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SENSITIVITY AND CONTROLLABILITY OF SYSTEMS GOVERNED BY INTEGRAL EQUATIONS VIA PROXIMAL ANALYSIS

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ABSTRACT In this paper, we are concerned with the basic problem defined in [9] Formulas for $\partial V(0)$ and $\partial^\infty V(0)$, respectively the generalized and asymptotic gradient of the value function at zero, corresponding to an L^2-additive perturbation of dynamics are given Under the normality condition, $\partial V(0)$ turns out to be a compact subset of L^2, formed entirely of arcs, and V is locally finite and Lipschitz at 0 Moreover, estimations of the generalized directional derivative and Dini's derivative of V at 0 are derived Supplementary conditions imply that Dini's derivative of V at 0 exists, and V is actually strictly differentiable at this point

1. **Introduction.**

1.1 *Proximal Analysis.* Let X be a real Banach space whose norm is denoted $\|\cdot\|$ and C a closed subset of X. The space X is denoted respectively H or $\mathbb{R}^n$ whenever it is a Hilbert space or finite-dimensional. When $X = H$ or $\mathbb{R}^n$, we denote its inner product by $\langle\cdot,\cdot\rangle$, which gives the norm $\|\cdot\|$. We define the distance function from C, $d_C(\cdot)$ by $d_C(x) := \inf\{\|c-x\| : c \in C\}$.

Proximal analysis started with the notion of perpendiculars introduced by Clarke in the finite-dimensional case [4], which has given rise to a formula for the generalized gradient of the distance function from C, $\partial d_C(\cdot)$. The fact that the normal cone of C, $N_C(\cdot)$ is the w^*-closed convex cone generated by $\partial d_C(\cdot)$, involves an important object which is called *proximal normal vector*.

In fact, when $X = H$, a vector $\xi \in X$ is said to be *perpendicular* to C at $x \in C$, and we write $\xi \perp C$ at x, if $\xi = x' - x$, where x is the unique nearest point from C to x'. The fact that $\xi \perp C$ at x or equivalently $d_C(x+\xi) = \|\xi\|$ is equivalent to the inequality

$$\langle \xi, c-x\rangle \leq (1/2)\|c-x\|^2 \quad \forall c \in C.$$

This inequality can be interpreted as the assertion that the point $x \in C$ minimizes over C the functional: $-\langle\xi,c\rangle + (1/2)\|c-x\|^2$ and in practice, this assertion leads to the study of a certain optimization problem, which characterizes the perpendiculars.

When X is not necessarily a Hilbert space, we define the perpendicularity as follows: a functional $\xi^* \in X^*$ is said to be *perpendicular* to C at x, and we write $\xi^* \perp C$ at x, if

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there exists $y \in X^* \setminus C$ such that:

$$d_C(y) = \|y - x\|,\ (\xi^*, y - x) = \|y - x\|^2, \text{ and } \|\xi^*\| = \|y - x\|.$$

That is $\xi^*/\|y - x\|$ supports the unit ball of X at $(y - x)/\|y - x\|$. We can easily check that this general perpendicularity is identical to that for $X = H$; it suffices to note the identity:

$$\|\xi - (y - x)\|^2 = \|\xi\|^2 + \|y - x\|^2 - 2\langle \xi, y - x\rangle \quad \text{for } \xi \in H.$$

The generalized gradient of the distance function $d_C(\cdot)$ at x, $\partial d_C(x)$ is defined in terms of perpendiculars to C at x by:

$$\partial d_C(x) = \overline{\text{co}}\{0\} \bigcup \left\{ w^*\text{-}\lim_\iota \frac{\xi_\iota}{\|\xi_\iota\|_*} : \xi_\iota \perp C \text{ at } x_\iota \text{ with } \xi_\iota \longrightarrow 0 \text{ and } x_\iota \longrightarrow x \text{ in } C \right\} \tag{1.3}$$

where w^*-lim denotes the weak star limit, which coincides with the weak limit if $X = H$ and with the strong limit if $X = \mathbb{R}^n$, in which case the closure in (1.3) is superfluous, and $\|\cdot\|_*$ denotes the dual norm corresponding to $\|\cdot\|$. Another alternative approach to calculate $\partial d_C(x)$ in the finite-dimensional case is given in [4]. It has been extended to infinite-dimensional case by Borwein-Giles [1].

The normal cone of C at x, $N_C(x)$ is defined by:

(1.4)

$$\begin{aligned} N_C(x) &:= \text{cl}\left\{ \bigcup_{\lambda \geq 0} \lambda \partial d_C(x) \right\} \\ &= \overline{\text{co}}\left\{ w^*\text{-}\lim_\iota \lambda_\iota \cdot \frac{\xi_\iota}{\|\xi_\iota\|_*} : \lambda_\iota \geq 0, \xi_\iota \perp C \text{ at } x_\iota \text{ with } x_\iota \longrightarrow x \text{ in } C \text{ and } \xi_\iota \longrightarrow 0 \right\}, \end{aligned}$$

where cl denotes the w^*-closures and $\overline{\text{co}}$ denotes the w^*-closed convex hull. To this normal cone, we associate another cone denoted $\bar{N}_C(x)$, which is called the *upper semicontinuous normal cone* to C and which is defined by:

$$\bar{N}_C(x) := \{\lim_\iota \xi_\iota : \xi_\iota \in N_C(x_\iota) \text{ and } x_\iota \longrightarrow x \text{ in } C\}.$$

Under mild regularity conditions on C, $N_C(\cdot)$ and $\bar{N}_C(\cdot)$ coincide.

An element $\xi^* \in X^*$ is said to be a *proximal normal functional* to C at x, if it is a scalar positive multiple of a perpendicular functional to C at x; that is:

there exist $y \in X/C$, and $\lambda > 0$ such that: $\xi^* = \lambda \xi_1^*$ with $d_C(y) = \|y - x\|$, $\|\xi_1^*\| = 1$, and $(\xi_1^*, y - x) = \|y - x\|$.

Note that when $X = H$, this last definition is equivalent to, say, that an element $\xi \in X$ (because $X^* = X$) is a proximal normal vector to C at x if there exists $\sigma > 0$ such that

$$\langle \xi, c - x\rangle \leq \sigma\|c - x\|^2 \quad \forall c \in C, \tag{1.5}$$

or equivalently $d_C(\xi/2\sigma + x) = (1/2\sigma)\|\xi\|$, and we find (1.1) by taking $\sigma = 1/2$. We denote by $\text{PN}_C(x)$ the set of all proximal normal vectors to C at x, which is a convex cone containing always 0 when $X = H$.

The best thing that proximals provide is the proximal normal formula:

$$N_C(x) = \overline{\text{co}}\{w^*\text{-}\lim_\iota \xi_\iota : \xi_\iota \in \text{PN}_C(x_\iota), x_\iota \longrightarrow x \text{ in } C\}, \tag{1.6}$$

which was first proved by Clarke [4] in the finite-dimensional case and extended by Borwein-Strojwas [2] in the infinite-dimensional setting, for X being any reflexive Banach space. For the particular case $X = H$, Loewen [6] has given a simple proof based essentially on geometrical properties in Hilbert space; in this case w^*-lim is replaced by w-lim.

1.2 *Clarke's derivative, Dini's derivative and strict derivative.* Let $f\colon X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l. s. c.) function and $x \in X$ a point at which f is finite. Then the epigraph of f, $\text{epi}\, f := \{(x, \lambda) : f(x) \leq \lambda\}$ is a closed subset of $X \times \mathbb{R}$, and $\big(x, f(x)\big) \in \text{epi}\, f$. We define the generalized gradient and generalized asymptotic gradient of f at x, respectively $\partial f(x)$ and $\partial^\infty f(x)$ as follows:

$$\partial f(x) := \Big\{\xi \in X^* : (\xi, -1) \in N_{\text{epi}\, f}\big(x, f(x)\big)\Big\} \tag{1.7}$$

$$\partial^\infty f(x) := \Big\{\xi \in X^* : (\xi, 0) \in N_{\text{epi}\, f}\big(x, f(x)\big)\Big\}, \tag{1.8}$$

Then $\partial f(x)$ is a w^*-closed convex subset of X^* and $\partial^\infty f(x)$ is a w^*-closed convex cone of X^* always containing 0. Thus, elements of $\partial f(x)$ and $\partial^\infty f(x)$ are captured by elements of $N_{\text{epi}\, f}\big(x, f(x)\big)$ via (1.7) and (1.8) and those of the latter are captured by normal proximals to epi f via (1.6) in a way that characterization and nature of $\partial f(x)$ and $\partial^\infty f(x)$ depend on $\text{PN}_{\text{epi}\, f}\big(x, f(x)\big)$. This diagram appears clearly in the proof of Theorem 4.1 in the sequel.

Recall that the support function of a nonempty subset C of X is the function $\sigma_C\colon X^* \longrightarrow \mathbb{R} \cup \{\infty\}$ defined by:

$$\sigma_C(\zeta) := \sup\{(\zeta, x) : x \in C\}.$$

A function $f\colon X \longrightarrow \mathbb{R}$ is said to be *locally Lipschitz* at x if it is Lipschitz in a neighbourhood of x, that is there exists an open set U in X containing x and for some positive constant K

$$|f(y) - f(z)| \leq K\|y - z\| \quad \forall y, z \in U.$$

Likewise, f is locally Lipschitz on an open subset A of X, if it is locally Lipschitz at each point of A. When f is locally Lipschitz at x, we prove that $\partial f(x)$ is exactly the set whose support function is $f^0(x; \cdot)$, given by:

$$f^0(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}, \quad \text{for all } v \in X. \tag{1.9}$$

The latter is known as Clarke's (or generalized) directional derivative of f at x. In this case $\partial f(x)$ is bounded and then it is w^*-compact.

The function f is said to be *regular* at x if its usual right directional derivative $f'_r(x; \cdot)$ exists and coincides with $f^0(x; \cdot)$.

The four Dini's derivatives of f at x are defined as follows:

$$f^+(x;v) := \limsup_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t},$$
$$f_+(x;v) := \liminf_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t},$$
$$f^-(x;v) := -f_+(x;-v), \text{ and}$$
$$f_-(x;v) := -f^+(x;-v), \quad \text{for all } v \in X.$$

We say that f has a right Dini's derivative at x, which is denoted $f_r'(x;\cdot)$, if $f^+(x;\cdot)$ and $f_+(x;\cdot)$ exist and coincide. The left Dini's derivative $f_l'(x;\cdot)$, if it exists, is defined in a similar way.

Recall that a Lipschitz function $f: X \longrightarrow \mathbb{R}$ is said to be *strictly differentiable* at x (in Clarke's terminology) with strict derivative $D_s f(x) \in X^*$, provided that for each $v \in X$, we have:

$$\lim_{\substack{x'\to x\\ t\downarrow 0}} \frac{f(x'+tv)-f(x')}{t} = \big(D_s f(x), v\big).$$

A practical characterization of strict differentiability of f at x is given by the condition that the generalized gradient of f at x, $\partial f(x)$ is reduced to a singleton (see [4], Proposition 2.2.4).

1.3 *Relaxed controls.* Let Ω denote a compact nonempty subset of R^m and $L^1\big(T, C(\Omega)\big)$ be the Banach space formed by classes of functions $f: T\times\Omega \longrightarrow \mathbb{R}$, $(t,u) \longmapsto f(t,u)$, such that $f(\cdot,u)$ is measurable, $f(t,\cdot)$ is continuous, and $\max\{|f(t,u)| : u \in \Omega\}$ is in $L^1(T,\mathbb{R})$. The norm in this space is taken as $\|f\| := \int_T \max\{|f(t,u)| : u\in\Omega\}\, dt$.

Let $\mathcal{B}(\Omega)$ denote the Borelian σ-algebra generated by Ω. Define the vector space:

$$\text{rpm}(\Omega) := \{\mu : \mathcal{B}(\Omega) \longrightarrow \mathbb{R} \text{ is a probability Radon measure}\}.$$

We define a *relaxed control* as a mapping $\nu: T \longrightarrow \text{prm}(\Omega)$ which associates to each $t\in T$, a probability Radon measure $\nu(t)$ in $\text{prm}(\Omega)$, such that for all $f \in L^1\big(T, C(\Omega)\big)$, the mapping $t \longmapsto \int_\Omega f(t,u)\, d\nu(t)(u)$ is (Lebesgue) measurable. Note that every ordinary control $u(t)$ can be identified with a relaxed control, namely the Dirac measure $\delta_{u(t)}$ concentrated at $u(t)$ for each $t \in T$. Each relaxed control ν can be regarded as a linear functional on $L^1\big(T, C(\Omega)\big)$ defined by

$$\langle \nu, f\rangle = \nu(f) := \int_T \int_\Omega f(t,u)\, d\nu(t)(u)\, dt \quad \text{for all } f \in L^1\big(T, C(\Omega)\big).$$

We denote by $\mathcal{R}$ the set of classes (modulo Lebesgue measure) of all such relaxed controls. If a function $f: T\times\Omega \longrightarrow \mathbb{R}^n$, $(t,u) \longmapsto f(t,u)$ is in $L^1\big(T, C(\Omega)\big)$, then we denote by $f\big(t,\nu(t)\big)$ the integral $\int_\Omega f(t,u)\, d\nu(t)(u)$, in such a way that $\langle\nu, f\rangle = \int_T f\big(t,\nu(t)\big)\, dt$ for $\nu \in \mathcal{R}$.

Two natural topologies can be defined on $\mathcal{R}$. The first one is the dual norm topology defined by $\|f\| = \sup\{|\nu(f)| : \|f\| \le 1\} = \text{ess sup}\{|\nu(t)|(\Omega) : t \in T\}$, where $|\nu(t)|$ is the total variation of $\nu(t)$. The second is the w^*-topology whose convergence is defined by: $\nu_\iota \longrightarrow \nu(w^*)$ iff $\langle \nu_i, f\rangle \longrightarrow \langle \nu, f\rangle$ for all $f \in L^1\big(T, C(\Omega)\big)$. It is demonstrated in [10] that $\mathcal{R}$ is convex, and that with this topology, $\mathcal{R}$ is compact and sequentially compact and is the closure of $\mathcal{U}$ (the class of ordinary controls). For more details concerning this part, we invite the reader to consult [10].

2. **The problem and its perturbation.** Let T denote the interval $[0,1]$. We are given functions $f\colon T \longrightarrow \mathbb{R}$, $F\colon T \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$, $\psi\colon T \longrightarrow \mathbb{R}^n$, $\phi\colon T \times T \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, two subsets Ω and C of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and a tube Ω_1 in $T \times \mathbb{R}^n$, which is by definition a subset of $T \times \mathbb{R}^n$ of the form

$$\big\{(t,x) \in T \times \mathbb{R}^n : |x - w(t)| < \varepsilon(t)\big\},$$

where $w(\cdot)$ and $\varepsilon(\cdot)$ are continuous functions with $\varepsilon(\cdot)$ positive.

For each $\alpha(\cdot) \in L^2(T, \mathbb{R}^n)$, we denote by $P(\alpha)$ the α-perturbed original problem, which consists in minimizing the cost functional

$$J(x,u) := f\big(x(1)\big) + \int_0^1 F\big(t, x(t), u(t)\big)\, dt, \tag{2.1}$$

over all state/control pairs (x,u) satisfying the constraints

$$\begin{cases} \dot{x}(t) = \dot{\psi}(t) + \phi\big(t,t,x(t),u(t)\big) + \int_0^t \phi_t\big(t,s,x(s),u(s)\big)\, ds + \alpha(t) \quad (t \in T), \\ x(0) = \psi(0), \end{cases} \tag{2.2}$$
$$u(t) \in \Omega \quad \text{(a.e.)-}t,\ x(1) \in C \text{ and } \big(t, x(t)\big) \in \bar{\Omega}_1 \text{ for each } t \in T.$$

The state x of $P(\alpha)$ is an absolutely continuous function from T to $\mathbb{R}^n$ and its control u is a measurable function from T to $\mathbb{R}^m$. In particular, $P(0)$ is the unperturbed original problem. We denote by $\tilde{P}(\alpha)$ $(\alpha \neq 0)$ and $\tilde{P}(0)$ respectively the relaxed α-perturbed problem and the relaxed unperturbed problem, in the sense of [9, Section 4]. More precisely, $\tilde{P}(\alpha)$ is defined as follows:

$$\tilde{P}(\alpha)\colon \quad J(x,\nu) := f\big(x(1)\big) + \int_0^1 F\big(t, x(t), \nu(t)\big)\, dt \longrightarrow \min,$$

under the constraints:

$$\begin{cases} \dot{x}(t) = \dot{\psi}(t) + \phi\big(t,t,x(t),\nu(t)\big) + \int_0^t \phi_t\big(t,s,x(s),\nu(s)\big)\, ds, \\ x(0) = \psi(0), \end{cases}$$
$$\nu \in \mathcal{R}, \quad x(1) \in C \text{ and } \big(t, x(t)\big) \in \bar{\Omega}_1 \text{ for each } t \text{ in } T.$$

We denote by $\tilde{Y}_\alpha$ the set of solutions of $\tilde{P}(\alpha)$ and we set in particular $\tilde{Y}_0 = \tilde{Y}$. We denote by $\mathrm{AC}^2(T, \mathbb{R}^n)$ the Hilbert space of functions $x\colon T \longrightarrow \mathbb{R}^n$ such that x is absolutely continuous and has derivative $\dot{x}$ in $L^2(T, \mathbb{R}^n)$. The inner product in this space is given by: $\langle x, y\rangle = \langle x(0), y(0)\rangle_{\mathbb{R}^n} + \langle \dot{x}, \dot{y}\rangle_{L^2}$. With the associated norm, $\mathrm{AC}^2(T, \mathbb{R}^n)$ is isometric to $L^2(T, \mathbb{R}^n) \times \mathbb{R}^n$.

The following assumptions remain in force along this paper:

(H1) f is locally Lipschitz.

(H2) F is measurable in t, continuously differentiable in x, and continuous in u; ϕ is measurable in s, continuously differentiable in x and t, continuous in (t, x, u), and ϕ_t is continuously differentiable in x.

(H3) For each compact subset Γ of $\mathbb{R}^n \times \mathbb{R}^m$, there exists a function $K(\cdot) \in L^2(T, \mathbb{R})$ such that for (a.e.)-s, $|f(t,s,x,u)|+|\phi_t(t,s,x,u)|+|\phi_{xt}(t,s,x,u)| \leq K(s), \forall (t,x,u) \in T \times \Gamma$; for (a.e.)-$t$, $|F(t,x,u)|\,|F_x(t,x,u)| \leq K(t), \forall (x,u) \in \Gamma$.

(H4) $\psi \in \mathrm{AC}^2(T, \mathbb{R}^n)$.

(H5) C is closed and Ω is compact.

(H6) The set of all admissibles state/control pairs for $\tilde{P}(0)$ is nonempty and all solutions of $\tilde{P}(0)$ remain in Ω_1; that is: $\big(t, x(t)\big) \in \Omega_1$ for each $t \in T$, for all x in $\tilde{Y}$.

In the sequel of this paper, the symbol "$\twoheadrightarrow$" will denote the uniform convergence on T. Existence for $\tilde{P}(\alpha)$ is an immediate consequence of the lemma below, provided that $\mathrm{Ad}\big(\tilde{P}(\alpha)\big)$, the set of admissibles pairs state/control for $\tilde{P}(\alpha)$, is nonempty.

LEMMA 2.1. *Let $(x_i, \nu_i) \in \mathrm{Ad}\big(\tilde{P}(\alpha_i)\big)$, where (α_i) is a sequence in $L^2(T, \mathbb{R}^n)$ converging strongly to α. Then there exists a subsequence of $\{(x_i, \nu_i)\}$, which we do not relabel and $x \in \mathrm{AC}^2(T, \mathbb{R}^n)$, $\nu \in \mathcal{R}$ such that*

$$x_i \twoheadrightarrow x,\ \dot{x}_1 \longrightarrow \dot{x}(w - L^2),\ \nu_i \longrightarrow \nu(w^*),\ J(x_i, \nu_i) \longrightarrow J(x, \nu),\ \textit{and}\ (x, \nu) \in \mathrm{Ad}\big(\tilde{P}(\alpha)\big).$$

PROOF. The proof is similar to that of [9, Lemma 4.1, where $\alpha_i = 0$ for each i], and thus is omitted. ■

Define the set of admissible states for the problem $\tilde{P}(\alpha)$ by:

$$\mathrm{Ad}_s\big(\tilde{P}(\alpha)\big) := \big\{x : T \longrightarrow \mathbb{R}^n \text{ absolutely continuous:} \exists \nu \in \mathcal{R} \text{ such that } (x, \nu) \in \mathrm{Ad}\big(\tilde{P}(\alpha)\big)\big\}.$$

For $x \in \mathrm{Ad}_s\big(\tilde{P}(\alpha)\big)$ and $\lambda \in \{0, 1\}$, we define the λ-multiplier set corresponding to x, denoted $M_\alpha^\lambda(x)$ as the set of all $p \in \mathrm{AC}^2(T, \mathbb{R}^n)$ satisfying the following conditions for a certain $\nu \in \mathcal{R}$:

(i) $-\dot{p}(t) = p(t)\big\{\phi_x\big(t, t, x(t), \nu(t)\big) + \int_0^t \phi_{xt}\big(t, s, x(s), \nu(s)\big)\, ds\big\} - \lambda F_x\big(t, x(t), \nu(t)\big)$ (a.e.),

(ii) $-p(1) \in \lambda \partial f\big(x(1)\big) + \bar{N}_C\big(x(1)\big)$,

(ii) $\max\Big\{p(t) \cdot \big\{\phi\big(t, t, x(t), \mu(t)\big) + \int_0^t \phi_t\big(t, s, x(s), \mu(s)\big)\, ds\big) - \lambda F\big(t, x(t), \mu(t)\big) : \mu \in \mathcal{R}\Big\}$ is attained (a.e.)-t in $\mu = \nu$.

The λ-multiplier set $M_\alpha^\lambda(\tilde{Y}_\alpha)$ corresponding to the problem $\tilde{P}(\alpha)$ is defined as $\bigcup\{M_\alpha^\lambda(x) : x \in \tilde{Y}_\alpha\}$. For $\alpha = 0$, it is simply denoted by $M^\lambda(\tilde{Y})$. The sets $M^1(\tilde{Y})$ and $M^0(\tilde{Y})$ are respectively called the *normal multiplier set* and the *abnormal multiplier set*. Note that 0 is always in the cone $M_\alpha^0(\tilde{Y})$. In [9], it is shown that $[M^0(\tilde{Y}) \backslash \{0\}] \cup M^1(\tilde{Y}) \neq \emptyset$. This condition rephrases Theorem 7.4 of [9].

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3. **Value function.** To each problem $\tilde{P}(\alpha)$, we associate its value, namely the quantity $V(\alpha) := \inf\{J(x,\nu) : (x,\nu) \in \mathrm{Ad}(\tilde{P}(\alpha))\}$, with the convention that $V(\alpha) = +\infty$ if $\mathrm{Ad}(\tilde{P}(\alpha)) = \emptyset$. In this way, we have defined a function $V: L^2(T,\mathbb{R}^n) \longrightarrow \mathbb{R} \cup \{+\infty\}$, $\alpha \longmapsto V(\alpha)$, which is called the *value function* corresponding to the perturbation of the last section. The behaviour of V near the point $\alpha = 0$ has a direct reflection on the problem $\tilde{P}(\alpha)$. The very finiteness of V near 0 corresponds to an important property, namely the local controllability of the system, in the sense that the set $\mathrm{Ad}(\tilde{P}(\alpha))$ is nonempty for α in a neighbourhood of 0. The local Lipschitz property of V near 0 measures the sensitivity rate of the problem value vis-à-vis small perturbations in dynamics (for these two properties, see Theorem 4.2). It is not always the case that the value function is differentiable at 0, except under some conditions (for this, see Corollaries 4.3, 4.4 and 4.5). However, the generalized gradient of V at 0, $\partial V(0)$, contains information about differential properties of V at 0. A formula for $\partial V(0)$ in terms of the multiplier sets $M^0(\tilde{Y})$ and $M^1(\tilde{Y})$ is derived in Theorem 4.1.

The following lemma asserts that for α too small, hypothesis (H6) remains true for the α-perturbed relaxed problem $\tilde{P}(\alpha)$.

LEMMA 3.1. *There exists $\delta > 0$ such that if $\|\alpha\|_{L^2} < \delta$ and $V(\alpha) \leq V(0) + \delta$, then each solution of $\tilde{P}(\alpha)$ remains in Ω_1.*

PROOF. If the lemma is false, then we can construct a sequence (α_i) converging to 0 in $L^2(T,\mathbb{R}^n)$, with $V(\alpha_i) \leq V(0) + 1/i$ for all $i \geq 1$, and a corresponding sequence $(x_i,\nu_i) \in \mathrm{Ad}(\tilde{P}(\alpha_i))$, with $x_i \in \bar{\Omega}_1 \setminus \Omega_1$ for all $i \geq 1$. Then the use of Lemma 2.1, leads to the existence of $(x,\nu) \in \mathrm{Ad}(\tilde{P}(0))$, with $V(\alpha_i) = J(x_i,\nu_i) \longrightarrow J(x,\nu)$ $(\geq V(0))$ for a certain subsequence of (i) which we don't relabel. Thus $V(0) = J(x,\nu)$, and by hypothesis (H6), x must remain in the tube Ω_1. According to Lemma 2.1, x is a uniform limit of (x_i), therefore $x \in \bar{\Omega}_1 \setminus \Omega_1$. Hence a contradiction. ∎

This lemma combined with Lemma 2.1 and the fact that $\tilde{Y}_\alpha \neq \emptyset$, once $\mathrm{Ad}(\tilde{P}(\alpha)) \neq \emptyset$ (which is a consequence of Lemma 2.1), give rise to the following lemma whose proof arguments can be found in [5, p. 538].

LEMMA 3.2. *The value function V is (strongly)* l. s. c. *near zero.* ∎

4. **Main result.** The principal result of this paper is the following theorem, where relations between the generalized and asymptotic gradients of V at 0, $\partial V(0)$ and $\partial V^\infty(0)$ and the normal and abnormal multiplier sets $M^1(\tilde{Y})$, $M^0(\tilde{Y})$ are established.

THEOREM 4.1. *Under the hypotheses (H1), (H2), (H3), (H4), (H5) and (H6), we have the two formulas:*

(1) $\partial V(0) = \overline{\mathrm{co}}\{[-M^1(\tilde{Y})] \cap \partial V(0) + [-M^0(\tilde{Y})] \cap \partial^\infty V(0)\}$;

(2) $\partial^\infty V(0) = \overline{\mathrm{co}}\{[-M^0(\tilde{Y})] \cap \partial^\infty V(0)\}$,

where the operation $\overline{\mathrm{co}}$ is taken in L^2 with the strong topology. ∎

In the remainder, we suppose that the problem $\tilde{P}(0)$ is normal *i.e.*, $M^0(\tilde{Y}) = \{0\}$.

THEOREM 4.2. *The value function is locally finite and Lipschitz at* 0. *Consequently,* $\partial V(0) \neq \emptyset$ *and for each* $\alpha \in \delta B_{L^2}$, $\tilde{Y}_\alpha \neq \emptyset$. *In addition, if* $\tilde{P}(0)$ *is free endpoint, i.e.,* $C = \mathbb{R}^n$, *then values of* $\tilde{P}(\alpha)$ *and* $P(\alpha)$ *are equal for each* $\alpha \in \delta B_{L^2}$. ■

Conditions on the set of solutions $\tilde{Y}$ and on the normal multiplier set $M^1(\tilde{Y})$, imply important consequences, which are expressed in the following corollaries:

COROLLARY 4.3. *We have the formulas:*

$$\partial V(0) = \overline{\mathrm{co}}\{[-M^1(\tilde{Y})] \cap \partial V(0)\},$$
$$\partial^\infty V(0) = \{0\}.$$

In addition, $\partial V(0)$ *is a compact subset of* $L^2(T, \mathbb{R}^n)$, *formed entirely by arcs.* ■

COROLLARY 4.4. *For each* $\beta \in L^2(T, \mathbb{R}^n)$, *we have the following estimations:*
(i) $V^0(0;\beta) \leq \sup_{x\in\tilde{Y}} \sup_{p\in M^1(x)} \langle p, \beta\rangle_{L^2}$,
(ii) $V^+(0;\beta) \leq \inf_{x\in\tilde{Y}} \sup_{p\in M^1(x)} \langle -p, \beta\rangle_{L^2}$,
(iii) $V_+(0;\beta) \geq \inf_{x\in\tilde{Y}} \inf_{p\in M^1(x)} \langle -p, \beta\rangle_{L^2}$,
(iv) $V_-(0;\beta) \geq \sup_{x\in\tilde{Y}} \inf_{p\in M^1(x)} \langle -p, \beta\rangle_{L^2}$,
(v) $V^-(0;\beta) \leq \sup_{x\in\tilde{Y}} \sup_{p\in M^1(x)} \langle -p, \beta\rangle_{L^2}$.
In particular, if for each $x \in \tilde{Y}$, $M^1(x)$ *is reduced to a singleton* $\{p^x\}$, *then* $V'_r(0;\cdot)$, $V'_1(0;\cdot)$ *exist, and are given by:*

$$V'_r(0;\beta) = \inf\{\langle -p^x, \beta\rangle : x \in \tilde{Y}\},$$
$$V'_1(0;\beta) = \sup\{\langle -p^x, \beta\rangle : x \in \tilde{Y}\} \text{ for each } \beta \in L^2(T, \mathbb{R}^n).$$

In addition, $-V$ *is regular at* 0. ■

COROLLARY 4.5. *If in addition to the hypotheses of the previous corollary, we suppose that* $\tilde{Y} = \{x\}$, *then by setting* $M^1(x) = \{p\}$, V *is strictly differentiable at* 0 *and its strict derivative is given by* $D_sV(0) = -p$. *Consequently, for* $\alpha \in L^2$ *too small,* $V(\alpha) \cong V(0) - \langle p, \alpha\rangle_{L^2}$. ■

5. **Proof of results.** Recall that the symbol "$\longrightarrow\!\!\!\!\rightarrow$" denotes the uniform convergence on T and we define the topology of the product space $L^2 \times \mathbb{R}^n$ by the inner product:

$$\langle [\alpha, x], [\alpha', x']\rangle := \langle \alpha, \alpha'\rangle_{L^2} + \langle x, x'\rangle_{\mathbb{R}^n},$$

for $[\alpha, x]$, $[\alpha', x'] \in L^2 \times \mathbb{R}^n$, and the associated norm.

5.1 *Proof of Theorem 4.1.* First of all, we prove the following lemma:

LEMMA 5.1. *Let* $[\beta, -\lambda] \in \mathrm{PN}_{\mathrm{epi}\, V}(\alpha, \gamma)$ *with* $\lambda \geq 0$ *and let* (x, ν) *be a solution of* $\tilde{P}(\alpha)$. *Then* $\beta \in \mathrm{AC}^2(T, \mathbb{R}^n)$ *and satisfies* $\beta \in -M^\lambda_\alpha(x)$.

PROOF. The fact $[\beta, -\lambda] \in \mathrm{PN}_{\mathrm{epi}\, V}(\alpha, \gamma)$ signifies the existence of $\sigma > 0$ such that for all $[\alpha', \gamma'] \in \mathrm{epi}\, V$

$$(5.1)\quad -\int_0^1 \beta(t)\cdot\alpha(t)\,dt + \lambda\gamma \leq -\int_0^1 \beta(t)\cdot\alpha'(t)\,dt + \lambda\gamma' + \sigma\int_0^1 |\alpha'(t) - \alpha(t)|^2\,dt + \sigma(\gamma - \gamma')^2.$$

FIRST CASE: $V(\alpha) = \gamma\big(= J(x,\nu)\big)$. For an arbitrary y in $\mathrm{AC}^2(T,\mathbb{R}^n)$ with $y(0) = \psi(0)$, $y(1) \in C$, and y remaining in $\bar{\Omega}_1$, and for any $\mu \in \mathcal{R}$; let $\alpha'(\cdot) \in L^2(T,\mathbb{R}^n)$ be given by:

$$\alpha'(t) = \dot{y}(t) - \dot{\psi}(t) - \phi\big(t,t,y(t),\mu(t)\big) - \int_0^t \phi_t\big(t,s,x(s),\nu(s)\big)\,ds.$$

Therefore $(y,\mu) \in \mathrm{Ad}\big(\tilde{P}(\alpha')\big)$, and consequently $[\alpha',\gamma'] \in \operatorname{epi} V$. Replacing $[\alpha',\gamma']$ and $[\alpha,\gamma]$ by their values in (5.1), we obtain

(5.2)

$$\begin{aligned}
\lambda J(x,\nu) + \int_0^1 \beta(t)\cdot\Big\{-\dot{x}(t) + \phi\big(t,t,x(t),\nu(t)\big) + \int_0^t \phi_t\big(t,s,x(s)\nu(s)\big)\,ds\Big\}\,dt \\
\leq \lambda J(y,\mu) + \sigma\big(J(y,\mu) - J(x,\nu)\big)^2 \\
+ \int_0^1 \Big\{\beta(t)\cdot\Big\{-\dot{y}(t) + \phi\big(t,t,y(t),\mu(t)\big) + \int_0^t \phi_t\big(t,s,y(s),\mu(s)\big)\,ds\Big\} \\
+ \sigma\Big|\dot{y}(t) - \dot{x}(t) - \phi\big(t,t,y(t),\mu(t)\big) - \int_0^t \phi_t\big(t,s,y(s),\mu(s)\big)\,ds \\
+ \phi\big(t,t,x(t),\nu(t)\big) + \int_0^t \phi_t\big(t,s,x(s),\nu(s)\big)\,ds\Big|^2\Big\}\,dt.
\end{aligned}$$

Then $(y,\mu) = (x,\nu)$ minimizes the RHS of this inequality over all possible (y,μ). Consider a second state $m(\cdot) = \mathrm{AC}^2(T,\mathbb{R})$ solution of $\dot{m}(t) = F\big(t,y(t),\mu(t)\big)$ (a.e.) and $m(0) = 0$, with y and μ chosen as above. We can consider $\dot{y}$ as a second control $v(\cdot)$ in $L^2(T,\mathbb{R}^n)$. Define the functional:

$$\begin{aligned}
\tilde{F}\colon T \times \mathrm{AC}^2(T,\mathbb{R}^n) \times \mathcal{R} \times L^2(T,\mathbb{R}^n) \longrightarrow \mathbb{R} \\
\big(t,y(\cdot),\mu(\cdot),v(\cdot)\big) \longmapsto \tilde{F}\big(t,y(\cdot),\mu(\cdot),v(\cdot)\big)
\end{aligned}$$

such that

$$\begin{aligned}
\tilde{F}\big(t,y(\cdot),\mu(\cdot),v(\cdot)\big) := \beta(t)\cdot\Big\{-v(t) + \phi\big(t,t,y(t),\mu(t)\big) + \int_0^t \phi_t\big(t,s,y(s),\mu(s)\big)\,ds \\
+ \sigma\Big|v(t) - \phi\big(t,t,y(t),\mu(t)\big) - \int_0^t \phi_t\big(t,s,y(s),\mu(s)\big)\,ds - \dot{x}(t) \\
+ \phi\big(t,t,x(t),\nu(t)\big) + \int_0^t \phi_t\big(t,s,x(s),\nu(s)\big)\,ds\Big|^2 \\
+ \lambda F\big(t,x(t),\nu(t)\big),
\end{aligned}$$

and the function $\tilde{f}\colon \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$, $(y,m) \longmapsto \tilde{f}(y,m) := \lambda f(y) + \sigma[f(y) + m - J(x,\nu)]^2$, and set for t in T:

$$m_0(t) := \int_0^t F\big(s,x(s),\nu(s)\big)\,ds.$$

From all this data, we deduce that the state/control pair $(y,m;\mu,v) = (x,m_0;\nu,\dot{x})$ is a solution of the following problem:

(P) $\tilde{J}(y,m;\mu,v) := \tilde{f}\big(y(1),m(1)\big) + \int_0^1 \tilde{F}\big(t,y(\cdot),\mu(\cdot),v(\cdot)\big)\,dt \longrightarrow \min$
$\dot{y}(t) = v(t)$ (a.e.), $y(0) = \psi(0)$, $y(1) \in C$ and y remains in $\bar{\Omega}_1$,
$\dot{m}(t) = F\big(t,y(t),\mu(t)\big)$ (a.e.), $m(0) = 0$, $(\mu,v) \in \mathcal{R} \times L^2(T,\mathbb{R}^n)$.

This problem is very similar to the basic problem $\tilde{P}(0)$, and the technique of [9, Proof of Theorem 7.4] can be applied to yield a $p \in \mathrm{AC}^2(T, \mathbb{R}^n)$, $\lambda^0 \in \{0,1\}$, and $\xi \in \lambda\partial f\big(x(1)\big)$, with $1 = \lambda^0 + \lambda^0|\xi| + |p(0)|$ and such that the following hold:

$$(5.3)\qquad \begin{aligned} -\dot{p}(t) = -\lambda^0\beta(t)\Big\{\phi_x\big(t,t,x(t),\nu(t)\big) + \int_0^t \phi_{xt}\big(t,s,x(s),\nu(s)\big)\,ds\Big\} \\ -\lambda^0\lambda F\big(t,x(t),\nu(t)\big) \text{ (a.e.)}, \end{aligned}$$

$$(5.4)\qquad -p(1) - \lambda^0\xi \in \bar{N}_C\big(x(1)\big),$$

(5.5)

$$\begin{aligned} \max\Big\{p(t)\cdot v(t) - \lambda^0\beta(t)\cdot\Big\{-v(t) + \phi\big(t,t,x(t),\mu(t)\big) \\ + \int_0^t \phi_t\big(t,s,x(s),\mu(s)\big)\,ds\Big\} - \lambda^0\lambda F\big(t,x(t),\mu(t)\big) : \mu \in \mathcal{R}, v \in L^2(T,\mathbb{R}^n)\Big\} \end{aligned}$$

is attained (a.e.) for $\mu = \nu$ and $v = \dot{x}$.

From this last conclusion, we deduce that $p(t) = -\lambda^0\beta(t)$ (a.e.), and, since p is continuous, $p \equiv -\lambda^0\beta$, and satisfies

(5.6)

$$\begin{aligned} \max\Big\{p(t)\cdot\Big\{\phi\big(t,t,x(t),\mu(t)\big) + \int_0^t \phi_t\big(t,s,x(s),\mu(s)\big)\,ds\Big\} \\ -\lambda\lambda^0 F\big(t,x,(t),\mu(t)\big) : \mu \in \mathcal{R}\Big\} \text{ is attained (a.e) for } \mu = \nu. \end{aligned}$$

We claim that λ^0 must be equal to 1. If not, then $|p(0)| = 1$ and therefore $p \not\equiv 0$, but $p \equiv -\lambda^0\beta = 0$. Hence a contradiction. In conclusion, $\beta \in -M_\alpha^\lambda(x)$, where $x \in \tilde{Y}_\alpha$.

SECOND CASE: $J(x,\nu) = V(\alpha) < \gamma$. In this case, if we put in (5.1) $\alpha' = \alpha$ and we consider the γ' which are near γ, we conclude that $\lambda = 0$ necessarily. Armed with this information, the proof given in the first case can be adapted to the present situation (it suffices to ignore all terms containing λ) in order to show that $-\beta \in M_\alpha^0(x)$ for a certain x in $\tilde{Y}_\alpha$. At this point, the proof of the lemma is achieved. ∎

The next lemma asserts that the λ-multiplier set related to $\tilde{P}(\alpha)$ has a sequential compactness property.

LEMMA 5.2. *Let* $(\alpha_\iota) \subset L^2(T,\mathbb{R}^n)$ *such that* $V(\alpha_\iota) < \infty$ *and* $\alpha_\iota \longrightarrow 0$ *in* L^2. *Let* $(x_\iota,\nu_\iota) \in \mathrm{Ad}\big(\tilde{P}(\alpha_\iota)\big)$ *and let* $p_\iota \in M_{\alpha_\iota}^{\lambda_\iota}(x_\iota)$, *where* $\{(p_\iota,\lambda_\iota)\}$ *is a bounded sequence in* $C(T,\mathbb{R}^n)\times\mathbb{R}$ *with* $\lambda_\iota \geq 0$. *Then, there exists a subsequence of* $\{(x_\iota,\nu_\iota)\}$, *which we don't relabel such that* $x_\iota \twoheadrightarrow x$, $\nu_\iota \longrightarrow \nu(w^*)$ *with* $(x,\nu) \in \mathrm{Ad}\big(\tilde{P}(0)\big)$. *There exists also a subsequence of* $\{(p_\iota,\lambda_\iota)\}$ *converging uniformly to* (p,λ), *where* $p \in M^\lambda(x)$. *If* $(x_\iota,\nu_\iota) \in \tilde{Y}_\alpha$, *then we can claim that* $(x,\nu) \in \tilde{Y}$ *and* $p \in M^\lambda(x)$.

PROOF. By virtue of Lemma 2.1, we can assume the existence of $x \in \mathrm{AC}^2(T,\mathbb{R}^n)$, $\nu \in \mathcal{R}$ such that for a certain subsequence of $\{(x_\iota,\nu_\iota)\}$, which we don't relabel, $x_\iota \twoheadrightarrow x$,

$\nu_\iota \longrightarrow \nu(w^*)$, and $(x,\nu) \in \mathrm{Ad}\big(\tilde{P}(0)\big)$. Since (λ_ι) is bounded, we can assert without loss of generality the existence of $\lambda \geq 0$ such that $\lambda_\iota \longrightarrow \lambda$. Let M be a constant such that $\lambda_\iota \leq M$ for all i. Then, from the inequality $|\dot{p}_\iota(t)| \leq K(t)\big(|p_\iota(t)| + M\big)$, where $K(\cdot) \in L^2(T,\mathbb{R})$, we can see that the sequence (p_ι) is absolutely equicontinuous and $(\dot{p}_\iota)$ is bounded in $L^2(T,\mathbb{R}^n)$. Consequently, there exists a subsequence of (p_ι) which we don't relabel and $p \in \mathrm{AC}^2(T,\mathbb{R}^n)$ such that $p_\iota \twoheadrightarrow p$, $\dot{p}_\iota \longrightarrow \dot{p}$ $(w - L^2)$. From the inclusion $p_\iota \in M^{\lambda_\iota}_{\alpha_\iota}(x_\iota)$, and by taking the limit as $i \longrightarrow +\infty$ and using the preliminary lemma of [9] and the fact that the multifunction $\bar{N}_C(\cdot)$ is closed, we can see that $p \in M^\lambda_\alpha(x)$. ∎

LEMMA 5.3. *Let* $[\beta,-\lambda] = w - \lim_\iota[\beta_\iota,-\lambda_\iota]$ *in* $L^2 \times \mathbb{R}$, *where* $[\beta,\lambda] \neq 0$, $\lambda_\iota \geq 0$ *and* $[\beta_\iota,-\lambda_\iota] \in \mathrm{PN}_{\mathrm{epi}\,V}(\alpha_\iota,\gamma_\iota)$ *with* $(\alpha_\iota,\gamma_\iota) \longrightarrow \big(0,V(0)\big)$ *in* $L^2 \times \mathbb{R}$. *Then* $\beta \in \mathrm{AC}^2(T,\mathbb{R}^n)$ *and satisfies*

(i) $\beta/\lambda \in -M^1(x) \cap \partial V(0)$ *if* $\lambda > 0$,

(ii) $\beta \in -M^0(x) \cap \partial^\infty V(0)$ *if* $\lambda = 0$, *for a certain* $x \in \tilde{Y}$.

PROOF. Applying Lemma 5.1, we conclude that $-\beta_\iota \in M^{\lambda_\iota}_{\alpha_\iota}(x_\iota)$ for a certain $x_\iota \in \tilde{Y}_{\alpha_\iota}$. Let $p_\iota := \beta_\iota$. Then we can show by using the boundedness of (β_ι) in L^2 that $\{p_\iota(0)\}$ is bounded. Applying Gronwall's lemma to the inequality

$$|p_\iota(t)| \leq |p_\iota(0)| + \int_0^t \Lambda(s)\cdot\big(|p_\iota(s)| + M\big)\,ds \quad \text{for some constant } M \geq \lambda_\iota (\forall i),$$

which is a consequence of the adjoint equation. Note that the previous lemma is applicable to the sequence $\{(p_\iota,\lambda_\iota)\}$. Therefore $\beta \in -M^\lambda(x)$ for some $x \in \tilde{Y}$, which is equivalent to $\beta/\lambda \in -M^1(x)$ for $\lambda > 0$. The fact that $\beta/\lambda \in \partial V(0)$ if $\lambda > 0$ and $\beta \in -\partial^\infty V(0)$ if $\lambda = 0$, follows immediately from proximal normal formula (1.6) with $C = \mathrm{epi}\,V$ and from (1.7) and (1.8) respectively. ∎

Now, by the proximal formula (1.6) for $C = \mathrm{epi}\,V$ and from the last lemma, we conclude the inclusion $N_{\mathrm{epi}\,V}\big(0,V(0)\big) \subset \overline{\mathrm{co}}\{N \cup N^\infty\}$, where $\overline{\mathrm{co}}$ denotes the closed convex hull in L^2,

$$N := \{\alpha(\xi,-1) : \alpha \geq 0, \xi \in -M^1(\tilde{Y}) \cap \partial V(0)\} \text{ and}$$
$$N^\infty := \{(\xi,0) : \xi \in -M^0(\tilde{Y}) \cap \partial^\infty V(0)\}.$$

In fact, the previous inclusion is an equality. This is due to the inclusion $N \cup N^\infty \subset N_{\mathrm{epi}\,V}\big(0,V(0)\big)$, which results from the definitions of $\partial V(0)$ and $\partial^\infty V(0)$ (see (1.7), (1.8)). At this point, a direct application of [6, Proposition 4.2] gives the formula (1) of Theorem 4.1.

In order to prove the second formula (2) of Theorem 4.1, it suffices to show the inclusion $\partial^\infty V(0) \subset \overline{\mathrm{co}}\{-M^0(\tilde{Y}) \cap \partial^\infty V(0)\}$. For this, we use the same technique as that of [7, pp. 83–84]. Let $\xi \in \partial^\infty V(0)$. Then according to proximal formula (1.6), it follows that for $\varepsilon > 0$ small enough, we can construct a sequence $\{[\beta_\iota,-\lambda_\iota]\}$ and an element $[q,s] \in L^2(T,\mathbb{R}^n) \times \mathbb{R}^n$ such that

$$[\xi,0] = \sum_{\iota \in I} \theta_\iota[\beta_\iota,-\lambda_\iota] + [q,s],$$

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where $\theta_\imath \geq 0$, $\sum_{\imath \in I} \theta_\imath = 1$ and $||q||_{L^2} + |s| < \varepsilon$, and for each $i \in I_\varepsilon$ $\beta_\imath/\lambda_\imath \in -M^1(\tilde{Y}) \cap \partial V(0)$ if $\lambda_\imath > 0$, $\beta_\imath \in -M^0(\tilde{Y}) \cap \partial^\infty V(0)$ if $\lambda_\imath = 0$ (see last lemma).

Define the index set $\Lambda_\varepsilon := \{i \in I_\varepsilon : \lambda_\imath > 0\}$. Then we can write

$$s = \sum \theta_\imath \lambda_\imath = \sum_{\imath \in \Lambda} \theta_\imath \lambda_\imath \text{ and } \xi = q + \sum_{\imath \in \Lambda_\varepsilon} \theta_\imath \beta_\imath + \sum_{\imath \in I_\varepsilon \setminus \Lambda_\varepsilon} \theta_\imath \beta_\imath.$$

If $\Lambda_\varepsilon = \emptyset$, then from the last line, the fact that $\beta_\imath \in -M^0(\tilde{Y}) \cap \partial^\infty V(0)$ and the convexity of this last set, it follows immediately that $\xi \in \overline{\text{co}}\{-M^0(\tilde{Y}) \cap \partial^\infty V(0)\}$, since ε is chosen arbitrarily positive. Now if $\Lambda_\varepsilon \neq \emptyset$, we can suppose without any loss of generality that the set of finite sequences $\{(\beta_\imath/\lambda_\imath) : i \in \Lambda_\varepsilon\}_{\varepsilon > 0} \subset -M^1(\tilde{Y})$ is uniformly bounded in L^2 relative to ε. That is; there exists a constant $M > 0$ such that

$$\beta_\varepsilon := \sum_{\imath \in \Lambda} \theta_\imath \beta_\imath = \sum_{\imath \in \Lambda_\varepsilon} \theta_\imath \lambda_\imath (\beta_\imath/\lambda_\imath) \in sM\bar{B}_{L^2}.$$

Thus, we can write

$$\xi \in q + sMB_{L^2} + \text{co}\{-M^0(Y) \cap \partial^\infty V(0)\},$$

with $||q||_{L^2} < \varepsilon$ and $|s| < \varepsilon$. The result then follows by taking the limit $\varepsilon \longrightarrow 0$ in this last inclusion.

5.2 *Proof of Theorem 4.2.* Define for $\alpha \in L^2(T, \mathbb{R}^n)$ the set

$$\mathbb{A}_\alpha := \left\{x \in \text{AC}^2(T, \mathbb{R}^n) : \exists \nu \in \mathcal{R} \text{ such that } (x, \nu) \in \text{Ad}\big(\tilde{P}(\alpha)\big)\right\}$$

In particular we denote $\mathbb{A}_0$ simply by $\mathbb{A}$. Let $M^0(\mathbb{A}_\alpha)$ denote the abnormal multiplier set associated to $\mathbb{A}_\alpha$; *i.e.*, $M^0(\mathbb{A}_\alpha) = \bigcup\{M^0(x) : x \in \mathbb{A}_\alpha\}$. We will first show the implication:

$$M^0(\mathbb{A}) = \{0\} \Rightarrow V \text{ is finite near } 0. \tag{5.7}$$

To this end, we define the set $W := \left\{\alpha \in L^2(T, \mathbb{R}^n) : \text{Ad}\big(\tilde{P}(\alpha)\big) \neq \emptyset\right\}$. Then $W \neq \emptyset$ once $\text{Ad}\big(\tilde{P}(0)\big) \neq \emptyset$, *i.e.*, $0 \in W$; moreover, W is closed by virture of Lemma 2.1. Therefore, showing (5.7) is equivalent to show

$$M^0(\mathbb{A}) = \{0\} \Rightarrow 0 \in \text{int}(W). \tag{5.8}$$

Suppose that $0 \notin \text{int}(W)$. Then $0 \in \text{Fr}(W)$ (boundary of F) and by [3, Theorem 5.1], which asserts in particular that the set $\left\{\alpha \in L^2 : \text{PN}_W(\alpha) \neq \{0\}\right\}$ is dense in $\text{Fr}(W)$ we can construct a sequence $(\alpha_\imath) \subset L^2$ such that

$$\text{PN}_W(\alpha_\imath) \neq \{0\} \text{ for all } i, \text{ and } \alpha_\imath \longrightarrow 0 \text{ in } L^2. \tag{5.9}$$

Fix α in W and let $\beta \neq 0$ in $\text{PN}_W(\alpha)$. Then there exists $\sigma > 0$ such that

$$\langle \beta, \gamma - \alpha \rangle \leq \sigma ||\alpha - \gamma||^2_{L^2} \quad \forall \gamma \in W. \tag{5.10}$$

Let $(x_0, \nu_0) \in \text{Ad}\big(\tilde{P}(\alpha)\big)$ (fixed). Treating the inequality (5.10) with the same arguments as those applied to the inequality (5.1) in Lemma 5.1 leads to the following conclusion:

CONCLUSION 1 $0 \neq \beta \in \mathrm{PN}_W(\alpha) \Rightarrow -\beta \in M^0(\mathbb{A}_\alpha)$

As a second step in the proof of (5 8) or equivalently (5 7), we combine (5 9) and this conclusion in order to construct sequences $(\beta_i) \subset \mathrm{AC}^2(T, \mathbb{R}^n)$, $\{(x_i, \nu_i)\}$ such that $\beta_i \neq 0$, $-\beta_i \in M^0(x_i)$, $(x_i, \nu_i) \in \mathrm{Ad}\big(\tilde{P}(\alpha_i)\big)$ for each i and $\alpha_i \longrightarrow 0$ in L^2 We can suppose without loss of generality that $\|\beta_i\|_{\sup} = 1$ for all i, and then a direct application of Lemma 5 2, leads to the existence of $\beta \neq 0$ with $-\beta \in M^0(x)$ for a certain $x \in \mathbb{A}$ Consequently

CONCLUSION 2 $0 \notin \mathrm{int}(W) \Rightarrow M^0(\mathbb{A}) \neq \{0\}$, hence (5 8) and therefore (5 7)

Now, if the problem $\tilde{P}(0)$ is free endpoint, then we have automatically $M^0(\mathbb{A}) = \{0\}$ Consequently, such problems have a locally finite value function, that is, there exists $\delta > 0$ such that for all α in δB_{L^2}, $V(\alpha) < \infty$

In this second part, we will show under the hypothesis $C = \mathbb{R}^n$ that V is locally Lipschitz on δB_{L^2} Let $\alpha, \beta \in \delta B_{L^2}$, then in view of the fact that $\tilde{Y}_\alpha \neq \emptyset$ as well as $V(\alpha) < \infty$, there exist (x, ν) and (y, μ), respectively solutions of $\tilde{P}(\alpha)$ and $\tilde{P}(\beta)$, i e $V(\alpha) = J(x, \nu)$ and $V(\beta) = J(y, \mu)$ We will estimate $|J(x,\nu) - J(y,\mu)|$ by $\|\alpha - \beta\|_L$ up to a multiplicative factor depending on δ But before, let us examine the quantity $|x(t) - y(t)|$ for t in T

$$\begin{aligned}
|x(t) - y(t)| &\leq \int_0^t \Big|\phi\big(t, s, x(s), \nu(s)\big) - \phi\big(t, s, y(s), \nu(s)\big)\Big|\, ds \\
&\qquad + \int_0^t \Big|\phi\big(t, s, y(s), \nu(s) - \mu(s)\big)\Big|\, ds + \int_0^t |\alpha(s) - \beta(s)|\, ds \\
&\leq \int_0^t \sup_{0<\lambda<1} \Big|\phi_x\big(t, s, \lambda x(s) + (1-\lambda) y(s), \nu(s)\big)\Big| \; |x(s) - y(s)|\, ds \\
&\qquad + \int_0^t 2K(s)\, ds + \|\alpha - \beta\|_{L^2} \\
&\leq \int_0^t M|x(s) - y(s)|\, ds + 2\|K\|_{L^2} + \|\alpha - \beta\|_L ,
\end{aligned}$$

(where the existence of the constant M is due to the fact that the set of all admissible arcs for $\tilde{P}(0)$ is uniformly bounded and ϕ is continuously differentiable in x)

By applying the Gronwall's lemma to this inequality, we obtain

$$\|x - y\|_{\sup} \leq (2\|K\|_{L^2} + \|\alpha - \beta\|_{L^2}) \;\; \exp(M) \tag{5 11}$$

Now we have

$$\begin{aligned}
|J(x,\nu) - J(y,\mu)| &\leq \Big|f\big(x(1)\big) - f\big(y(1)\big)\Big| + \int_0^1 \Big|F\big(t, x(t), \nu(t)\big) - F\big(t, y(t), \nu(t)\big)\Big|\, dt \\
&\qquad + \int_0^1 \Big|F\big(t, y(t), \nu(t) - \mu(t)\big)\Big|\, dt \\
&\leq K_0 |x(1) - y(1)| \\
&\qquad + \int_0^1 \sup_{0<\lambda\leq 1} \Big|F_x\big(t, \lambda x(t) - (1-\lambda) y(t), \nu(t)\big)\Big| \; |x(t) - y(t)|\, dt \\
&\qquad + \int_0^1 2K(t)\, dt \qquad (K_0 \text{ is the Lipschitz constant for } f) \\
&\leq K_0 \|x - y\|_{\sup} + M_1 \|x - y\|_{\sup} + 2\|K\|_{L^2} \quad (M_1 \text{ is a constant})
\end{aligned}$$

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Taking (5.11) in into account, we obtain

$$|J(x,\nu) - J(y,\mu)| \leq \exp(M)\cdot(K_0 + M_1)\cdot(2\|K\|_{L^2} + \|\alpha-\beta\|_{L^2}) + 2\|K\|_{L^2}$$

From this inequality, we can see easily that it is possible to find a constant $\ell = \ell(\delta)$ such that the *LHS* is $\leq \ell\|\alpha-\beta\|_{L^2}$ for all $\alpha,\beta \in \delta B_{L^2}$. Thus, V is locally Lipschitz at 0.

In order to prove the same result in the presence of the endpoint constraint and when $\tilde{P}(0)$ is normal, we can reduce $\tilde{P}(0)$ to a problem which is free endpoint and has the same set of solutions and the same value function as $\tilde{P}(0)$. In fact, there exists a constant $\bar{K}$ such that $\tilde{P}(0)$ is equivalent to the problem $\tilde{P}_1(0)$ of minimizing the functional $I(x,\nu) := J(x,\nu) + \bar{K}d_C\big(x(1)\big)$ over the state/control pairs (x,ν) verifying the dynamics of $\tilde{P}(0)$ with $x(0) = \psi(0)$ and x remaining in $\bar{\Omega}_1$. The problem $\tilde{P}_1(0)$ is automatically normal, and the previous result is applicable to this one to assert that $V_1 = V$ is locally finite and Lipschitz at 0 (see [8] for a proof).

Examine now the rest of the assertions: the assertion that $\tilde{Y}_\alpha \neq \emptyset$ for each α in δB_{L^2} is an immediate consequence of Lemma 2.1 and the conclusion that $V(\alpha) < \infty$ for α in δB_{L_2}, and the fact that $P(\alpha)$ and $\tilde{P}(\alpha)$ have the same value for α in δB_{L^2} when $P(0)$ is free endpoint, can be shown using the same arguments as those of the proof of [9, Theorem 5.1]. ■

5.3 *Proof of Corollary 4.3.* In order to prove the corollary, we need the following lemma whose proof arguments may be found in [5, Lemma 6] and where we have to use Lemma 2.1 of this paper instead of Lemma 4 of [5].

LEMMA 5.4. *There exist $K(\cdot) \in L^2(T,\mathbb{R})$ and a constant M such that for all $p \in M^1(\tilde{Y})$.*

$$|\dot{p}(t)| \leq K(t)\cdot\big(|p(t)|+1\big) \text{ (a.e.) and } |p(1)| \leq M.$$

Consequently, normality of $\tilde{P}(0)$ implies that $M^1(\tilde{Y})$ is bounded in $C(T,\mathbb{R}^n)$. ■

Now the proof of the corollary is the same as the proof given in [5, after the proof of Lemma 6]. ■

5.4 *Proof of Corollaries 4.4 and 4.5.* For proof, we refer the reader to [5, Corollaries 1, 2 and 3]. ■

6. **Example.** We consider in this example a projectile launched from rest at time $t = 0$ with a constant thrust $\tau > g$, employed along and making a steering variable angle $u(t)$ with the horizontal. Here g denotes the gravitational acceleration. The equations of motion are:

$$\begin{cases} x_1(t) = \int_0^t x_3(s)\,ds, \\ x_2(t) = \int_0^t x_4(s)\,ds, \\ x_3(t) = \int_0^t \tau\cos u(s)\,ds, \\ x_4(t) = \int_0^t \{\tau\sin u(s) - g\}\,ds, \end{cases}$$

where (x_1, x_2) and (x_3, x_4) are respectively the position vector and the velocity vector of the projectile If in addition, we suppose that interior factors to the projectile act on it by the vector

$$\Big(\int_0^t r_1(t-s)\,x_3(s)\,ds,\ \int_0^t r_2(t-s)\,x_4(s)\,ds\Big),$$

where r_1, r_2 are C^1 real functions satisfying $r_1(0) = 0 = r_2(0)$, then the third and the fourth equations become

$$x_3(t) = \int_0^t \{\tau\cos u(s) + r_1(t-s)\,x_3(s)\}\,ds, \quad x_4(t) = \int_0^t \{\tau\sin u(s) - g + r_2(t-s)\,x_4(s)\}\,ds$$

We suppose that fuel runs out at $t = T$, so that thereafter the projectile is subject only to gravity until it lands (*i e* , until $x_2 = 0$ occurs) The problem is to maximize total horizontal range, which is equal to $x_1(T)$ plus the horizontal distance traveled after fuel runs out This is equivalent to minimizing the negative of the range, which can be calculated and is given by

$$f = -x_1(T) - \big(x_3(T)/g\big)\{x_4(T) + [x_4(T)^2 + 2gx_2(T)]^{1/2}\}$$

We recognize this problem as a special case of the basic problem $P(0)$ of Section 2, with the data $f, F \equiv 0, \psi = 0, \phi(t,s,x,u) = [x_3, x_4, \tau\cos u + r_1(t-s)\,x_3, \tau\sin u - g + r_2(t-s)\,x_4]$, $C = \mathbb{R}^4$, and $\Omega = [0, \pi/2]$ (say)

Our main goal is to prove that the optimal control can be constant only in the case where $r_1 \equiv r_2$ Let (x,u) be an ordinary solution of the problem Since we are in presence of a free endpoint problem, by [9, Theorem 5 1] (x,u) is also a solution of the relaxed problem and then by [9, Theorem 7 4], we conclude that there exists $p = (p_1, p_2, p_3, p_4) \in$ $\mathrm{AC}^2(T, \mathbb{R}^4)$ such that

(a) $p_1 = 0, p_2 = 0, -p_3(t) = p_1(t) + p_3(t)\int_0^t r_1(t-s)\,ds, -p_4(t) = p_2(t) + \int_0^t r_2(t-s)\,ds$,

(b) $p_1(T) = 1$, $p_2(T) = x_3(T)/m$, $p_3(T) = \big(x_3(T) + m\big)/g$, $p_4(T) = \big(x_3(T)/gm\big)\big(x_4(T) + m\big)$, where $m = [x_4(T)^2 + 2gx_2(T)]^{1/2}$,

(c) For (a e)-t in $[0,T]$, $p_3(t)\cos\nu(t) + p_4(t)\sin\nu(t) \le p_3(t)\cos u(t) + p_4(t)\sin u(t)$ for all $\nu \in \mathcal{R}$, which is valid also for ν replaced by any point v in Ω

From (a), (b) we conclude that $p_1 \equiv 1, p_2 \equiv x_3(T)/m$, and that p_3, p_4 are solutions of the linear differential equations $p_3 + r_1 p_3 = -1, p_4 + r_2 p_4 = -p_2$ with $p_3(T)$ and $p_4(T)$ respectively given The condition (c) implies that u is characterized by $tg(u) = p_4/p_3$ This quotient is constant iff its derivative is zero, or equivalently iff $(r_2 - r_1)p_4 p_3 =$ $p_4 - p_2 p_3$ or $p_4/p_3 = p_2 + (r_1 - r_2)p_4$ This last equality can be written in the form

$$(\#) \qquad [r_2(t) - r_1(t)]F(r_1)(t)\,F(r_2)(t) = F(r_2)(t) - F(r_1)(t), \text{ where}$$

$$F(r)(t) = \exp\Big(-\int_0^t r(s)\,ds\Big)\ \Big[p_3(T)\exp\Big(\int_0^T r(s)\,ds\Big) + \int_t^T \exp\Big(\int_0^s r(\tau)\,d\tau\Big)\,ds\Big]$$

Note that for each choice of r as before and for all t in $[0,T]$, $F(r)(t) > 0$ We conclude from (#) that when u is constant, then $r_1 \equiv r_2$ iff $F(r_1)(t) = F(r_2)(t)\ \forall t \in [0,T]$ From this, we conclude that u is constant only if $r_1 \equiv r_2$

From now on, we take $r_1(t) \equiv r_2(t) = -\sigma t$, where σ is a given positive constant. The state (x_1, x_2, x_3, x_4) is given by:

$$x_4(t) = (\alpha/\sqrt{\sigma}) \sin \sqrt{\sigma} t,$$
$$x_3(t) = (\beta/\sqrt{\sigma}) \sin \sqrt{\sigma} t,$$
$$x_2(t) = (\alpha/\sigma)(1 - \cos \sqrt{\sigma} t),$$
$$x_1(t) = (\beta/\sigma)(1 - \cos \sqrt{\sigma} t),$$
$$\text{where } \alpha = \tau \sin u - g, \beta = \tau \cos u.$$

From this and the expressions of $tg(u)$ and m, we conclude that (u, m) is the solution of the system:

$$m^2 = (\alpha^2/\sqrt{\sigma}) \sin^2 \sqrt{\sigma} T + 2(g\alpha/\sigma)(1 - \cos \sqrt{\sigma} T),$$
$$tg(u) = \big(\beta \sin(\sqrt{\sigma} T)\big)/\sqrt{\sigma} T.$$

In order to find results of [5, Example E], it suffices to take the limit in the particular case $r_1(t) = r_2(t) = -\sigma t$ as σ tends to 0.

Suppose now that the dynamics of the problem is perturbed by a vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. The perturbation disappears in the beginning of the second phase of motion (*i.e.*, after fuel runs out). First let us consider only perturbation in velocity, that is $(\alpha_3, \alpha_4) = (0, 0)$, $\alpha_1(t) = 0 = \alpha_2(t)$ for $t \geq T$, and

$$\dot{x}_1 = x_3 + \alpha_1(t),$$
$$\dot{x}_2 = x_4 + \alpha_2(t).$$

Corollary 4.5 asserts that the value function V has derivative equal to $-p(\cdot)$ and to first order approximation we have

$$V(\alpha) \cong V(0) - \int_0^T \alpha_1(t)\, dt - \big(x_3(T)/m\big) \int_0^T \alpha_2(t)\, dt.$$

From this, we deduce that in order to have a maximal range of the perturbed projectile, it suffices that there exists a positive constant k such that $\alpha_2 = k\big(x_3(T)/m\big)\alpha_1$.

If the perturbation is in the thrust, *i.e.*, $\alpha_3 = 0 = \alpha_4$, then we find

$$V(\alpha) \cong V(0) - \int_0^T p_3(t)[\alpha_3(t) + p_2\alpha_4(t)]\, dt.$$

If we ask for the perturbation well chosen so as to make the range maximal under the constraint that $\|\alpha_3\|_{L^2}^2 + \|\alpha_4\|_{L^2}^2 \leq \delta$, where δ is a given small number, then by virtue of the last formula, this is equivalent to maximize $\int_{[0,T]} p_3\alpha_3 + \int_{[0,T]} p_2 p_3$ subject to the given constraint. By looking at this as a mathematical programming problem, we find that there must exist a constant $k > 0$ such that $\alpha_3 = kp_3$ and $\alpha_4 = kp_2p_3$, that is α_3 and α_4 are in the proportion $1 : p_2$.

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