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A NOTE ON K-SPACES

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Abstract

In this paper, it is shown that every compact Hausdorff K-space has countable tightness. This result gives a positive answer to a problem posed by Malykhin and Tironi ['Weakly Fréchet–Urysohn and Pytkeev spaces', *Topology Appl.* **104** (2000), 181–190]. We show that a semitopological group G that is a K-space is first countable if and only if G is of point-countable type. It is proved that if a topological group G is a K-space and has a locally paracompact remainder in some Hausdorff compactification, then G is metrisable.

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1. Introduction

All topological spaces considered in this paper are supposed to be Hausdorff. Recall that a topological space is said to be a K-space [8] if for any subset A of X and any point $x \in X$ such that $x \in \overline{A} \setminus A$, there exists a countable infinite disjoint (CID) family $\mathcal{F}$ of compact subsets of A such that for every neighbourhood V of x the subfamily $\{F \in \mathcal{F} : F \cap V = \emptyset\}$ is finite. For convenience, we denote the relation of x and $\mathcal{F}$ above by $x(K)\mathcal{F}$.

The K-property is a generalisation of the wFU-property [8], where one substitutes compact subsets in a CID family for finite subsets in a CID family. In [11], Wang and He proved that a regular K-space X is a wFU-space if and only if the tightness of X is countable.

In [8], Malykhin and Tironi posed the following question.

PROBLEM 1.1. Must a compact K-space have countable tightness?

The answer is 'yes' in the category of topological groups (see [11]). Now we answer this question completely by means of free sequences, where a sequence $\{x_\alpha : 0 \le \alpha < \kappa\}$ in a space X is a free sequence of length κ if for all $\beta < \kappa$, $\overline{\{x_\alpha : 0 \le \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \ge \beta\}} = \emptyset$.

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We also investigate the K-property in the category of topological groups and the category of semitopological groups. Recall that a semitopological group is a group G endowed with a topology τ such that the product map is separately continuous. A topological group is a group G endowed with a topology τ such that the product map is jointly continuous and the inverse map is also continuous.

For other terms and symbols we refer to [4].

2. Main results

The following theorem gives a positive answer to Problem 1.1 posed by Malykhin and Tironi [8, Question 6.4].

Theorem 2.1. *Every compact K-space has countable tightness.*

Proof. Suppose to the contrary that there exists a compact Hausdorff K-space X whose tightness is not countable. Due to a famous result by Shapirovskii [10], the hereditary π-character coincides with the tightness for every compact Hausdorff space Y, that is, $h\pi\chi(Y) = t(Y)$. Then it follows that $h\pi\chi(X) = t(X) > \omega$. By [7, Theorem 7.10], if Y is compact and $h\pi\chi(Y) > \kappa$, then Y has a free sequence of length κ^+. This implies that X has a free sequence $\{x_\alpha : 0 \le \alpha < \omega_1\}$ of length ω_1. Since X is compact, it follows that the subset $\{x_\alpha : 0 \le \alpha < \omega_1\}$ has a complete accumulation point $x \in X$, that is, each neighbourhood U of x in X contains a subset $A_U \subset \{x_\alpha : 0 \le \alpha < \omega_1\}$ such that $|A_U| \ge \omega_1$. The condition that $\{x_\alpha : 0 \le \alpha < \omega_1\}$ is a free sequence implies that $x \in \overline{\{x_\alpha : 0 \le \alpha < \omega_1\}} \setminus \{x_\alpha : 0 \le \alpha < \omega_1\}$.

Since X is a K-space, it follows that there is a CID family $\mathcal{F} = \{K_n : n \in \omega\}$ of compact subsets of $\{x_\alpha : 0 \le \alpha < \omega_1\}$ such that $x(K)\mathcal{F}$. Clearly, $x \in \overline{\bigcup \mathcal{F}}$. Since each K_n is compact and $\{x_\alpha : 0 \le \alpha < \omega_1\}$ is discrete, it follows that K_n is finite for each $n \in \omega$. Hence $\bigcup \mathcal{F}$ is countable. Then there is a $\beta < \omega_1$ such that $\bigcup \mathcal{F} \subset \{x_\alpha : 0 \le \alpha < \beta\}$. Therefore it follows that $x \in \overline{\{x_\alpha : 0 \le \alpha < \beta\}}$. On the other hand, from the fact that x is a complete accumulation point of $\{x_\alpha : 0 \le \alpha < \omega_1\}$ in X one can conclude that $x \in \overline{\{x_\alpha : \beta \le \alpha < \omega_1\}}$. This is a contradiction since $\{x_\alpha : 0 \le \alpha < \omega_1\}$ is a free sequence. Therefore, every compact Hausdorff K-space has countable tightness. □

Theorem 2.2. *If there exists a closed continuous mapping $f : X \to Y$ of a K-space X onto a space Y, then Y is a K-space.*

Proof. Suppose that $A \subset Y$ and $y \in Y$ such that $y \in \overline{A} \setminus A$. For each $a \in A$ fix a point $x_a \in f^{-1}(a)$. Assume that $B = \{x_a : a \in A\}$. Since f is closed, it follows that $f^{-1}(y) \cap \overline{B} \neq \emptyset$. Fix a point x from $f^{-1}(y) \cap \overline{B}$. Then there is a CID family $\mathcal{F}$ of compact subsets of B such that $x(K)\mathcal{F}$. Put $\mathcal{L} = \{f(F) : F \in \mathcal{F}\}$. It is clear that $\mathcal{L}$ is a CID family of compact subsets of A satisfying that $y(K)\mathcal{L}$. Therefore, we conclude that Y is a K-space. □

Corollary 2.3. *Let G be a topological group and H be a locally compact subgroup of G. If G is a K-space, then the quotient space G/H of all left cosets aH of H in G is a K-space.*

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Proof. Suppose that $A \subset G/H$ and $y \in G/H$ such that $y \in \overline{A} \setminus A$. Since G/H is a homogeneous space, we can assume that y is $\pi(e)$, where $\pi : G \to G/H$ is the quotient mapping and e is the neutral element of G. By [1], there exists an open neighbourhood U of e such that the restriction of π to $\overline{U}$ is a perfect mapping of $\overline{U}$ onto the subspace $\pi(\overline{U})$. Since $\overline{U}$ is a K-space, it follows from Theorem 2.2 that $\pi(\overline{U})$ is a K-space. Notice that π is an open mapping [3, Theorem 1.5.1]. It follows that $\pi(U)$ is open in G/H. Then the homogeneity of G/H guarantees that G/H is a K-space. □

Lemma 2.4. *Suppose that K is a compact subset of a Hausdorff space X such that K has a countable base $\{U_n : n \in \omega\}$ of open neighbourhoods in X and $s \in K$. If K has a countable π-base $\{V_n : n \in \omega\}$ at s, then X has countable π-character at s.*

Proof. For each $n \in \omega$ fix a sequence $\{O_n^i : i \in \omega\}$ of open subsets of X such that $O_n^i \subset U_i \cap (X \setminus (K \setminus V_n))$ and $\overline{O_n^{i+1}} \subset O_n^i$ for each $i \in \omega$. Put $K_n = \bigcap_{i\in\omega} O_n^i$ for every $n \in \omega$. Clearly $K_n \subset V_n$. We claim that $\{O_n^i : i \in \omega\}$ is a π-base of X at K_n. Take an open neighbourhood W of K_n in X. Then there is an O_n^i contained in W. Otherwise, take an x_i from $O_n^i \setminus W$ for each $i \in \omega$. Since $x_i \in U_i$, it follows that the subset $\{x_i : i \in \omega\}$ has a cluster x. Clearly $x \in \bigcap_{i\in\omega} \overline{O_n^i} = K_n$. This is a contradiction.

We show that the family $\{O_n^i : n, i \in \omega\}$ is a countable π-base of X at s. Take an open neighbourhood O of s in X. Since $\{V_n : n \in \omega\}$ is a π-base of K at s, there is some V_n contained in O. Since $K_n \subset V_n$, it follows that there exists an O_n^i contained in O. Therefore X has countable π-character at s. □

Recall that a space X is of (point-) countable type if every (point) compact subspace of X is contained in a compact subspace of X with countable character in X.

Theorem 2.5. *A semitopological group G that is a K-space is first countable if and only if G is of point-countable type.*

Proof. Obviously, every first countable space is of point-countable type. So it is only necessary to prove the 'if' part. Fix a point x of G. Then there is a compact subset F of G such that $x \in F$ and F has a countable base of open neighbourhoods in G. Since F is a K-space, it follows from Theorem 2.1 that F has countable tightness. Thus the π-character of F is countable [10]. By Lemma 2.1, G has a countable π-base at x. Since G is homogeneous, it follows that G has countable π-character. According to [3, Corollary 5.7.5], every semitopological group with countable π-character has a G_δ-diagonal, so G has a G_δ-diagonal. This implies that F has a G_δ-diagonal. Since each compact space with a G_δ-diagonal is metrisable [5, Theorem 2.13], it follows that F is metrisable. Particularly, F is first countable. Notice that F has a countable base of open neighbourhoods in G. It follows that G is first countable at x. □

Theorem 2.6. *If G is a topological group that is a K-space and H is a locally compact subgroup of G such that the quotient space G/H is first countable, then G is metrisable and G/H is locally metrisable.*

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Proof. Obviously, H is a K-space. Since H is locally compact, it is of countable type. Then it follows from Theorem 2.3 that H is first countable. Taking into account that G/H is first countable, one can conclude that G is first countable [3, Corollary 1.5.21]. Therefore, G is metrisable since it is a topological group. Let $\pi : G \to G/H$ be the quotient mapping. Then there exists an open neighbourhood U of the neutral element e such that the restriction of π to $\overline{U}$ is a perfect mapping of $\overline{U}$ onto the subspace $\pi(\overline{U})$. Since the image of a metrisable space under a perfect mapping is metrisable, it follows that $\pi(\overline{U})$ is metrisable. Since $\pi(U)$ is an open subset of G/H, it follows that G/H is locally metrisable. □

Theorem 2.7. *Suppose that G is a semitopological group which is a paracompact p-space. If G is a K-space, then G is metrisable.*

Proof. Since each p-space is of countable type, it follows from Theorem 2.3 that G is first countable. Therefore G has a G_δ-diagonal [3, Corollary 5.7.5]. Since every paracompact p-space with a G_δ-diagonal is metrisable, so is G. □

Theorem 2.8. *Let G be a topological group and bG be a Hausdorff compactification of G such that the remainder $Y = bG \setminus G$ is locally paracompact. If G is a K-space, then G is metrisable.*

Proof. We consider two cases.

Case 1. G is locally compact. In this case we can conclude that G is a paracompact p-space since G is a topological group [9]. By Theorem 2.5, G is metrisable.

Case 2. G is nonlocally compact. Then G is nowhere locally compact since it is homogeneous. Thus Y is dense in bG. Since Y is regular and locally paracompact, for each $y \in Y$ one can take an open neighbourhood U of y in Y such that $\overline{U}^Y$ is paracompact. By [2], every remainder of a topological group in any compactification is either Lindelöf or pseudocompact. We show that Y cannot be pseudocompact. Suppose to the contrary that Y is pseudocompact. Then $\overline{U}^Y$ is pseudocompact since $\overline{U}^Y$ is a regular closed subset of Y. Thus $\overline{U}^Y$ is compact since it is also paracompact. Then it follows that Y is locally compact. Therefore, Y is an open subset of bG since Y is dense in bG. This implies that G is compact, which is a contradiction. Hence Y is Lindelöf. According to a famous result due to Henriksen and Isbell [6] that a Tychonoff space X is of countable type if and only if the remainder in any (or some) Hausdorff compactification of X is Lindelöf, one can conclude that G is of countable type. Notice that G is a topological group. Then it follows that G is a paracompact p-space [9]. By Theorem 2.5, G is metrisable. □

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