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THE PRODUCT OF PRE-RADON MEASURES

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Let μ and ν be non-σ-finite pre-Radon measures on topological spaces X and Y respectively. Then there exists a unique pre-Radon measure λ on the product space $X \times Y$ which satisfies $\lambda(A \times B) = \mu(A)\nu(B)$ for all Borel sets A in X and B in Y such that $\mu(A) < \infty$ and $\nu(B) < \infty$.

1. Introduction

Let X be a topological space, $\mathcal{O}(X)$ the family of open subsets of X , $\mathcal{F}(X)$ the family of closed subsets, and $\mathcal{B}(X)$ the Borel field, that is, the σ-algebra generated by $\mathcal{O}(X)$. A Borel measure μ is said to be a *pre-Radon measure* if it satisfies the following conditions:

(i) for each $x \in X$, there is an open neighbourhood U of x such that $\mu(U) < \infty$;

(ii) for each $B \in \mathcal{B}(X)$ with $\mu(B) < \infty$,

$$\mu(B) = \sup\{\mu(F) : B \supset F \in \mathcal{F}(X)\} ;$$

(iii) for each $B \in \mathcal{B}(X)$,

$$\mu(B) = \inf\{\mu(U) : B \subset U \in \mathcal{O}(X)\} ;$$

(iv) for each increasing net $\{U_\alpha\} \subset \mathcal{O}(X)$,

$$\mu\left(\bigcup_\alpha U_\alpha\right) = \sup_\alpha \mu\left(U_\alpha\right) .$$

A Borel measure satisfying (i) is called locally-bounded. If a Borel

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measure satisfies (ii) (respectively (iii)), then we call it inner (respectively outer) regular. In particular, an inner and outer regular Borel measure is said to be regular.

A locally bounded measure μ is called a Radon measure if $\mu(B) = \sup\{\mu(K) : K \subset B \text{ and } K \text{ is compact}\}$ for every $B \in \mathcal{B}(X)$. Note that every Radon measure is semi-finite. Recall that a measure ν on a measurable space $(Y, \mathcal{B})$ is called semi-finite if

$$\nu(A) = \sup\{\nu(B) : A \supset B \in \mathcal{B},\ \nu(B) < \infty\}\ ,\quad A \in \mathcal{B}\ .$$

A measure space $(X, \mathcal{B}, \mu)$ is called locally determined if μ is semi-finite and any subset E of X satisfying that $E \cap F \in \mathcal{B}$ for all $F \in \mathcal{B}$ with $\mu(F) < \infty$ belongs to $\mathcal{B}$.

Suppose that X is a topological space. A measure space $(X, \mathcal{A}, \mu)$ is said to be a quasi-Radon measure space if it satisfies the following:

(i) $(X, \mathcal{A}, \mu)$ is complete and locally determined;

(ii) $\mathcal{A} \supset \mathcal{B}(X)$;

(iii) if $E \in \mathcal{A}$ and $\mu(E) < \infty$, then there is a $G \in \mathcal{O}(X)$ such that $\mu(G) < \infty$ and $\mu(E \cap G) > 0$;

(iv) $\mu(E) = \sup\{\mu(F) : E \supset F \in \mathcal{F}(X)\}$ for all $E \in \mathcal{A}$;

(v) for every increasing net $\{U_\alpha\} \subset \mathcal{O}(X)$,

$$\mu\Big(\bigcup_\alpha U_\alpha\Big) = \sup_\alpha \mu(U_\alpha)\ .$$

As for its details, see Fremlin [3, §72]. The relationship between pre-Radon measures and quasi-Radon measures is given by Amemiya, Okada and Okazaki [1, pp. 131-132].

From now on, all topological spaces are supposed to be Hausdorff.

In this paper, we study the product of two non-σ-finite pre-Radon measures.

In the σ-finite case, the following theorem is proved by Amemiya, Okada and Okazaki [1, §9].

THEOREM 1.1. *Let* μ *and* ν *be* σ*-finite pre-Radon measures on topological spaces* X *and* Y *respectively. Then there exists a unique* *pre-Radon measure* λ *on* $X \times Y$ *such that* $\lambda(A \times B) = \mu(A)\nu(B)$ *for every* $A \in \mathcal{B}(X)$ *and every* $B \in \mathcal{B}(Y)$. *Moreover, for a non-negative, extended real-valued Borel measurable function* f *on* $X \times Y$,

(i) $x \mapsto \int_Y f(x, y) d\nu(y)$ *is* $\mathcal{B}(X)$*-measurable,*

(ii) $y \mapsto \int_X f(x, y) d\mu(x)$ *is* $\mathcal{B}(Y)$*-measurable,*

(iii) $$\int_X d\mu(x) \int_Y f(x, y) d\nu(y) = \int_Y d\nu(y) \int_X f(x, y) d\mu(x) = \int_{X \times Y} f(x, y) d\lambda(x, y) .$$

In the case of Radon measures, Bourbaki [2, §2, no. 6] has shown the following theorem.

THEOREM 1.2. *Let* μ *and* ν *be Radon measures on topological spaces* X *and* Y *respectively. Then there is a unique Radon measure* λ *on* $X \times Y$ *such that* $\lambda(A \times B) = \mu(A)\nu(B)$ *for every* $A \in \mathcal{B}(X)$ *and* $B \in \mathcal{B}(Y)$. *Furthermore, for a non-negative, lower semi-continuous function* f *on* $X \times Y$,

(i) $x \mapsto \int_Y f(x, y) d\nu(y)$ *and* $y \mapsto \int_X f(x, y) d\mu(x)$ *are lower semi-continuous on* X *and* Y *respectively, and*

(ii) $$\int_X d\mu(x) \int_Y f(x, y) d\nu(y) = \int_Y d\nu(y) \int_X f(x, y) d\nu(x) = \int_{X \times Y} f(x, y) d\lambda(x, y) .$$

Fremlin [4, Proposition 4.2] has shown the following theorem.

THEOREM 1.3. *Let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *be two quasi-Radon measure spaces. Then there is a unique quasi-Radon measure* λ *on* $X \times Y$ *such that* $\lambda(A \times B) = \mu(A)\nu(B)$ *for each* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$.

In §2, we shall show that the statement in Theorem 1.1 does not hold for non-σ-finite pre-Radon measures, in general; but we have a unique pre-Radon measure λ on $X \times Y$ which satisfies the condition $\lambda(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ such that $\mu(A) < \infty$ and $\nu(B) < \infty$.

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2. The product of pre-Radon measures

Let X be a set. A family $\mathcal{U}$ of subsets of X is said to be a paving if it satisfies the following conditions:

(i) $\emptyset \in \mathcal{U}$;

(ii) $\bigcup_{U \in \mathcal{U}} U = X$;

(iii) if $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2$, $U_1 \cup U_2 \in \mathcal{U}$.

We denote by $R[\mathcal{U}]$ the ring generated by $\mathcal{U}$.

The proof of the following lemma is straightforward.

LEMMA 2.1. *Let X be a set and $\mathcal{U}$ a paving of subsets of X . Then, for a subset E of X , $E \in R[\mathcal{U}]$ if and only if there are $V_i, W_i \in \mathcal{U}$ $(i = 1, 2, \ldots, n)$ such that*

(i) $V_i \subset W_i$ $(i = 1, 2, \ldots, n)$,

(ii) $(W_i - V_i) \cap (W_j - V_j) = \emptyset$ *if* $i \neq j$,

(iii) $E = \bigcup_{i=1}^{n} (W_i - V_i)$.

Let $(X, \mathcal{B}, \mu)$ be a measure space. For $A \in \mathcal{B}$, we can define a measure μ_A on $A \cap \mathcal{B}$ as follows:

$$\mu_A(A \cap B) = \mu(A \cap B) , \quad B \in \mathcal{B} ,$$

where $A \cap \mathcal{B} = \{A \cap B : B \in \mathcal{B}\}$. We call μ_A the restriction of μ to A .

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \mu)$ be two totally finite measure spaces. Then we denote by $\mu \otimes \nu$ the product measure of μ and ν . This $\mu \otimes \nu$ is defined on the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ and satisfies the condition $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$.

The following lemma is a fundamental tool.

LEMMA 2.2 (Amemiya, Okada and Okazaki [1, Theorem 3.1]). *Let X be a topological space and $\mathcal{U}$ a paving generated by an open base of X . Let a non-negative, real-valued, finitely additive set function m on*

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$R[\mathcal{U}]$ *satisfy the following conditions:*

(i) *for every net* $\{U_\alpha\} \subset \mathcal{U}$ *increasing to a* $U \in \mathcal{U}$,

$$\lim_\alpha m(U_\alpha) = m(U) \ ;$$

(ii) *for each* $U \in \mathcal{U}$,

$$m(U) = \sup\{m(F) : U \supset F \in R[\mathcal{U}] \cap F(X)\} .$$

Then m *can be extended to a unique pre-Radon measure on* X .

THEOREM 2.3. *Let* μ *and* ν *be pre-Radon measures on topological spaces* X *and* Y *respectively. Then there is a unique pre-Radon measure* λ *on* $X \times Y$ *satisfying the condition* $\lambda(A \times B) = \mu(A)\nu(B)$ *for all* $A \in B(X)$ *and* $B \in B(Y)$ *such that* $\mu(A) < \infty$ *and* $\nu(B) < \infty$.

Proof. Let $\mathcal{U}$ be the paving generated by the open base $V = \{U \times V : U \in O(X),\ V \in O(X),\ \mu(U) < \infty,\ \nu(V) < \infty\}$. It follows from Lemma 2.1 that, for every $E \in R[\mathcal{U}]$, there is a $U \times V \in V$ such that $E \subset U \times V$. Hence we can define a set function m on $R[\mathcal{U}]$ by

$$m(E) = \left(\mu_U \otimes \nu_V\right)(E) \ .$$

We claim that $m(E)$ is independent of the choice of $U \times V$. In fact, suppose that $E \subset U' \times V' \subset U \times V$ for another $U' \times V' \in V$, then it follows from the definition of product measures that $\left(\mu_U \otimes \nu_V\right)_{U' \times V'} = \mu_{U'} \otimes \nu_{V'}$, which implies that $\left(\mu_{U'} \otimes \nu_{V'}\right)(E) = \left(\mu_U \otimes \nu_V\right)(E)$.

Given an increasing net $\{W_\alpha\} \subset \mathcal{U}$ such that $\bigcup_\alpha W_\alpha = W \in \mathcal{U}$, there exists a set $U \times V \in V$ such that $W \subset U \times V$. It follows from Theorem 1.1 that $m(W) = \left(\mu_U \otimes \nu_V\right)(W) = \sup_\alpha \left(\mu_U \otimes \nu_V\right)\left(W_\alpha\right) = \sup_\alpha m\left(W_\alpha\right)$.

Given $W = \bigcup_{i=1}^{n} \left(U_i \times V_i\right) \in \mathcal{U}$ with $U_i \times V_i \in V$, and $\varepsilon > 0$, there are $F_i \in F(X)$ with $F_i \subset U_i$, $G_i \in F(X)$ with $G_i \subset V_i$ such that $\mu\left(U_i - F_i\right) < \varepsilon/n\left(\nu\left(V_i\right)+1\right)$ and $\nu\left(V_i - G_i\right) < \varepsilon/n\left(\mu\left(U_i\right)+1\right)$ for all $i = 1, 2, \ldots, n$ since both μ and ν are regular. Let

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$F = \bigcup_{i=1}^{n} (F_i \times G_i) \in F(X \times Y)$; then

$$m(W-F) \le m\left(\bigcup_{i=1}^{n} (U_i \times V_i - F_i \times G_i)\right)$$
$$\le \sum_{i=1}^{n} \left(m(U_i \times V_i) - m(F_i \times G_i)\right)$$
$$\le \sum_{i=1}^{n} 2\varepsilon/n = 2\varepsilon .$$

Then it follows from Lemma 2.2 that there is a unique pre-Radon measure λ on $X \times Y$ such that $\lambda = m$ on $R[U]$. For each $A \in B(X)$ with $\mu(A) < \infty$, and each $B \in B(Y)$ with $\nu(B) < \infty$, there exists a $U \times V \in V$ such that $U \times V \supset A \times B$. Then $\lambda_{U\times V}$ is a pre-Radon measure on $U \times V$ by Amemiya, Okada and Okazaki [1, Theorem 5.2]. If we denote by $\overline{\mu_U \otimes \nu_V}$ the pre-Radon extension of $\mu_U \otimes \nu_V$ on $U \times V$, then $\lambda_{U\times V}$ coincides with $\overline{\mu_U \otimes \nu_V}$ on $R[U_{U\times V}]$, where $U_{U\times V}$ is the paving generated by $\{U_1 \times V_1 : U_1 \in O(U), V_1 \in O(V)\}$. By Lemma 2.2, $\lambda_{U\times V} = \overline{\mu_U \otimes \nu_V}$ on $U \times V$, so that

$$\lambda(A \times B) = \lambda_{U\times V}(A \times B)$$
$$= \overline{\mu_U \otimes \nu_V}(A \times B)$$
$$= \mu_U(A)\nu_V(B)$$
$$= \mu(A)\nu(B) .$$

The uniqueness of λ is obvious. This completes the proof.

REMARK 2.4. In the above theorem, if both μ and ν are semi-finite, then $\lambda(A \times B) = \mu(A)\nu(B)$ even when $A \in B(X)$ with $\mu(A) = \infty$ and $B \in B(Y)$ with $\nu(B) = \infty$. In fact, given a natural number N, there are an $A_0 \in B(X)$ with $A_0 \subset A$ and a $B_0 \in B(Y)$ with $B_0 \subset B$ such that $\mu(A_0) > N$ and $\mu(B_0) > N$. Then

$$\lambda(A \times B) \ge \lambda(A_0 \times B_0) = \mu(A_0)\nu(B_0) > N^2 ,$$

which implies $\lambda(A \times B) = \mu(A)\nu(B) = \infty$.

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The following example shows that we cannot have $\lambda(A \times B) = \mu(A)\nu(B)$ for some $A \in \mathcal{B}(X)$ with $\mu(A) = 0$ and $B \in \mathcal{B}(Y)$ with $\mu(B) = \infty$.

EXAMPLE 2.5. Let $X = [0, 1]$ with the usual topology and $Y = [0, 1]$ with the discrete topology. Suppose that $C \subset Y$ is not Lebesgue measurable. We define a measure ν on Y by

$$\nu(B) = \text{the cardinality of } B \cap C$$

for $B \subset Y$. Then ν is a Radon measure as well as a semi-finite pre-Radon measure. Let μ be the Lebesgue measure on X. By Theorem 2.3, there is a unique pre-Radon measure λ on $X \times Y$ satisfying the condition $\lambda(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ such that $\mu(A) < \infty$ and $\nu(B) < \infty$. Fix an $x_0 \in X$. Then $\mu(\{x_0\})\nu(Y) = 0.\infty = 0$. On the other hand, take any open subset W of $X \times Y$ which includes the set $\{x_0\} \times Y$. Let

$$Y_n = \{y \in Y : \mu(W(y)) \geq 1/n\}, \quad n = 1, 2, \ldots,$$

where $W(y) = \{x \in X : (x, y) \in W\}$. Since Y is equal to the union of $\{Y_n : n = 1, 2, \ldots\}$, there exists a natural number n for which $Y_n \cap C$ is an infinite set. Hence $\lambda(W) = \infty$ since W includes the union of the family $\{W(y) \times \{y\} : y \in Y_n \cap C\}$. Since λ is outer regular, we have $\lambda(\{x_0\} \times Y) = \infty$; in other words, $\lambda(\{x_0\} \times Y) \neq \mu(\{x_0\})\nu(Y)$.

Furthermore, this example shows that the statement in Theorem 1.1 does not always hold for the non-σ-finite case. In fact, let f be the characteristic function of the Borel subset $E = \{(x, x) : x \in [0, 1]\}$ of $X \times Y$. Then the function $x \mapsto \int_Y f(x, y)d\nu(y)$ is not μ-measurable.

REMARK 2.6. In the above example, there is a Radon measure ρ on $X \times Y$ such that $\rho(A \times B) = \mu(A)\nu(B)$ for each $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ by Theorem 1.2. Further, note that $\left(X \times Y, \overline{\mathcal{B}(X \times Y)}, \rho^*\right)$ is a quasi-Radon measure space, where ρ^* is the outer measure derived from ρ, $\overline{\mathcal{B}(X \times Y)}$ is the completion of $\mathcal{B}(X \times Y)$ with respect to ρ. For every compact set K, there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ of Y such that $K = \bigcup_{i=1}^{n} K(y_i) \times \{y_i\}$. So $\lambda(K) = \sum_{i=1}^{n} \mu(K(y_i))\nu(\{y_i\})$. Thus,

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given a Borel subset B of $X \times Y$, we have

$$\rho(B) = \sup\{\rho(K) : K \subset B \text{ and } K \text{ is compact}\}$$
$$= \sup\{\lambda(K) : K \subset B \text{ and } K \text{ is compact}\} \leq \lambda(B) .$$

We claim that ρ is different from λ. Indeed, $\rho(E) = 0$ while $\lambda(E) > 0$ for the diagonal set E.

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