any $i \in \mathbb{Z}$. Since $\sigma(r_i) = r_i$, we have

$$\sigma(rr_i) = \sigma(r)r_i, \quad \text{for any } i \in \mathbb{Z}. \tag{4.2}$$

Last, for any $r, s \in R$,

$$\begin{aligned} 0 &= \left(r, \textstyle\sum_{i\in\mathbb{Z}} r_i X^i, s\right) = \textstyle\sum_{i\in\mathbb{Z}} \left(r, r_i X^i, s\right) = \textstyle\sum_{i\in\mathbb{Z}} \left(r\left(r_i X^i\right)\right) s - r\left(\left(r_i X^i\right) s\right) \\ &= \textstyle\sum_{i\in\mathbb{Z}} ((rr_i)\sigma^i(s) - r(r_i\sigma^i(s)))X^i, \\ 0 &= (r, s, \textstyle\sum_{i\in\mathbb{Z}} r_i X^i) = \textstyle\sum_{i\in\mathbb{Z}} \left(r, s, r_i X^i\right) = \textstyle\sum_{i\in\mathbb{Z}} (rs)\left(r_i X^i\right) - r\left(s\left(r_i X^i\right)\right) \\ &= \textstyle\sum_{i\in\mathbb{Z}} ((rs)r_i - r(sr_i))X^i, \end{aligned}$$

and since σ is surjective, so is σ^i, and hence

$$r_i \in N_m(R) \cap N_r(R). \tag{4.3}$$

We now prove that the conditions (4.1)–(4.3) are also sufficient. First, we claim that (4.2) is equivalent to

$$\sigma^j(rr_i) = \sigma^j(r)r_i, \quad \text{forany} i, j \in \mathbb{Z}. \tag{4.4}$$

By letting $j = 1$, we see that (4.4) implies (4.2). We show the opposite implication by induction on j. The base case $j = 0$ follows from the definition. We now split the induction step into two cases. First, assume that $j \geq 0$ and that the induction hypothesis and (4.2) hold. Then $\sigma^{j+1}(rr_i) = \sigma\left(\sigma^j(rr_i)\right) = \sigma\left(\sigma^j(r)r_i\right) = \sigma\left(\sigma^j(r)\right) r_i = \sigma^{j+1}(r)r_i$. Now assume that $j \leq 0$, that the induction hypothesis and (4.2) hold, and set $p = -j$. From (4.2), $rr_i = \sigma^{-1}(\sigma(r)r_i)$, so $\sigma^{-1}(r)r_i = \sigma^{-1}\left(\sigma\left(\sigma^{-1}(r)\right) r_i\right) = \sigma^{-1}(rr_i)$. Hence $\sigma^{-(p+1)}(rr_i) = \sigma^{j-1}(rr_i) = \sigma^{-1}\left(\sigma^j(rr_i)\right) = \sigma^{-1}\left(\sigma^j(r)r_i\right) = \sigma^{-1}\left(\sigma^j(r)\right) r_i = \sigma^{j-1}(r)r_i = \sigma^{-(p+1)}(r)r_i$. Moreover, by letting $r = 1$ in (4.4),

$$\sigma^j(r_i) = r_i, \quad \text{for any } i \in \mathbb{Z}. \tag{4.5}$$

Now assume that the conditions (4.1)–(4.3) hold. As noted above, then (4.4) and (4.5) also hold. We wish to show that $\sum_{i\in\mathbb{Z}} r_i X^i \in Z(S)$. We note that it is sufficient to show that $r_i X^i \in Z(S)$ for any $i \in \mathbb{Z}$. By (iii) in Proposition 2.1, $Z(S) = C(S) \cap N_m(S) \cap N_r(S)$, so it is sufficient to show that $\left[r_i X^i, s_j X^j\right] = \left(s_j X^j, t_k X^k, r_i X^i\right) = \left(s_j X^j, r_i X^i, t_k X^k\right) = 0$ hold for any $s_j, t_k \in R$ and $i, j, k \in \mathbb{Z}$. We have

$$\begin{aligned} \left[r_i X^i, s_j X^j\right] &= \left(r_i X^i\right)\left(s_j X^j\right) - \left(s_j X^j\right)\left(r_i X^i\right) = \left(r_i\sigma^i(s_j) - s_j\sigma^j(r_i)\right) X^{i+j} \\ &\overset{(4.1)}{=} (s_j r_i - s_j r_i)X^{i+j} = 0. \end{aligned}$$

Moreover, for any $r, s, t \in R$ and $i, j, k \in \mathbb{Z}$,

$$\begin{aligned} \left(rX^i, sX^j, tX^k\right) &= \left(\left(rX^i\right)\left(sX^j\right)\right)\left(tX^k\right) - \left(rX^i\right)\left(\left(sX^j\right)\left(tX^k\right)\right) \\ &= \left(\left(r\sigma^i(s)\right) X^{i+j}\right)\left(tX^k\right) - \left(rX^i\right)\left(\left(s\sigma^j(t)\right) X^{j+k}\right) \\ &= \left(\left(r\sigma^i(s)\right) \sigma^{i+j}(t) - r\sigma^i\left(s\sigma^j(t)\right)\right) X^{i+j+k}, \end{aligned} \tag{4.6}$$

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

$$\begin{aligned}
\left(s_jX^j, t_kX^k, r_iX^i\right) &\overset{(4.6)}{=} \left(\left(s_j\sigma^j(t_k)\right)\sigma^{j+k}(r_i) - s_j\sigma^j\left(t_k\sigma^k(r_i)\right)\right)X^{i+j+k}\\
&\overset{(4.5)}{=} \left(\left(s_j\sigma^j(t_k)\right)r_i - s_j\sigma^j(t_kr_i)\right)X^{i+j+k}\\
&\overset{(4.4)}{=} \left(\left(s_j\sigma^j(t_k)\right)r_i - s_j\left(\sigma^j(t_k)r_i\right)\right)X^{i+j+k}\\
&\overset{(4.3)}{=} \left(\left(s_j\sigma^j(t_k)\right)r_i - \left(s_j\sigma^j(t_k)\right)r_i\right)X^{i+j+k} = 0,\\
\left(s_jX^j, r_iX^i, t_kX^k\right) &\overset{(4.6)}{=} \left(\left(s_j\sigma^j(r_i)\right)\sigma^{i+j}(t_k) - s_j\sigma^j\left(r_i\sigma^i(t_k)\right)\right)X^{i+j+k}\\
&\overset{(4.5)}{=} \left((s_jr_i)\sigma^{i+j}(t_k) - s_j\sigma^j\left(r_i\sigma^i(t_k)\right)\right)X^{i+j+k}\\
&\overset{(4.1)}{=} \left((s_jr_i)\sigma^{i+j}(t_k) - s_j\sigma^j(t_kr_i)\right)X^{i+j+k}\\
&\overset{(4.4)}{=} ((s_jr_i)\sigma^{i+j}(t_k) - s_j\left(\sigma^j(t_k)r_i\right))X^{i+j+k}\\
&\overset{(4.1)}{=} \left((s_jr_i)\sigma^{i+j}(t_k) - s_j\left(r_i\sigma^{i+j}(t_k)\right)\right)X^{i+j+k}\\
&\overset{(4.3)}{=} \left((s_jr_i)\sigma^{i+j}(t_k) - (s_jr_i)\sigma^{i+j}(t_k)\right)X^{i+j+k} = 0.
\end{aligned}$$

□

Using Proposition 4.2, the following result is immediate:

Corollary 4.1. *Let R be a non-associative ring with an additive bijection σ that respects 1. Then $Z(R[X^{\pm};\sigma]) \subseteq R$ holds if and only if there do not exist a non-zero $s \in N_m(R) \cap N_r(R)$ and a non-zero $n \in \mathbb{Z}$, such that for all $r \in R$, the following equalities hold:*

(i) $s\sigma^n(r) = rs$;
(ii) $\sigma(rs) = \sigma(r)s$.

Theorem 4.1 *Let R be a non-associative ring with an additive bijection σ that respects 1. Then $R[X^{\pm};\sigma]$ is simple if and only if R is σ-simple and there do not exist $u \in N_m(R) \cap N_r(R) \cap R^{\times}$ and a non-zero $n \in \mathbb{Z}$, such that for all $r \in R$, the following equalities hold:*

(i) $\sigma^n(r) = u^{-1}ru$;
(ii) $\sigma(ru) = \sigma(r)u$.

Proof. We note that $S := R[X^{\pm};\sigma]$ is a faithfully $\mathbb{Z}$-graded ring with $\mathrm{Supp}(S) = \mathbb{Z}$, and that $\mathbb{Z}$ is a torsion-free hypercentral group. Hence, by Corollary 2.1, Corollary 4.1 and Proposition 4.1, we need only show that if R is σ-simple and there is an $s \in R$ satisfying the conditions in Corollary 4.1, then s belongs to $R^{\times}$. To show this, consider the ideal I of R generated by s. Since $s \in N_m(R)$, we have $I = RsR$. By using (i) in Corollary 4.1, we see that $I = Rs$. Hence, by (ii) in Corollary 4.1, we have $\sigma(I) = I$. Since R is σ-simple and I is non-zero, it follows that $I = R$, and so there exist $t \in R$ such that $ts = 1$. This implies that $s\sigma^n(t) = 1$ by (i) in Corollary 4.1 and that $\sigma^n(t)s = 1$ by (ii) in Corollary 4.1, so $s \in R^{\times}$. □

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

By using the above theorem, we may deduce when the examples in § 3 are simple, and when they are not.

Corollary 4.2. *The following assertions hold:*

(i) $\mathbb{C}[X^{\pm};\sigma_q]$ *in Example 3.1 is simple if and only if* $q \neq \pm 1$*;*
(ii) *If* T *is simple, then the non-associative quantum torus* $T[Y^{\pm}][X^{\pm};\sigma_q]$ *in Example 3.2 is simple if and only if* q *is not a root of unity;*
(iii) $K[[Y]][X^{\pm};\sigma_g]$ *in Example 3.3 is simple;*
(iv) $M_n(R)[X^{\pm};\sigma]$ *in Example 3.4,* $M_n(\mathbb{C})[X^{\pm};*]$ *in Example 3.5 and* $A[X^{\pm};*]$ *in Example 3.6 are not simple.*

Proof.

(i) Since $\mathbb{C}$ is simple, it is also σ_q-simple. If $q = \pm 1$, then $\sigma_q^2 = \mathrm{id}_{\mathbb{C}}$, so (i) and (ii) in Theorem 4.1 hold with $u = 1$. Hence $\mathbb{C}[X^{\pm};\sigma_q]$ is not simple. If $q \neq \pm 1$, then $\sigma_q^n(r) \neq r = u^{-1}ur = u^{-1}ru$ for any $r \in \mathbb{C}$ and $u \in \mathbb{C}^{\times}$. By Theorem 4.1, $\mathbb{C}[X^{\pm};\sigma_q]$ is simple.
(ii) The proof is an adaptation of that of [3, Corollary 1.18] to the non-associative setting. Let $R = T[Y^{\pm}]$ and $S = R[X^{\pm};\sigma_q]$. If q is a root of unity, say $q^n = 1$ for some non-zero $n \in \mathbb{N}$, then $\sigma_q^n(Y) = q^nY = Y$ and $\sigma_q^n(Y^{-1}) = q^{-n}Y^{-1} = Y^{-1}$, so $\sigma_q^n(r) = r$ for all $r \in R$. Hence (i) and (ii) in Theorem 4.1 hold with $u = 1$, so S is not simple. Conversely, assume that q is not a root of unity. Let $u \in N_m(R) \cap N_r(R) \cap R^{\times}$ and assume that $n \in \mathbb{Z}$ is non-zero. Then $\sigma_q^n(Y) = q^nY \neq Y = u^{-1}uY = u^{-1}Yu$, so (i) in Theorem 4.1 does not hold. We note that by Theorem 4.1, $R = T[Y^{\pm};\mathrm{id}_T]$ is not simple. We claim, however, that it is σ_q-simple. To this end, let I be a non-zero σ_q-ideal of R. We must show that $I = R$. We observe that $I \cap T[Y]$ is non-zero, and so we can choose a non-zero $p \in I \cap T[Y]$ of minimal degree, say $p = t_mY^m + \cdots + t_0$ for some $m \in \mathbb{N}$ and $t_m, \ldots, t_0 \in T$ where $t_m \neq 0$. If we can show that $p = t_mY^m$, then we are done: $I \cap T$ is an ideal of T, and if $p = t_mY^m$, then $t_m = pY^{-m} \in I$, so $I \cap T$ is non-zero. Since T is simple, we must have $I \cap T = T$. In particular, $1 \in T = I \cap T \subseteq I$, so $I = R$. If $m = 0$, then clearly $p = t_mY^m$. Hence, assume that m is positive. Since I is a σ_q-ideal, we have $\sigma_q(p) \in I \cap T[Y]$. Now, $\sigma_q(p) = q^mt_mY^m + \cdots + t_0$, and so $\sigma_q(p) - q^mp$ is in $I \cap T[Y]$ and of degree at most $m - 1$. By the minimality of m, we must have $\sigma_q(p) - q^mp = 0$, from which it follows that $q^it_i = q^mt_i$, that is, $(q^{m-i} - 1)t_i = 0$, for any $i \in \{1, \ldots, m\}$. Since q is not a root of unity, $q^{m-i} - 1$ is non-zero and therefore an element of $Z(T)\backslash\{0\}$ for any $i \in \{1, \ldots, m-1\}$. By Proposition 2.2, $Z(T)$ is a field, so $q^{m-i} - 1$ is invertible for any $i \in \{1, \ldots, m-1\}$. In particular, $(q^{m-i} - 1)t_i = 0$ implies $t_i = 0$ for any $i \in \{1, \ldots, m-1\}$. Consequently, $p = t_mY^m$.
(iii) Let $R = K[[Y]]$ where K is a field. Then R is not simple, but we claim it is σ_g-simple. To show this, let I be a non-zero ideal of R. It is well-known that I is generated by some element Y^m where $m \in \mathbb{N}$. Hence I contains an element Y^k, where k is a power of 2. However, then it follows that if I is σ_g-invariant, then it must contain Y, and hence also Y^3. Then $1 = Y^3 + 1 - Y^3 = \sigma_g(Y) - Y^3 \in I$, so $I = R$.

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press

Clearly $\sigma_g^n(Y) \neq Y = u^{-1}uY = u^{-1}Yu$ for any $u \in R^\times$ and non-zero $n \in \mathbb{N}$. By Theorem 4.1, $R[X^\pm;\sigma]$ is simple.

(iv) For $M_n(R)[X^\pm;\sigma]$ in Example 3.4, σ may be viewed as a permutation on a finite set, and since any permutation on a finite set has finite order, there is some non-zero $n \in \mathbb{Z}$ such that $\sigma^n = \mathrm{id}_{M_n(R)}$. Moreover, for any involution $*$ on a non-associative ring R, we have $*^2 = \mathrm{id}_R$. Hence we can conclude that for all three examples, (i) and (ii) in Theorem 4.1 hold with $u=1$, so the corresponding generalized Laurent polynomial rings are not simple.

□

Let R be a non-associative ring with pairwise commuting additive bijections $\sigma_1,\dots,\sigma_n$ respecting 1. Then we may construct an iterated generalized Laurent polynomial ring $R[X_1^\pm;\sigma_1]\cdots[X_n^\pm;\widehat{\sigma}_n]$, denoted by $R[X_1,\dots,X_n;\sigma_1,\dots,\sigma_n]$, in the same way as described in § 2.3. We note that this is a generalization of the non-associative skew Laurent polynomial rings introduced in [9]; the construction in loc. cit. corresponds exactly to the case when $\sigma_1,\dots,\sigma_n$ are automorphisms. Moreover, we can now formulate a generalization of Theorem 2.2:

Theorem 4.2 *Let R be a non-associative ring with pairwise commuting additive bijections $\sigma_1,\dots,\sigma_n$ respecting 1. Then $R[X_1^\pm,\dots,X_n^\pm;\sigma_1,\dots,\sigma_n]$ is simple if and only if R is $(\sigma_1,\dots,\sigma_n)$-simple and there do not exist $u \in N_m(R)\cap N_r(R)\cap R^\times$ and a non-zero $(m_1,\dots,m_n)\in\mathbb{Z}^n$, such that for all $r \in R$ and $i \in \{1,\dots,n\}$, the following equalities hold:*

(i) $(\sigma_1^{m_1}\circ\cdots\circ\sigma_n^{m_n})(r) = u^{-1}ru$;
(ii) $\sigma_i(ru) = \sigma_i(r)u$.

Proof. The proof follows the proof of Corollary 4.1 and Theorem 4.1 closely. First, we see that the arguments in Proposition 4.2 can easily be adapted to the current case. If $S = R[X_1^\pm,\dots,X_n^\pm;\sigma_1,\dots,\sigma_n]$, then it follows immediately that $Z(S)\subseteq R$ holds if and only if there do not exist a non-zero $s \in N_m(R)\cap N_r(R)$ and a non-zero $(m_1,\dots,m_n)\in\mathbb{Z}^n$, such that for all $r\in R$ and $i \in \{1,\dots,n\}$, the following equalities hold:

(i) $s(\sigma_1^{m_1}\circ\cdots\circ\sigma_n^{m_n})(r) = rs$;
(ii) $\sigma_i(rs) = \sigma_i(r)s$.

That graded simplicity of S is equivalent to $(\sigma_1,\dots,\sigma_n)$-simplicty of R is clear. The proof is then finished by an argument similar to the proof of Theorem 4.1. □

Acknowledgements. We would like to thank the anonymous referee for valuable comments on the manuscript.

Competing interests. The authors declare that they have no conflict of interest.

References

(1) J. C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39**(2): (2002), 145–205. Errata, [2].

(2) J. C. Baez, Errata for "The octonions", *Bull. Amer. Math. Soc.* **39**(2): (2005), 213–213.

(3) K. R. Goodearl and J. R. Warfield, *An introduction to noncommutative Noetherian rings*, 2nd edition (Cambridge University Press, Cambridge, UK, 2004).

(4) D. Hilbert, *Grundlagen der geometrie* (B. G. Teubner, Leipzig, 1903).

(5) D. A. Jordan, Simple skew Laurent polynomial rings, *Comm. Algebra* **12**(1–2): (1984), 135–137.

(6) T. Y. Lam, *A first course in noncommutative rings*, 2nd edition (Springer, New York, 2001).

(7) J. C. McConnell and J. C. Robsonwith the cooperation of L. W. Small, *Noncommutative Noetherian rings*, revised edition (American Mathematical Society, Providence, RI, 2001).

(8) K. McCrimmon, *A taste of Jordan algebras* (Springer, New York, 2004).

(9) P. Nystedt and J. Öinert, Simple graded rings, nonassociative crossed products and Cayley-Dickson doublings, *J. Algebra Appl.* **19**(12): (2020), 2050231.

(10) P. Nystedt, J. Öinert and J. Richter, Non-associative Ore extensions, *Isr. J. Math.* **224**(1): (2018), 263–282.

(11) P. Nystedt, J. Öinert and J. Richter, Simplicity of Ore monoid rings, *J. Algebra* **530** (2019), 69–85.

(12) O. Ore, Theory of non-commutative polynomials, *Ann. Math.* **34**(3): (1933), 480–508.

(13) R. D. Schafer, *An introduction to nonassociative algebras* (Academic Press, New York, 1966).

(14) M. R. Voskoglou, Simple skew Laurent polynomial rings, *Boll. Un. Mat. Ital. A (7)* **1**(2): (1987), 255–260.

https://doi.org/10.1017/S0013091524000683 Published online by Cambridge University Press