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# MAPS INTO DYNKIN DIAGRAMS ARISING FROM REGULAR MONOIDS

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**Abstract**

It has been shown by one of the authors that the system of idempotents of monoids on a group $G$ of Lie type with Dynkin diagram $\Gamma$ can be classified by the following data: a partially ordered set $\mathscr{U}$ with maximum element 1 and a map $\lambda\colon \mathscr{U} \to 2^{\Gamma}$ with $\lambda(1) = \Gamma$ and with the property that for all $J_1, J_2, J_3 \in \mathscr{U}$ with $J_1 < J_2 < J_3$, any connected component of $\lambda(J_2)$ is contained in either $\lambda(J_1)$ or $\lambda(J_3)$. In this paper we show that $\lambda$ comes from a regular monoid if and only if the following conditions are satisfied:

(1) $\mathscr{U}$ is a $\wedge$-semilattice;

(2) If $J_1, J_2 \in \mathscr{U}$, then $\lambda(J_1) \cap \lambda(J_2) \subseteq \lambda(J_1 \wedge J_2)$;

(3) If $\theta \in \Gamma$, $J \in \mathscr{U}$, then $\max\{J_1 \in \mathscr{U} \mid J_1 \leq J, \theta \in \lambda(J_1)\}$ exists;

(4) If $J_1, J_2 \in \mathscr{U}$ with $J_1 < J_2$ and if $X$ is a two element discrete subset of $\lambda(J_1) \cup \lambda(J_2)$, then $X \subseteq \lambda(J)$ for some $J \in \mathscr{U}$ with $J_1 \leq J \leq J_2$.

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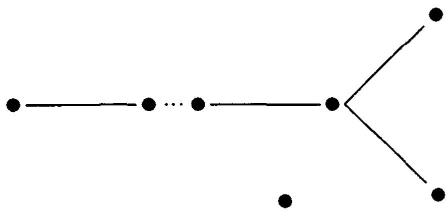
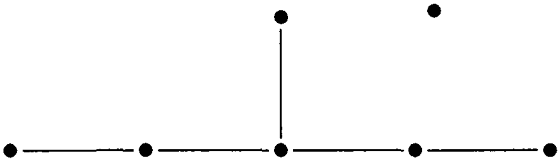
By a *Coxeter group* $W = (W, \Gamma)$ is meant a group $W$ generated by a subset $\Gamma$ of elements of order 2, such that $W$ has a presentation by the relations $(\sigma\theta)^{m(\sigma,\theta)} = 1$, for $\sigma, \theta \in \Gamma$. We assume that the *rank* $|\Gamma| < \infty$. If $\sigma, \theta \in \Gamma$, define $\sigma \overset{m}{-} \theta$ if $m = m(\sigma, \theta) \geq 3$. In this way $\Gamma$ becomes a graph, called the *Coxeter graph* of $W$. Note that $\sigma, \theta$ are not adjacent in the graph if and only if $\sigma\theta = \theta\sigma$. It is customary to write $\sigma - \theta$ to mean $m = 3$, $\sigma = \theta$ to mean $m = 4$ and $\sigma \equiv \theta$ to mean $m = 6$. The possible graphs for finite $W$ were determined by Coxeter (see [8]). Coxeter groups arise in much of algebra as

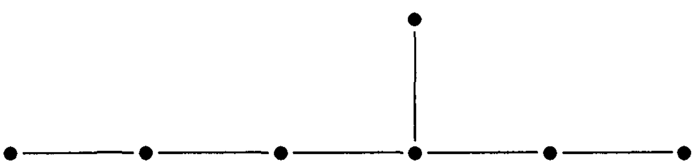
Weyl groups related to root systems. The possible connected Coxeter graphs are then

$A_n$ :

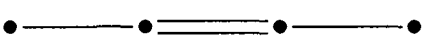

$B_n$ or $C_n$ :

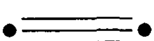

$D_n$ :


$E_6$ :


$E_7$ :


$E_8$ :


$F_4$ :


$G_2$ :


These graphs are closely related to the Dynkin diagrams of root systems.

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Let $W$ be a Coxeter group, $\sigma \in W$. Then $\sigma = \theta_1 \cdots \theta_k$ for some $\theta_1, \dots, \theta_k \in \Gamma$. If $k$ is minimal, then the *length* $l(\sigma)$ is defined to be $k$. Coxeter groups are characterized by Matsumoto's exchange condition [8, Theorem 4.4].

THEOREM (Exchange condition). *Let $\theta_1, \dots, \theta_k \in \Gamma$, $\sigma = \theta_1 \cdots \theta_k$, $l(\sigma) = k$. If $\theta \in \Gamma$, then either $l(\theta\sigma) = k + 1$ or else $l(\theta\sigma) = k - 1$ and $\theta\sigma = \theta_1 \cdots \hat{\theta}_i \cdots \theta_k$ for some $i = 1, \dots, k$.*

REMARK. The exchange condition implies the following.

(i) If $\theta_1, \dots, \theta_n \in \Gamma$, $\sigma = \theta_1 \cdots \theta_n$, $l(\sigma) = k$, then $\sigma = \theta_{i_1} \cdots \theta_{i_k}$ for some $i_1 < \cdots < i_k$.

(ii) If $\theta_1, \dots, \theta_n$, $\theta'_1, \dots, \theta'_n \in \Gamma$, $\sigma = \theta_1 \cdots \theta_n = \theta'_1 \cdots \theta'_n$ and $l(\sigma) = n$, then $\{\theta_1, \dots, \theta_n\} = \{\theta'_1, \dots, \theta'_n\}$.

If $I \subseteq \Gamma$, let $W_I = \langle I \rangle$ denote the subgroup of $W$ generated by $I$. If $I, I' \subseteq \Gamma$, then $W_{I \cap I'} = W_I \cap W_{I'}$ and $W_I = W_{I'}$ if and only if $I = I'$.

Let $\mathscr{U}$ be a partially ordered set with maximum element 1, $\lambda\colon \mathscr{U} \to 2^\Gamma$ such that $\lambda(1) = \Gamma$. If $J \in \mathscr{U}$, we write $W_J$ for $W_{\lambda(J)}$. Let $\mathscr{W}(\lambda) = \{(J, W_J\sigma) | J \in \mathscr{U}, \sigma \in W\}$. Define $(J_1, W_{J_1}\sigma) \leq (J_2, W_{J_2}\alpha)$ if $J_1 \leq J_2$ and $W_{J_1}\sigma \cap W_{J_2}\alpha \neq \varnothing$. Define $\lambda$ to be *transitive* if $\leq$ is transitive on $\mathscr{W}(\lambda)$. Define $\lambda$ to be *regular* if $(\mathscr{W}(\lambda), \leq)$ is a $\wedge$-semilattice. Then it can be seen [5] that $\lambda$ is transitive if and only if for all $J_1, J_2, J_3 \in \mathscr{U}$ with $J_1 \geq J_2 \geq J_3$, any connected component of $\lambda(J_2)$ is contained in either $\lambda(J_1)$ or $\lambda(J_3)$. The main goal of this paper is to obtain a similarly usable characterization of regularity.

Before proceeding, we explain the motivation for the above considerations. The basic motivation comes from the theory of linear algebraic monoids ([3], [4], [6], [7]). It has been shown by L. Renner and one of the authors [8] that for a connected regular linear algebraic monoid $M$ with zero, the system of idempotents (biordered set in the sense of Nambooripad [2]) is determined by a 'type map' $\lambda$ from the finite lattice $\mathscr{U}$ of principal ideals of $M$ into $2^\Gamma$, where $\Gamma$ is the Dynkin diagram of the group of units of $M$. One of the authors [5] considered the more general situation of monoids on a group $G$ with a $BN$-pair. Again the system of idempotents is characterized by a type map $\lambda\colon \mathscr{U} \to 2^\Gamma$. Moreover it was shown in [5] that an abstract map $\lambda\colon \mathscr{U} \to 2^\Gamma$ arises if and only if it is transitive. It was further shown in [5], that $\lambda$ comes from a regular monoid on $G$ if and only if $\lambda$ is regular.

For monoids $M$ on a group $G$ of Lie type, the partially ordered ser $\mathscr{W}(\lambda)$ is isomorphic to the partially ordered set of 'diagonal idempotents' of $M$. We illustrate with an example. Let $G = GL(4, F)$ where $F$ is a field. Then one monoid on $G$ is $\mathscr{M}_4(F)$, the monoid of all $4 \times 4$ matrices over $F$. In this case the Weyl group of $G$ is the group $S_4$ of all $4 \times 4$ permutation matrices and $\Gamma$

can be chosen to be

$$\Gamma = \left\{ \theta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \theta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \theta_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

The graph structure for $\Gamma$ is

$$\theta_1 \text{——} \theta_2 \text{——} \theta_3$$

The standard idempotent representatives for matrices in $\mathscr{M}_4(F)$ of different ranks are given by the linearly ordered set

$$\mathscr{U} = \left\{ I, e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0 \right\}.$$

The corresponding regular map $\lambda\colon \mathscr{U} \to 2^\Gamma$ is given by $\lambda(e) = \{\theta \in \Gamma | e\theta = \theta e\}$, for all $e \in \mathscr{U}$. Thus $\lambda(I) = \Gamma$, $\lambda(e_1) = \{\theta_2, \theta_3\}$, $\lambda(e_2) = \{\theta_1, \theta_3\}$, $\lambda(e_3) = \{\theta_1, \theta_2\}$ and $\lambda(0) = \Gamma$. The lattice $\mathscr{W}(\lambda)$ is a sixteen element Boolean lattice isomorphic to the lattice of diagonal idempotents of $\mathscr{M}_4(F)$.

We now fix a Coxeter group $W = W(\Gamma)$. Before stating the main theorem, we prove some lemmas.

LEMMA 1. *Let $\sigma_1, \dots, \sigma_k$, $\theta \in \Gamma$, $\theta \neq \sigma_i$, $i = 1, \dots, k$. Let $\sigma = \sigma_1 \cdots \sigma_k$. Suppose $l(\sigma) = k$ and $\sigma\theta = \theta\overline{\sigma}$ for some $\overline{\sigma} \in W$ with $l(\overline{\sigma}) = k$ and $\theta$ not appearing in $\overline{\sigma}$. Then $\sigma_i\theta = \theta\sigma_i$ for $i = 1, \dots, k$.*

PROOF. We prove this by induction on $k$. By the exchange condition

$$\sigma_2 \cdots \sigma_k \theta = \sigma_1 \theta \overline{\sigma} = \theta \overline{\sigma}' \tag{1}$$

with $l(\overline{\sigma}') = k - 1$, and $\theta$ does not appear in $\overline{\sigma}'$. So by the induction hypothesis, $\theta\sigma_i = \sigma_i\theta$ for $i = 2, \dots, k$. So if $\sigma_1 \in \{\sigma_2, \dots, \sigma_k\}$ we are done. So assume $\sigma_1 \notin \{\sigma_2, \dots, \sigma_k\}$. Now $\overline{\sigma} = u\sigma_i v, \overline{\sigma}' = uv$ for some $i \in \{2, \dots, k\}$. Since $\sigma_1$ does not appear in the left side of (1), we see that $\sigma_1$ does not appear in $uv = \overline{\sigma}'$. Hence $\sigma_1 = \sigma_i$ and $\sigma_1\theta u\sigma_1 v = \theta uv$. So $\sigma_1\theta u = \theta u\sigma_1$; and hence $u\sigma_1 = \theta\sigma_1\theta u$. Since $\theta$ does not appear in $u\sigma_1$, we see by the exchange condition that $\theta\sigma_1\theta u = \sigma_1 u$. So $\theta\sigma_1\theta = \sigma_1$ and $\theta\sigma_1 = \sigma_1\theta$. This completes the proof.

LEMMA 2. *Let $J_1, J_2 \subseteq \Gamma$, $\sigma \in W_{J_2}$, $\alpha \in W_{J_1}$, $\alpha \notin W_{J_2}$. Let $l(\sigma) = k$, $\sigma = \sigma_1 \cdots \sigma_k$, $\sigma_i \in \Gamma$. Suppose that $\sigma$ is of minimal length in $W_{J_1}\sigma W_{J_1}$ and* *that $W_{J_1}\sigma \cap W_{J_2}\alpha^{-1} \neq \varnothing$. Then there exists $\theta \in J_1 \backslash J_2$ such that $\theta\sigma_i = \sigma_i\theta$, $i = 1, \ldots, k$.*

PROOF. We prove this by induction of $l(\alpha)$. By the exchange condition

$$l(\sigma x) = l(\sigma) + l(x) = l(x\sigma) \quad \text{for all } x \in W_{J_1}. \tag{2}$$

In particular, $l(\sigma\alpha) = l(\sigma) + l(\alpha)$. Now $\sigma\alpha \in W_{J_1}W_{J_2}$. So by the exchange condition, $\sigma\alpha = uv$ for some $u \in W_{J_1}$ and $v \in W_{J_2}$ such that $l(uv) = l(u) + l(v)$. Since $\alpha \notin W_{J_2}$, $u \neq 1$. So $u = \theta u_1$ for some $u_1 \in W_{J_1}$, $\theta \in J_1$, $l(u_1) = l(u) - 1$. So $u_1 v = \theta\sigma\alpha$. By (2), $l(\theta\sigma) = l(\sigma) + 1$. So by the exchange condition, $\theta\sigma\alpha = \sigma\alpha_1$ for some $\alpha_1 \in W_{J_1}$ with $l(\alpha_1) = l(\alpha) - 1$. So $\sigma\alpha_1 = u_1 v \in W_{J_1}W_{J_2}$. If $\alpha_1 \notin W_{J_2}$, we are done, by the induction hypothesis. So assume $\alpha_1 \in W_{J_2}$. Since $\alpha \notin W_{J_2}$, we see that $\alpha = \alpha_2\pi\alpha_3$, $\alpha_1 = \alpha_2\alpha_3$ with $\pi \in J_1\backslash J_2$. If $\pi \neq \theta$ then $\pi$ appears in $u_1 v = \sigma\alpha_1$, a contradiction. Hence $\pi = \theta$ and $\theta\sigma\alpha_2\theta\alpha_3 = \sigma\alpha_2\alpha_3$. Then $\theta\sigma\alpha_2\theta = \sigma\alpha_2$. So $\sigma\alpha_2\theta = \theta\sigma\alpha_2$, $\theta$ does not appear in $\sigma\alpha_2$. We are now done, by Lemma 1.

THEOREM 1. *Let $\mathscr{U}$ be a partially ordered set with maximum element* 1 *and let $\lambda: \mathscr{U} \to 2^\Gamma$ be a transitive map such that $\lambda(1) = \Gamma$. Then $\lambda$ is regular if and only if*

(i) *$\mathscr{U}$ is a $\wedge$-semilattice,*
(ii) *if $J_1, J_2 \in \mathscr{U}$, then $\lambda(J_1) \cap \lambda(J_2) \subseteq \lambda(J_1 \wedge J_2)$,*
(iii) *if $J \in \mathscr{U}$, $\theta \in \Gamma$, then* $\max\{J_1 \in \mathscr{U} | J_1 \leq J, \theta \in \lambda(J_1)\}$ *exists,*
(iv) *if $J_1, J_2 \in \mathscr{U}$, $J_1 \geq J_2$ and $X$ is a two element discrete subset of $\lambda(J_1) \cup \lambda(J_2)$, then $X \subseteq \lambda(J)$ for some $J \in \mathscr{U}$ with $J_1 \geq J \geq J_2$.*

PROOF. First we prove necessity. So assume that $\lambda$ is regular. So $(\mathscr{W}(\lambda), \leq)$ is a $\wedge$-semilattice. Let $J_1, J_2 \in \mathscr{U}$, such that $(J_1, W_{J_1}) \wedge (J_2, W_{J_2}) = (J, W_J\alpha)$. Let $J' \in \mathscr{U}$ with $J_1 \geq J'$, $J_2 \geq J'$. Then $(J_i, W_{J_i}) \geq (J', W_{J'})$, $i = 1, 2$. So $(J, W_J\alpha) \geq (J', W_{J'})$. So $J \geq J'$. Hence $J = J_1 \wedge J_2$. Also $(J, W_J\alpha) \geq (J, W_J)$ whereby $W_J\alpha = W_J$. If $\theta \in \lambda(J_1) \cap \lambda(J_2)$, then $\theta \in W_{J_1} \cap W_{J_2}$. So $(J, W_J\theta) = (J_1, W_{J_1}\theta) \wedge (J_2, W_{J_2}\theta) = (J_1, W_{J_1}) \wedge (J_2, W_{J_2}) = (J, W_J)$. So $\theta \in W_J$ and hence $\theta \in \lambda(J)$. This proves (i) and (ii).

Next let $\theta \in \Gamma\backslash\lambda(J)$, $(J, W_J) \wedge (J, W_J\theta) = (J_1, W_{J_1}\alpha)$. So $J_1 \leq J$, $\theta \in W_J W_{J_1}\alpha, \alpha \in W_{J_1}W_J$. So $\theta \in W_J W_{J_1} W_J$. Since $\theta \notin \lambda(J)$, $\theta \in \lambda(J_1)$. So $(J, W_J) \geq (J_1, W_{J_1})$, $(J, W_J\theta) \geq (J_1, W_{J_1})$. So $(J_1, W_{J_1}\alpha) \geq (J_1, W_{J_1})$ whereby $W_{J_1}\alpha = W_{J_1}$. Let $J_2 \in \mathscr{U}$ with $\theta \in \lambda(J_2)$. Then $(J, W_J) \geq (J_2, W_{J_2})$ and $(J, W_J\theta) \geq (J_2, W_{J_2})$. So $(J_1, W_{J_1}) \geq (J_2, W_{J_2})$ and hence $J_1 \geq J_2$. This proves (iii).

Finally we prove (iv). We can assume that $X \nsubseteq \lambda(J_1)$, $X \nsubseteq \lambda(J_2)$. So $X = \{\theta, \pi\}$ with $\pi \in \lambda(J_1)\backslash\lambda(J_2)$ $\theta \in \lambda(J_2)\backslash\lambda(J_1)$, $\theta\pi = \pi\theta$. Let $J = \max\{J_3 | J_3 \leq$

$J_1$, $\theta \in \lambda(J_3)\}$. Then as above $(J_1, W_{J_1}) \wedge (J_1, W_{J_1}\theta) = (J_3, W_{J_3})$. Since $\pi \in \lambda(J_1)$, $(J_1, W_{J_1}) \geq (J_2, W_{J_2}\pi)$. Since $\theta\pi = \pi\theta$, $W_{J_1}\theta \cap W_{J_2}\pi \neq \varnothing$. So $(J_1, W_{J_1}\theta) \geq (J_2, W_{J_2}\pi)$, whence $(J, W_J) \geq (J_2, W_{J_2}\pi)$. Hence $J \geq J_2$ and $\pi \in W_{J_2}W_J$. Since $\pi \notin \lambda(J_2)$, $\pi \in \lambda(J)$. So $\theta, \pi \in \lambda(J)$.

Conversely assume that (i), (ii), (iii) and (iv) are valid. First we claim that for any $J \in \mathscr{U}$, $X \subseteq \Gamma$,

$$\max\{J_1 \in \mathscr{U} \mid J_1 \leq J, X \subseteq \lambda(J_1)\} \text{ exists.} \tag{3}$$

We prove this by induction on $|X|$. If $X \subseteq \lambda(J)$, there is nothing to prove. Otherwise there exists $\theta \in X \backslash \lambda(J)$. By (iii), $J_0 = \max\{J_1 \leq J \mid \theta \in \lambda(J_1)\}$ exists. By the induction hypothesis, $J_2 = \max\{J_1 \leq J_0 \mid X \backslash \{\theta\} \subseteq \lambda(J_1)\}$ exists. Now $J_2 \leq J_0 \leq J$, $\theta \in \lambda(J_0)$, $\theta \notin \lambda(J)$. So by transitivity $\theta \in \lambda(J_2)$. So $X \subseteq \lambda(J_2)$. Now let $J_1 \leq J$ such that $X \subseteq \lambda(J_1)$. Then $\theta \in \lambda(J_1)$. So $J_1 \leq J_0$ and then $J_1 \leq J_2$. Hence (3) holds.

Next we claim that if $J_1, J_2 \in \mathscr{U}$, $J_2 \leq J_1$, $\sigma_1, \dots, \sigma_k \in \lambda(J_2)$, $\pi \in \lambda(J_1)$, then $\pi\sigma_i = \sigma_i\pi$, $i = 1, \dots, k$, implies that there exists $J \in \mathscr{U}$ with

$$J_2 \leq J \leq J_1, \qquad \pi, \sigma_1, \dots, \sigma_k \in \lambda(J). \tag{4}$$

We prove this by induction on $k$. If $\sigma_i \in \lambda(J_1)$ for all $i$, there is nothing to prove. So assume $\sigma_1 \notin \lambda(J_1)$. By condition (iv), there exists $J_3 \in \mathscr{U}$, $J_2 \leq J_3 \leq J_1$ such that $\pi, \sigma_1 \in \lambda(J_3)$. By the induction hypothesis, there exists $J \in \mathscr{U}$, $J_2 \leq J \leq J_3$ such that $\pi, \sigma_2, \dots, \sigma_k \in \lambda(J)$. Now $J \leq J_3 \leq J_1$, $\sigma_1 \in \lambda(J_3)$, $\sigma_1 \notin \lambda(J_1)$. So by transitivity $\sigma_1 \in \lambda(J)$. So $\pi, \sigma_1, \dots, \sigma_k \in \lambda(J)$. This proves (4).

Let $(J_1, W_{J_1}\sigma_1), (J_2, W_{J_2}\sigma_2) \in \mathscr{W}(\lambda)$. We need to show that $(J_1, W_{J_1}\sigma_1) \wedge (J_2, W_{J_2}\sigma_2)$ exists in $\mathscr{W}(\lambda)$. If $\pi \in W$, then $(J, W_J\sigma) \to (J, W_J\sigma\pi)$ is an automorphism of $\mathscr{W}(\lambda)$. For this reason we need only show that $(J_1, W_{J_1}) \wedge (J_2, W_{J_2}\sigma)$ exists where $\sigma \in W$ is such that it is an element of minimum length in $W_{J_2}\sigma W_{J_1}$. Then by the exchange condition $l(\delta\sigma) = l(\delta) + l(\sigma)$, $l(\sigma\gamma) = l(\sigma) + l(\gamma)$ for all $\delta \in W_{J_2}$, $\gamma \in W_{J_1}$. There exists a maximum $J_3 \leq J_1 \wedge J_2$ such that $\sigma \in W_{J_3}$. We claim that $(J_1, W_{J_1}) \wedge (J_2, W_{J_2}\sigma) = (J_3, W_{J_3})$. So let $(J_1, W_{J_1}) \geq (J_4, W_{J_4}\alpha)$, $(J_2, W_{J_2}\sigma) \geq (J_4, W_{J_4}\alpha)$. We can assume that $\alpha$ is of minimum length in $W_{J_4}\alpha$. Now $\alpha \in W_{J_4}W_{J_1}$ and hence $\alpha \in W_{J_1}$. Also $\sigma \in W_{J_2}W_{J_4}\alpha \subseteq W_{J_2}W_{J_4}W_{J_1}$. Hence $\sigma \in W_{J_4}$. Therefore $J_4 \leq J_3$. There exists $u \in W_{J_4}$, $v \in W_{J_2}$ such that $u\alpha = v\sigma$. So $u = v\sigma\alpha^{-1} \in W_{J_2}W_{J_3}W_{J_1} \cap W_{J_4}$. By the exchange condition $u = abc$ for some $a \in W_{J_2} \cap W_{J_4}$, $b \in W_{J_3} \cap W_{J_4}$, $c \in W_{J_1} \cap W_{J_4}$. Now

$$(J_3, W_{J_3}c\alpha) \geq (J_4, W_{J_4}c\alpha) = (J_4, W_{J_4}\alpha). \tag{5}$$

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Also,

$$(6) \qquad \begin{aligned}(J_3, W_{J_3}c\alpha) &= (J_3, W_{J_3}bc\alpha) \leq (J_2, W_{J_2}bc\alpha) \\ &= (J_2, W_{J_2}abc\alpha) = (J_2, W_{J_2}u\alpha) = (J_2, W_{J_2}v\sigma) \\ &= (J_2, W_{J_2}\sigma).\end{aligned}$$

Moreover

$$(7) \qquad (J_3, W_{J_3}c\alpha) \leq (J_1, W_{J_1}c\alpha) = (J_1, W_{J_1})$$

By (5), (6) and (7), it is clear that without loss of generality we can assume that $J_3 = J_4$.

First we consider the case $J_1 = J_2$. We assume $\alpha \notin W_{J_3}$ and obtain a contradiction. Let $l(\sigma) = k$, $\sigma = \sigma_1 \cdots \sigma_k$, $\sigma_i \in \Gamma$. Then $\sigma_1, \ldots, \sigma_k \in \lambda(J_3)$, $\alpha \in W_{J_1}$. By Lemma 1 there exists $\theta \in \lambda(J_1)\backslash\lambda(J_3)$ such that $\theta\sigma_i = \sigma_i\theta$, $i = 1, \ldots, k$. By (4), there exists $J \in \mathscr{U}$, $J_3 \leq J \leq J_1$ such that $\sigma_1, \ldots, \sigma_k$, $\theta \in \lambda(J)$. So $\sigma \in W_J$ and $J = J_3$. So $\theta \in \lambda(J_3)$, a contradiction.

Next we consider the case where $J_1 \geq J_2$. Since $\alpha \in W_{J_1}$, we have $(J_3, W_{J_3}\alpha) \leq (J_2, W_{J_2}\alpha) \leq (J_1, W_{J_1})$. Also $\alpha \in W_{J_3}W_{J_2}\sigma \subseteq W_{J_3}W_{J_2}W_{J_3}$. Since $\alpha$ is of minimum length in $W_{J_3}\alpha$, $\alpha \in W_{J_2}W_{J_3}$. So $(J_3, W_{J_3}) \leq (J_2, W_{J_2}\alpha)$. By the above, $(J_2, W_{J_2}\sigma) \wedge (J_2, W_{J_2}\alpha) = (J_0, W_{J_0}\beta)$ exists. Then $(J_0, W_{J_0}\beta) \leq (J_2, W_{J_2}\sigma)$, $(J_0, W_{J_0}\beta) \leq (J_1, W_{J_1})$. So as before $\sigma \in W_{J_0}$. Hence $J_3 \geq J_0$. But $(J_0, W_{J_0}\beta) \geq (J_3, W_{J_3})$ and $(J_0, W_{J_0}\beta) \geq (J_3, W_{J_3}\alpha)$. So $J_3 = J_0$ and $W_{J_3} = W_{J_3}\beta = W_{J_3}\alpha$.

Finally we consider the general case. Now $\alpha \in W_{J_3}W_{J_2}\sigma \subseteq W_{J_3}W_{J_2}W_{J_3}$. Since $\alpha$ is of minimum length in $W_{J_3}\alpha$, $\alpha \in W_{J_2}W_{J_3}$. Since also $\alpha \in W_{J_1}$ we see by the exchange condition that $\alpha = ab$ for some $a \in W_{J_1} \cap W_{J_2}$, $b \in W_{J_1} \cap W_{J_3}$. Let $J = J_1 \wedge J_2$. Then $a \in W_J$ by (ii). Now $W_{J_1} \cap W_{J_2} \subseteq W_J$. So $(J, W_J\alpha) = (J, W_Jb) \leq (J_1, W_{J_1})$. Also $(J, W_J\alpha) = (J, W_Jb) \geq (J_3, W_{J_3}), (J_3, W_{J_3}\alpha)$. Since $J_2 \geq J$, we see by the above that $(J_2, W_{J_2}\sigma) \wedge (J, W_J\alpha) = (J_0, W_{J_0}\beta)$ exists. Then $(J_0, W_{J_0}\beta) \leq (J_1, W_{J_1}), (J_2, W_{J_2}\sigma)$. So as above $\sigma \in W_{J_0}$. Hence $J_0 \leq J_3$. But $(J_0, W_{J_0}\beta) \geq (J_3, W_{J_3}), (J_3, W_{J_3}\alpha)$. So $J_0 = J_3$ and $W_{J_0}\beta = W_{J_3} = W_{J_3}\alpha$. This completes the proof of sufficiency.

COROLLARY 1. *If $\mathscr{U}$ is a finite linearly ordered set, then a transitive map $\lambda$ is regular if and only if for all $J_1, J_2 \in \mathscr{U}$, $X$ a two element discrete subset of $\lambda(J_1) \cup \lambda(J_2)$, $X \subseteq \lambda(J)$ for some $J$ between $J_1$ and $J_2$.*

If $\lambda\colon \mathscr{U} \to 2^\Gamma$, $X \subseteq \Gamma$, then let $\lambda_X\colon \mathscr{U} \to 2^X$ where for $J \in \mathscr{U}$, $\lambda_X(J) = \lambda(J) \cap X$.

COROLARY 2. *Let $\mathscr{U}$ be a partially ordered set with a maximum element 1 and let $\lambda\colon \mathscr{U} \to 2^\Gamma$ be such that $\lambda(1) = \Gamma$. Then $\lambda$ is transitive (respectively* *regular) if and only if $\lambda_X$ is transitive (respectively regular) for all rank $\leq 2$ subgraphs $X$ of* $\Gamma$.

In [1] a universal transitive map $u\colon \mathbf{U}(\Gamma) \to 2^\Gamma$ was constructed. It has the property that for any transitive map $\lambda\colon \mathscr{U} \to 2^\Gamma$, there is an order preserving map $\gamma\colon \mathscr{U} \to \mathbf{U}(\Gamma)$ such that $\lambda = u \circ \gamma$. The partially ordered set $\mathbf{U} = \mathbf{U}(\Gamma)$ was constructed as follows:

$$\mathbf{U} = \mathbf{U}(\Gamma) = \{(A,B) \mid A,B \in 2^\Gamma,\ A \cap B = \varnothing \text{ and each connected component of } A \cup B \text{ is either contained in } A \text{ or contained in } B\}.$$

For $(A,B),(A',B') \in \mathbf{U}$ we define $(A,B) \leq (A',B')$ if $A \subseteq A'$ and $B' \subseteq B$. Then $(\mathbf{U},\leq)$ is a distributive lattice with $(A,B) \vee (A',B') = (A \cup B, B \cap B')$ and $(A,B) \wedge (A',B') = (A \cap A', B \cup B')$.

Corollary 3. *The map* $u\colon \mathbf{U}(\Gamma) \to 2^\Gamma$, *where* $u(A,B) = A \cup B$, *is regular.*

Proof. Clearly $\mathbf{U}(\Gamma)$ is a $\wedge$-semilattice. Let $J_1 = (A_1,B_1)$, $J_2 = (A_2,B_2) \in \mathbf{U}(\Gamma)$. Then

$$\begin{aligned} u(J_1) \cap u(J_2) &= (A_1 \cup B_1) \cap (A_2 \cup B_2) \\ &= (A_1 \cap A_2) \cup (A_1 \cap B_2) \cup (B_1 \cap A_2) \cup (B_1 \cap B_2) \\ &\subseteq (A_1 \cap A_2) \cup B_1 \cup B_2 \\ &= u(J_1 \wedge J_2). \end{aligned}$$

Take any $J = (A,B) \in \mathbf{U}(\Gamma)$ and $\theta \in \Gamma$. Then $\max\{J' \in \mathbf{U}(\Gamma) \mid J' \leq J, \theta \in u(J')\} = \bigvee\{J' \in \mathbf{U}(\Gamma) \mid J' \leq J, \theta \in u(J')\}$ exists since $\mathbf{U}(\Gamma)$ is a finite lattice.

Let $J_1 = (A_1,B_1) \geq J_2 = (A_2,B_2)$ and $X$ be a 2-element discrete subset of $\Gamma$ such that

$$X \subseteq u(J_1) \cup u(J_2) = (A_1 \cup B_1) \cup (A_2 \cup B_2) = A_1 \cup B_2.$$

Then $X = (X \cap A_1) \cup (X \backslash A_1)$ with $X \backslash A_1 \subseteq B_2$. Take $J = (C,D)$ where $C = A_2 \cup (X \cap A_1)$, $D = B_1 \cup (X \backslash A_1)$. Then $C \cap D = \varnothing$. Now $B_1 \subseteq B_2$ and $X$ is discrete. So every connected component of $C \cup D$ is contained in $C$ or in $D$. Thus $J \in \mathbf{U}(\Gamma)$. Also $J_1 \geq J \geq J_2$ and $X \subseteq u(J)$. This completes the proof.

## References

[1] K. C. Misra, M. S. Putcha and D. S. Singh, 'Transitive maps from posets to Dynkin diagrams', to appear.
[2] K. S. S. Nambooripad, 'Structure of regular semigroups I.', *Mem. Amer. Math. Soc.* **224** (1979).
[3] M. S. Putcha, 'A semigroup approach to linear algebraic groups', *J. Algebra* **80** (1983), 164–185.
[4] ——, *Linear Algebraic Monoids*, (London Math. Soc. Lecture Note Series, No. 133, Cambridge Univ. Press, 1988).
[5] ——, 'Monoids on groups with BN-pairs', *J. Algebra* **120** (1989), 139–169.
[6] M. S. Putcha and L. E. Renner, 'The system of idempotents and the lattices of $\mathscr{T}$-classes of reductive algebraic monoids', *J. Algebra* **116** (1988), 385–399.
[7] L. E. Renner, 'Analogue of the Bruhat decomposition for algebraic monoids', *J. Algebra* **101** (1986), 303–338.
[8] M. Suzuki, *Group Theory* I, (Springer-Verlag, 1982).

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