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Fibrations and Grothendieck topologies

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Given a site T, that is, a category equipped with a fixed Grothendieck topology, we provide a definition of fibration for morphisms of the presheaves on T. We verify that the notion is well-behaved with respect to composition, base change, and exponentiation, and is trivial on the topos of sheaves. We compare our definition to that of Kan fibration in the semi-simplicial setting. Also we show how we can obtain a notion of fibration on our ground site T and investigate the resulting notion in certain ring-theoretic situations.

1. Introduction

Let T be a site; that is, a category equipped with a fixed Grothendieck topology. We have the adjoint pair

$$S \underset{sh}{\rightleftarrows} [T^0, \mathcal{Sets}]$$

where sh is the associated sheaf functor and S is the full topos of sheaves with respect to the topology. We define a notion of fibration for morphisms of presheaves that is well behaved with respect to composition, base change and exponentiation, and trivializes on the topos S. We investigate how our notion compares with that of Kan fibrations, when $T = \mathcal{Ord}$, the category of finite ordered sets equipped with an appropriate topology. We then observe we can pull our notion of fibration back to the ground site T and we investigate it in certain ring-theoretic situations.

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2. Basic notions

Let $p : E \to B$ be a map (that is, natural transformation) of presheaves. We define

DEFINITION. The map p is a (*weak*) *fibration* if the following diagram in *Sets* is (weak) cartesian:

$$\begin{array}{ccc} E(U) & \longrightarrow & B(U) \\ \downarrow & & \downarrow \\ \ker\left(\prod_i E(U_i) \rightrightarrows \prod_{i,j} E(U_i \times U_j)\right) & \to & \prod_i B(U_i) \end{array}$$

for every covering $\{U_i \to U\}$ in T.

As usual we also define

DEFINITION. X is a (*weak*) *fibrant* object in $[T^0, \mathit{Sets}]$ if $X \to e$ is a (weak) fibration. (e is the final object of $[T^0, \mathit{Sets}]$; $e(U) = \{*\}$ for all U in $\mathrm{ob}(T)$.)

We have three immediate trivialities.

FACTS. 1 Every isomorphism is a fibration.

2 A morphism of sheaves is a fibration.

3 X is (weak) fibrant iff X is a (weak) sheaf.

Weak sheaf is the "dual" notion to separated presheaf; that is, it means the canonical map of sets

$$X(U) \to \ker\left(\prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j)\right) = H^0(\{U_i \to U\}, X)$$

is epic for all coverings $\{U_i \to U\}$ in T. (We freely use the above cohomological abbreviation in the following.)

We now check the desired stability properties.

PROPOSITION 1. *If* $p : E \to B$ *is a (weak) fibration, and* $f : B' \to B$ *is arbitrary then* $p' : E \times_B B' \to B'$ *is a (weak) fibration where*

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$$\begin{array}{ccc} E \times_B B' & \xrightarrow{f'} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[f]{} & B \end{array}$$

is a cartesian square in $[T^0, \mathit{Sets}]$.

Proof. Let $\left\{U_i \xrightarrow{u_i} U\right\}$ be a covering; we check

$$\begin{array}{ccc} (E \times_B B')(U) & \longrightarrow & B'(U) \\ \downarrow & & \downarrow \\ H^0(\{U_i \to U\}, E \times_B B') & \to & \prod_i B'(U_i) \end{array}$$

is (weak) cartesian in Sets .

First observe that pullbacks in $[T^0, \mathit{Sets}]$ are computed pointwise; so $(E \times_B B')(V) = E(V) \times_{B(V)} B'(V)$, for V in $\mathrm{ob}(T)$ and induced maps are the obvious projections. Let s be in $B'(U)$ and (v_i, w_i) be in $H^0(\{U_i \to U\}, E \times_B B')$ where $w_i = p'(U_i)(v_i, w_i) = B'(u_i)(s)$. Consider $f(U)(s)$ in $B(U)$ and $\{v_i\}$ in $\prod_i E(U_i)$. We first observe that

$$B(u_i)f(U)(s) = f(U_i)B'(u_i)(s) = f(U_i)p'(U_i)(v_i, w_i) =$$
$$= p(U_i)f'(U_i)(v_i, w_i) = p(U_i)(v_i) ,$$

since f' is a projection onto the first factor at each "point". Since p is a (weak) fibration, there exists a (unique) t in $E(U)$ such that

(1) $p(U)(t) = f(U)(s)$, and

(2) $E(u_i)(t) = v_i$.

Consider (t, s) in $(E \times_B B')(U)$, by dint of (1) above. Certainly $p'(U)(t, s) = s$ and

$$(E \times_B B')(u_i)(t, s) = (E(u_i)(t), B'(u_i)(s)) = (v_i, w_i)$$

by (2) above. This completes the proof.

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PROPOSITION 2. *Let* $q : X \to E$, $p : E \to B$ *be (weak) fibrations; then* $pq : X \to B$ *is a (weak) fibration.*

Proof. Let $\left\{U_i \xrightarrow{u_i} U\right\}$ be a covering in T and consider

$$\begin{array}{ccc} X(U) & \xrightarrow{pq(U)} & B(U) \\ \downarrow & & \downarrow \\ H^0(\{U_i \to U\}, X) & \longrightarrow & \prod_i B(U_i) \end{array} .$$

Let s be in $B(U)$ and $\{w_i\}$ be in $H^0(\{U_i \to U\}, X)$ such that $B(u_i)(s) = (pq)(U_i)(w_i)$.

Certainly $q(U_i)(w_i)$ is in $H^0(\{U_i \to U\}, X)$ and is compatible with s in the obvious sense. So since $p : E \to B$ is a (weak) fibration, there exists a (unique) t in $E(U)$ such that

(1) $p(U)(t) = s$, and

(2) $E(u_i)(t) = q(U_i)(w_i)$.

Since $q : X \to E$ is a (weak) fibration and the second equality gives us "compatibility", there exists a (unique) z in $X(U)$ such that $X(u_i)(z) = w_i$ and $q(U)(z) = t$. So then

$$(pq)(U)(z) = p(U)q(U)(z) = p(U)(t) = s .$$

This completes the proof.

(Note that Fact 1 and Propositions 1 and 2 verify the (isolated) properties of a fibration in the sense of Quillen's model categories [4].)

We recall now the notion of exponentiation in our functor category $[T^0, Sets]$. Categorically one defines $(-)^Y$ as the right adjoint to the functor $(-) \times Y$. Along with the Yoneda Lemma, this forces the definition in the category of presheaves

$$X^Y(U) \cong \text{nat}\left[\hom_T(-, U), X^Y\right] \cong \text{nat}\left(\hom_T(-, U)\times Y, X\right) .$$

We then have

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PROPOSITION 3. *If* $p : E \to B$ *is a fibration and* K *is a presheaf, then* $p^K : E^K \to B^K$ *is a fibration.*

Proof. See Appendix.

COROLLARY. *If* E *is a sheaf and* K *a presheaf then* E^K *is a sheaf.* (This is well-known; see [6], p. 258.)

3. Semi-simplicial application

We now consider a particular situation. Let $T = \mathcal{O}rd$, the category whose objects are finite ordered sets and the morphisms are weakly monotone maps. It is customary to consider the obvious countable skeletal subcategory whose objects are denoted $n = \{0 < 1 < 2 < \ldots < n\}$. As usual, the simplicial sets are the set-valued presheaves on this category. We describe a Grothendieck topology on $\mathcal{O}rd$ and investigate the resulting notions of fibration and fibrant object. First we define a modified notion of topology.

DEFINITION. A *weak Grothendieck topology* is a category with a notion of covering which satisfies all but the composition axiom for Grothendieck topologies.

A sheaf with respect to a weak Grothendieck topology has the obvious meaning. Certainly it also makes sense to speak of the (weak) Grothendieck topology generated by a partial collection of "coverings". Hence consider the set C;

$$C = \{n \xrightarrow{1} n\} \cup \left\{ n \begin{array}{c} \xrightarrow{d_{i_0}} \\ \vdots \\ \xrightarrow[d_{i_q}]{} \end{array} n+1;\ 0 \le i_0 \le i_1 \le \ldots \le i_r \le q+1,\ r \le q \right\}.$$

We thus obtain a (weak) Grothendieck topology generated by C. We call $\mathcal{O}rd$ with this topology the (weak) combinatorial site.

PROPOSITION 4. X *is a Kan fibration iff* X *is a weak fibration on the weak combinatorial site.*

First we have an easy lemma.

LEMMA. *The following square is cartesian in* $\mathcal{O}rd$ *if* $i < j$;

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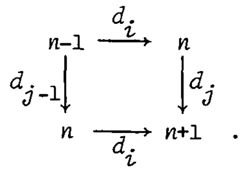
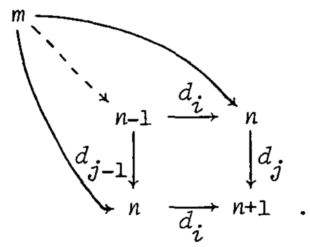


Proof. The diagram commutes by the usual "simplicial" identities in $\mathcal{O}rd$. Suppose we want to fill in the dotted arrow in the following commutative diagram;



Let l be in $m = \{0 < 1 < 2 < \ldots < m\}$. First we claim $e(l) \neq j - 1$. Otherwise $d_j(e'(l)) = d_i(e(l)) = d_i(j-1) = j$. But j is never in the image of d_j . Similarly $e'(l) \neq i$. Hence there exist $x, y < n$ such that $d_{j-1}(x) = e(l)$ and $d_i(y) = e'(l)$. We must show $x = y$. We have two cases.

Case 1. Suppose $e'(l) < i$. Then $e'(l) < j$; so

$$d_i(e(l)) = d_j(e'(l)) = e'(l) < i .$$

Hence $e(l) = e'(l) < i < j$; thus $e(l) < j - 1$ and (in the notation above) $y = e'(l)$, $x = e(l)$; so $x = y$.

Case 2. Suppose $e'(l) > i$. This splits up into two subcases.

(a) Suppose $j \leq e'(l)$. Then

$$d_i(e(l)) = d_j(e'(l)) = e'(l) + 1 > i .$$

Hence $e(l) = (e'(l)+1) - 1 = e'(l)$. So $y = e'(l) - 1$, and $x = y$.

(b) Suppose $e'(l) < j$. Then

$$d_i(e(l)) = d_j(e'(l)) = e'(l) > i .$$

Hence $e(l) = e'(l) - 1$. Since $e'(l) > i$, $y = e'(l) - 1$. Also $e(l) = e'(l) - 1 < j - 1$. So $x = e(l)$ and $x = y$.

This completes Case 2 and the proof.

Proof of Proposition 4. (ONLY IF) Let s_j be in $\prod_{\substack{0 \le j \le n \\ j \ne k}} E(n)$ and t in $B(n+1)$ such that $\partial_i(s_j) = \partial_j(s_{i-1})$, $i < j$, $i, j \ne k$, and $\partial_i(t) = p(n)(s_i)$. We consider the covering $\left\{ n \overset{d_0}{\underset{d_{n+1}}{\overset{\partial_k}{\rightrightarrows}}} n+1 \right\}$ and the hypothesis gives us

$$\begin{array}{ccc} E(n+1) & \longrightarrow & B(n+1) \\ \downarrow & & \downarrow \\ \ker\left(\prod_i E(n) \rightrightarrows \prod_{i,j} E(n \times_{n+1} n)\right) & \rightarrow & \prod_i B(n) \end{array}$$

is weak cartesian.

The lemma identifies $n \times_{n+1} n$ and the maps; hence our assumption implies $(s_0, \ldots, \hat{s}_k, \ldots, s_{n+1})$ is in ker . We thus obtain the desired $(n+1)$-simplex in E from the diagram.

(IF) The converse follows from a standard fact about Kan fibrations (see [3], p. 26) and the fact that C is closed under fibre products; hence is itself the weak Grothendieck topology. To prove the latter claim we first recall the unique factorization of morphisms in $\mathcal{O}rd$ as strings of d_i's and s_j's (see [3], p. 4). Since juxtapositions of cartesian squares are cartesian, it suffices to check closure under fibre products induced by the d_i's and s_j's individually. This is tedious and left to the reader.

COROLLARY. *X is a Kan complex iff X is a weak sheaf on the weak combinatorial site.*

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4. Fibrations on T : examples

It is also possible to obtain a notion of fibration on our ground site T . We have the fully faithful Yoneda embedding

$$T \xrightarrow{h} [T^0, \mathit{Sets}]$$

along which we can in some sense "pull back". Suppose $p_* : \hom_T(-, E) \to \hom_T(-, B)$ is a morphism of representable presheaves induced by $p : E \to B$. By definition, p_* is a fibration if the following square is cartesian:

$$\begin{array}{ccc}
\hom_T(U, E) & \longrightarrow & \hom_T(U, B) \\
\downarrow & & \downarrow \\
H^0(\{U_i \to U\}, \hom_T(-, E)) & \to & \prod_i \hom(U_i', B)
\end{array}$$

for every covering $\{U_i \to U\}$ in T .

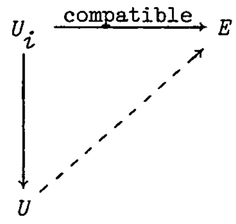
In other words we have the following lifting property;

$$\begin{array}{ccc}
U_i & \xrightarrow{\text{compatible}} & E \\
\downarrow & \nearrow & \downarrow p \\
U & \longrightarrow & B
\end{array}$$

where "compatible" means the diagram

$$\begin{array}{ccccc}
 & & U_i & & \\
 & \nearrow & & \searrow & \\
U_i \times_U U_j & & & & E \\
 & \searrow & & \nearrow & \\
 & & U_j & &
\end{array}$$

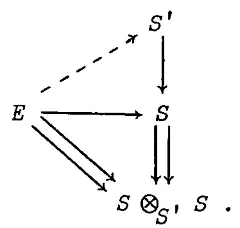
commutes for all i and j . Similarly E in $\mathrm{ob}(T)$ is fibrant if the following diagram can always be completed:



Numerous sites appear in algebro-geometric contexts. To consider a particularly simple example let T be the category of affine schemes over $\mathrm{spec}(R)$; that is, the opposite of the category of commutative R-algebras and declare a covering to be a single faithfully flat morphism $\mathrm{spec}(B) \to \mathrm{spec}(A)$. (These "affine" sites appear in Dobbs [1] under the name R-based topologies.) What are the fibrant R-algebras? We have the following observation.

PROPOSITION 5. *E is a fibrant R-algebra iff for any faithfully flat morphisms $S' \to S$, and homomorphism $f : E \to S$, for every e in E , $f(e) \otimes 1 = 1 \otimes f(e)$ in $S \otimes_{S'} S$.*

Proof. By faithfully flat descent the lifting below exists iff the bottom oblique arrows are equal;



Also the following is true.

PROPOSITION 6. *If E is fibrant then $B \to E$ is always a fibration.*

Proof. Suppose we have the diagram

$$\begin{array}{ccc} S & \leftarrow & E \\ \uparrow & & \uparrow \\ S' & \leftarrow & B \end{array} .$$

Since E is fibrant there exists a map $E \to S'$ making the resulting upper triangle commute. But since $S' \to S$ is a monomorphism the lower triangle also commutes.

For simplicity let us suppose $R = \mathbb{Z}$, so we are considering the category of commutative rings. We have three properties of fibrations.

PROPOSITION 7. *If* $p_i : B_i \to E_i$, $i = 1, 2$, *are fibrations then so is* $p_1 \otimes p_2 : B_1 \otimes B_2 \to E_1 \otimes E_2$.

COROLLARY. *Fibrant rings are closed under tensor product (equals corpoduct).*

PROPOSITION 8. *Epimorphisms of rings are fibrations.*

COROLLARY. *Fibrant rings are closed under homomorphic images.*

PROPOSITION 9. *If* A *is a ring,* S *a multiplicatively closed subset of* A *, then the localization map* $A \to S^{-1}A$ *is a fibration.*

COROLLARY. *Fibrant rings are closed under taking rings of fractions.*

PROPOSITION 10. *Fibrant rings are rigid (that is, have no non-trivial automorphisms).*

We now can produce many examples and non-examples of fibrant rings. $\mathbb{Z}$ is trivially fibrant, and all subrings of the rationals are fibrant by Corollary 3. The finite cyclic rings are fibrant by Corollary 2. The rings $R \times R$, for R arbitrary, and the complex numbers are non-examples by Proposition 4. If R is noetherian, $R[t]$ is never fibrant by considering the faithfully flat morphism $R \to R[[t]]$. We provide a representative proof.

Proof of Proposition 9. Suppose we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & R \\ \downarrow & & \downarrow i \\ S^{-1}A & \xrightarrow[g]{} & R' \end{array}$$

with i faithfully flat. We must check $f(S)$ is contained in $R^{\bullet}$, the invertible elements of R . Let s be in S . Consider the R-module $R/f(s)R$. We claim that $R' \otimes_R (R/f(s)R) = 0$. We compute

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$$s \otimes \left(r+f(s)R\right) = g\left(\frac{s}{1}\right)g\left(\frac{1}{s}\right)s \otimes \left(r+f(s)R\right)$$
$$= if(s)g\left(\frac{1}{s}\right)s \otimes \left(r+f(s)R\right)$$
$$= g\left(\frac{1}{s}\right)s \otimes f(s)\left(r+f(s)R\right) = 0 .$$

Hence by faithful flatness, $R = f(s)R$; so $1 = f(s)r$ for some r in R . The desired map $S^{-1}A \to R$ can now be constructed.

There exist two other examples where we can identify the fibrations.

EXAMPLE 1. Let T be an arbitrary category with topology defined by the universally effective epimorphisms; that is, $\{U_i \to U\}$ is a covering in T iff for all objects X of T ,

$$\hom_T(U, X) \xrightarrow{\approx} H^0\left(\{U_i \to U\}, \hom_T(-, X)\right)$$

is an isomorphism. Then since the definition forces every representable functor to be a sheaf, by Fact 2 above, every morphism is a fibration.

EXAMPLE 2. Let R be a commutative ring. If S is an R-algebra and M an S-module, Quillen [5] defines a cohomology theory $D^*(S/R, M)$ based on a Grothendieck topology on the category of S-algebras where a covering is a single S-algebra epimorphism with nilpotent kernel. Since all our notions are dualized, $p : B \to E$ is a fibration iff for every commutative square the dotted arrow exists;

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle \text{nilpotent kernel}}\downarrow & \nwarrow & \downarrow \\ A' & \longrightarrow & E \end{array} .$$

In the terminology of Grothendieck [2] we conclude the fibrations are precisely the formally unramified morphisms.

Appendix

We provide here a detailed proof of Proposition 4 ("Exponentiation").

Proof. We must check the following square is cartesian;

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$$\begin{array}{ccc} E^K(U) & \longrightarrow & B^K(U) \\ \downarrow & & \downarrow \\ H^0\left(\{U_i \to U\},\, E^K\right) & \to & \prod_i B^K(U_i) \end{array}$$

for an arbitrary covering $\left\{U_i \xrightarrow{u_i} U\right\}$. Letting $(-, -)$ denote $\hom_T(-, -)$, this square becomes

$$\begin{array}{ccc} \mathrm{nat}\left(K(-)\times(-,U)\to E(-)\right) & \longrightarrow & \mathrm{nat}\left(K(-)\times(-,U)\to B(-)\right) \\ \downarrow & & \downarrow \\ H^0\left(\{U_i\to U\},X\mapsto \mathrm{nat}\left(K(-)\times(-,X)\to E(-)\right)\right) & \to & \prod_i \left(\mathrm{nat}\left(K(-)\times\left(-,U_i\right)\to B(-)\right)\right) \end{array} .$$

So consider some $h : K(-) \times (-, U) \to B(-)$ and a compatible collection $\left\{t_i : K(-)\times\left(-, U_i\right) \to E(-)\right\}$ of natural transformations such that

(*) $$p\circ t_i = h\circ\left(1\times\left(u_i\right)_*\right) .$$

We want a natural transformation $t : K(-) \times (-, U) \to E(-)$ such that

(1) $$p\circ t = h$$

and

(2) $$t\circ\left(1\times\left(u_i\right)_*\right) = t_i .$$

Let X be an object of T and consider (s, f) in $K(X) \times (X, U)$. We have a cartesian square in T,

$$\begin{array}{ccc} X\times_U U_i & \xrightarrow{g_i} & U_i \\ {\scriptstyle e_i}\downarrow & & \downarrow{\scriptstyle u_i} \\ X & \xrightarrow[f]{} & U \end{array} .$$

By the fibre-product axiom for Grothendieck topologies we have $\left\{X \times_U U_i \to X\right\}$ is a covering of X. Since $p : E \to B$ is a fibration we have the following cartesian square;

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$$(3)\qquad \begin{array}{ccc} E(X) & \longrightarrow & B(X) \\ \downarrow & & \downarrow \\ H^0(\{X \times_U U_i \to X\},\, E) & \to & \prod_i B(X \times_U U_i) \end{array} .$$

Consider $h(X)(s, f)$ in $B(X)$ and $t_i(X \times_U U_i)(Ke_i(s), g_i)$ in $E(X \times_U U_i)$. We claim

$$(4)\qquad (Be_i)(h(X)(s, f)) = p(X \times_U U_i)(t_i(X \times_U U_i)(Ke_i(s), g_i)$$

and

$$(5)\qquad t_i(X \times_U U_i)(Ke_i(s), g_i) \text{ is in } H^0(\{X \times_U U_i \to X\}, E) .$$

Proof of (4). By naturality of h and (*),

$$\begin{aligned} (Be_i)(h(X)(s, f)) &= h(X \times_U U_i)(Ke_i \times e_i^*)(s, f) \\ &= h(X \times_U U_i)(1\times(u_i)_*)(Ke_i(s), g_i) \\ &= p(X \times_U U_i)t_i(X \times_U U_i)(Ke_i(s), g_i) . \end{aligned}$$

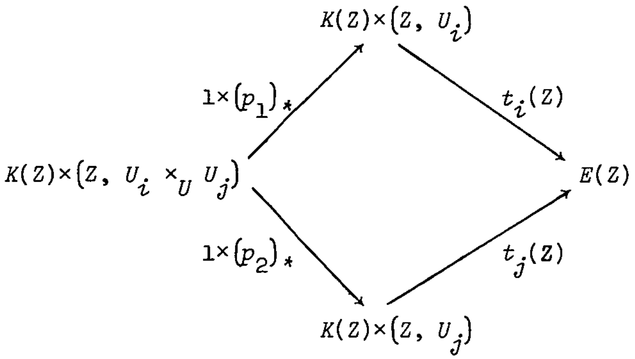
Proof of (5). This requires verifying

$$E(1\times p_1)(t_i(X \times_U U_i)(Ke_i(s), g_i)) = E(1\times p_2)(t_j(X \times_U U_j)(Ke_j(s), g_j))$$

where we have maps

$$\begin{array}{ccc} & & X \times_U U_i \\ & \overset{1\times p_1}{\nearrow} & \\ Z = X \times_U U_i \times_U U_j & \xrightarrow{\;q\;} & U_i \times_U U_j . \\ & \underset{1\times p_2}{\searrow} & \\ & & X \times_U U_j \end{array}$$

Using the compatibility of the t_i's we know the following diagram commutes;

(6) 

By naturality of t_i ,

$$E(1\times p_1)\big(t_i(X\times_U U_i)\big)\big(Ke_i(s),\ g_i\big) = t_i(Z)\big(K(1\times p_1)\times(1\times p_1)^*\big(Ke_i(s),\ g_i\big)\big)$$
$$= t_i(Z)\big(K\big(e_i\circ(1\times p_1)\big)(s),\ g_i\circ(1\times p_1)\big)$$
$$= t_i(Z)\big(K\big(e_i\circ(1\times p_1)\big)(s),\ p_1\circ q\big)\ .$$

By considering $K\big(e_i\circ(1\times p_1)(s),\ q\big)$ in $K(Z)\times(Z,\ U_i\times U_j)$ appearing in diagram (6) we can continue our computation;

$$= t_j(Z)\big(K\big(e_j\circ(1\times p_2)\big)(s),\ g_j\circ(1\times p_2)\big)$$
$$= t_j(Z)\big(K(1\times p_2)\times(1\times p_2)^*\big)\big(Ke_j(s),\ g_j\big)$$
$$= E(1\times p_2)\big(t_j(X\times_U U_j)\big)\big(Ke_u(s),\ g_j\big)\ ,$$

by the naturality of t_j . This completes the proof of (5).

Now by our cartesian square (3), there exists a unique z in $E(X)$ such that

(7) $$p(X)(z) = h(X)(s,\ f)$$

and

(8) $$(Ee_i)(z) = t_i(X\times_U U_i)\big(Ke_i(s),\ g_i\big)\ .$$

We then define $t(X)(s,\ f) = z$. First we claim that

(9) $$t : K(-)\times(-,\ U)\to E(-)$$

is a natural transformation.

Proof of (9). Let $F : Y\to X$ be a morphism in T and consider the diagram

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$$\begin{array}{ccc} K(X)\times(X,\,U) & \xrightarrow{t(X)} & E(X) \\ {\scriptstyle KF\times F^*}\downarrow & & \downarrow{\scriptstyle EF} \\ K(Y)\times(Y,\,U) & \xrightarrow[t(Y)]{} & E(Y)\ . \end{array}$$

We must show that $(EF)\big(t(X)(s,\,f)\big) = t(Y)\big((KF)(s),\,fF\big)$.

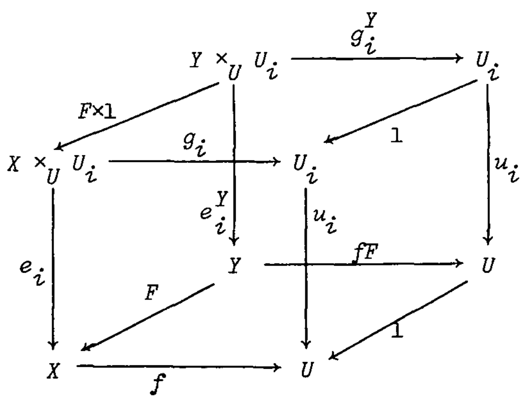
By our definition of t this requires showing

(10) $$p(Y)\Big((EF)\big(t(X)(s,\,f)\big)\Big) = h(Y)\big(KF(s),\,fF\big)$$

and

(11) $$\left(Ee_i^Y\right)\Big((EF)\big(t(X)(s,\,f)\big)\Big) = t_i\left(Y\times U_i\right)\left(Ke_i^Y\big(KF(s)\big),\,g_i^Y\right)$$

where the maps mentioned appear in the following cube:



Proof of (10).

$$\begin{aligned} p(Y)(EF)\big(t(X)(s,\,f)\big) &= (BF)p(X)\big(t(X)(s,\,f)\big) && \text{by naturality of } p \\ &= (BF)\big(h(X)(s,\,f)\big) && \text{by definition of } t \\ &= h(Y)(Ke\times e^*)(s,\,f) && \text{by naturality on } h \\ &= h(Y)\big(Ke(s),\,fF\big)\ . \end{aligned}$$

Proof of (11).

$$\begin{aligned} \left(Ee_i^Y\right)(EF)\big(t(X)(s,\,f)\big) &= E\left(Fe_i^Y\right)\big(t(X)(s,\,f)\big) \\ &= E\Big(e_i\big(F\times 1_{U_i}\big)\Big)\big(t(X)(s,\,f)\big) \end{aligned}$$

by cube. Now using our definition of t and the naturality of t_i ,

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$$= E\left(F\times 1_{U_i}\right)\left(Ee_i\right)\left(t(X)(s,\ f)\right) = E\left(F\times 1_{U_i}\right)\left(t_i\left(X\times U_i\right)\left(Ke_i(s),\ g_i\right)\right)$$
$$= t_i\left(Y\times_U U_i\right)\left(K\left(F\times 1_{U_i}\right)\times\left(F\times 1_{U_i}\right)^*\left(Ke_i(s),\ g_i\right)\right)$$
$$= t_i\left(Y\times_U U_i\right)\left(Ke_i^Y\left(KF(s)\right),\ g_i^Y\right) .$$

This completes the proof of (11) and, hence, (9). We now assert that t satisfies our original two requirements, (1) and (2).

Proof of (1). This follows immediately from (7).

Proof of (2). This statement translates into

$$t_i(X)(s,\ f) = t(X)\left(s,\ u_i\circ f\right) ,$$

where s is in $K(X)$ and f is in $\left(X,\ U_i\right)$. By definition of t this requires showing that

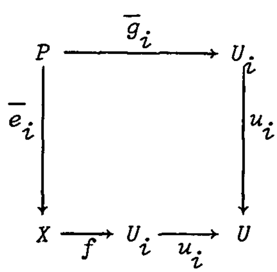
(12) $$p(X)\left(t_i(X)(s,\ f)\right) = h(X)\left(s,\ u_i\circ f\right)$$

and

(13) $$\left(E\overline{e}_i\right)\left(t_i(X)(s,\ f)\right) = t_i(P)\left(K\overline{e}_i(s),\ \overline{g}_i\right) .$$

Proof of (12). This follows immediately from (*).

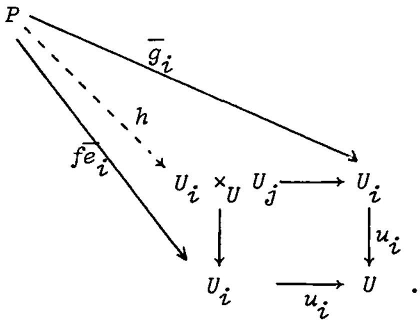
Proof of (13). The maps mentioned in (13) come from the following cartesian square,



By the naturality of t_i ,

(14) $$\left(E\overline{e}_i\right)\left(t_i(X)(s,\ f)\right) = t_i(P)\left(K\overline{e}_i\times\overline{e}_i^*\right)(s,\ f) = t_i(P)\left(K\overline{e}_i(s),\ f\circ\overline{e}_i\right) .$$

Consider the following dotted arrow h ,



In diagram (6), let $i = j$, $Z = P$ and consider $(K\bar{e}_i(s), h)$ in $K(P) \times (P, U_i \times_U U_j)$. This gives the equality

$$t_i(P)(K\bar{e}_i(s), f\circ\bar{e}_i) = t_i(P)(K\bar{e}_i(s), \bar{g}_i) .$$

Together with computation (14), this completes the proof of (13) and thus, Proposition 4.

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