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Defining relations for the Held-Higman-Thompson simple group

John J. Cannon and George Havas

A set of defining relations for the Held-Higman-Thompson simple group of order 4 030 387 200 is given.

In [3], Held gives properties of a possible new simple group H of order $2^{10}.3^3.5^2.7^3.17$, whose character table was constructed by Thompson. This simple group is characterised as having the same centralizer of an involution as M_{24} and $L_5(2)$. Unpublished work of Graham Higman and McKay established that the smallest degree permutation representation of H is on 2058 letters. The stabilizer of a point of this representation is $Sp_4(4)$, extended by an automorphism of order 2. Let K be a subgroup of H isomorphic to $Sp_4(4)$ and K^* a subgroup isomorphic to $Sp_4(4)$ extended by the automorphism.

Using the fact that H contains $Sp_4(4)$, Higman constructed defining relations for a group H^* that contains H as a subgroup of index 2. Thus the existence of H was established. Higman's relations for H^* are (McKay [5]):

$$a^2 = b^3 = c^2 = d^4 = e^2 = (ad^2)^2 = (cd^2)^2 = (bd^2)^2 = (ad)^8 = (bdb^{-1}d)^2 =$$
$$[a, b] = adbdadb^{-1}db^{-1} = [c, b] = (d^{-1}bdc)^3 = ((ad)^3cd)^2 = (ac)^3 =$$
$$(adbdcdbd)^5 = (cadcda)^2 = (cdad)^4 = (cd)^4adcadcdada = (de)^2 =$$
$$[c, e] = (ad^2e)^3 = (adadae)^3 = b^{-1}dbdebdbde = 1 .$$

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Using a program which determines all subgroups of a finitely presented group up to a specified index (Lepique [4]), we found a unique subgroup of index 2, generated by

$$a,\ b,\ c,\ d^2,\ e,\ d^{-1}ad,\ d^{-1}cd$$

which can only be H. Further investigation revealed that the first six of these words suffice to generate H. However, contrary to Higman's earlier belief, the first five words generate a proper subgroup of H.

The next step was to apply a Reidemeister-Schreier program [2] to present this subgroup. This program constructed a seven generator, forty-five relator presentation for H. Further simplification of this presentation by hand resulted in the following six generator, thirty-seven relator presentation:

$$\begin{aligned} u^2 &= v^2 = w^2 = x^2 = y^2 = z^3 = (vw)^2 = (wy)^2 = [x,\ z] = [y,\ z] = (uv)^3 = \\ &(xy)^3 = (wx)^3 = (vz)^3 = (uvuy)^2 = (uz)^2\left(uz^{-1}\right)^2 = (vxyx)^2 = (ux)^4 = \\ &(uxuy)^2 = (uxvx)^2 = (uy)^4 = (vx)^4 = (uzux)^2 = uzuz^{-1}wuzuw = \\ &uvuzuzvz^{-1}uz^{-1} = uvxvyvyxuy = (uwux)^2wuwx = (uxwx)^3 = (uw)^6 = \\ &(uz)^6 = \left(v(uw)^3\right)^2 = \left(x(uw)^3\right)^2 = \left(y(uw)^3\right)^2 = \left(z(uw)^3\right)^2 = \\ &\left(uyz^{-1}uz^{-1}\right)^3 = \left(uzvz^{-1}\right)^5 = \left(xuzuyuz^{-1}u\right)^5 = 1 . \end{aligned}$$

The relationship between these generators and the generators for H^* is as follows:

$$\begin{aligned} u &= a\ ; & x &= dad^{-1}\ ; \\ v &= c\ ; & y &= dcd^{-1}\ ; \\ w &= e\ ; & z &= b\ . \end{aligned}$$

This presentation involves redundant generators. Indeed H is generated by the two elements x and $uyxz^{-1}wz^{-1}$.

We now proceed to identify subgroups K and K^*. Using the so-called random coincidence procedure in conjunction with the Todd-Coxeter algorithm (Cannon and Havas [1]) it was found that

$$K^* = \langle z^{-1}yvz^{-1}xvz,\ wuwuw \rangle$$

and

$$K = \langle u, v, (uz)^2, z^{-1}yvz^{-1}xvz \rangle .$$

Other subgroups located using this procedure were

$$L = \langle u, v, (uz)^2, z^{-1}yvxwvzx, wuwuw \rangle$$

of index 8 330 , and

$$M = \langle u, v, (uz)^2, z^{-1}yvxwvzx \rangle$$

of index 16 600 .

It is probable that L and M are related to the centralizer of the involution t in H (see [3]).

Enumerating cosets of K^* in H gives the permutation representation of H on 2058 letters. This permutation group was generated using the Sydney group theory system, GROUP, and found to have correct order of 4 030 387 200 . Further application of these programs showed that K^* has five orbits of lengths 1, 136, 136, 425 and 1360 . Thus H is a rank 5 group.

References

[1] John J. Cannon and George Havas, "Applications of the Todd-Coxeter algorithm", (Computer-aided Mathematics Project, Technical Report, 6. Department of Pure Mathematics, University of Sydney, Sydney, 1974).

[2] George Havas, "A Reidemeister-Schreier program", *Proc. Second Internat. Conf. Theory of Groups*, Canberra, 1973, 347-356 (Lecture Notes in Mathematics, 372. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[3] Dieter Held, "The simple groups related to M_{24} ", *J. Algebra* 13 (1969), 253-296.

[4] Evelyn Lepique, "Ein Programm zur Berechnung von Untergruppen von gegebenem Index in endlich präsentierten Gruppen" (Diplomarbeit, Rheinisch-Westfälische Technische Hochschule, Aachen, 1972).

[5] John McKay, "Computing with finite simple groups", *Proc. Second Internat. Conf. Theory of Groups*, Canberra 1973, 448-452 (Lecture Notes in Mathematics, 372. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

Department of Pure Mathematics,
University of Sydney,
Sydney,
New South Wales;

School of Information Sciences,
Canberra College of Advanced Education,
Canberra, ACT.