CORRESPONDENCE.

GRADUATION.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—After an interval of three years, the readers of the *Journal* may perhaps bear with some further observations on the well-worn subject of graduation.

Dr. Sprague has convinced us that graduation by a formula correct only to third differences distorts a table of mortality. On the other hand I think we may agree with him and Mr. Woolhouse in regarding the error as practically unimportant. Still, it is desirable to get rid of the error, and such is the object of the present communication, in which Δ will everywhere mean ΔU_0.

If, by the formula in vol. xxv of the *Journal*, page 22, correct to third differences, I graduate a series in seventh differences, I bring out successive results deficient as follows:

$$6{\cdot}4\Delta^4+38{\cdot}4\Delta^5+107{\cdot}6\Delta^6+186{\cdot}0\Delta^7$$

6·4	+44·8	+146·0	+293·6
6·4	+51·2	+190·8	+439·6
6·4	+57·6	+242·0	+630·4
&c.	&c.	&c.	&c.

These errors form a regular series in which the fourth differences vanish, and consequently, if graduated by the same formula or by one of similar capacity, they will be reproduced without alteration. Then, adding to the first graduation the graduated error (which is the same thing as the ungraduated error), I revert to the series in seventh differences with which I started.

Applying this to a mortality table, I graduate first the data, then the differences between the data and my results, and when the two are combined the work has been done correctly to the seventh order of differences.

Dr. Sprague's test (*J.I.A.*, xxix, 235) is a complete investigation of Mr. Woolhouse's formula as hitherto used, but in pursuance of what has now been written, I proceed to graduate the error or differences in l_x which Dr. Sprague demonstrates. I use my own formula above quoted because it is smoother than the other and more simple, while agreeing closely with it in result. If I may be allowed to repeat myself I would like to state more clearly than before how this formula is arrived at.

Let S represent the result of summing four times in fives.

Σ the result of summing thrice in fives, then in fours and twos.

U_c (being U_8) the central or ninth term of the seventeen in summation.

$S=\dfrac{(1+\Delta)^5-1}{\Delta}$ multiplied thrice by itself (*J.I.A.*, xxv, 246) expanding and multiplying.

$$S=625U_0+5{,}000\Delta^1+20{,}000\Delta^2+52{,}500\Delta^3+, \&c.$$

Compare $625U_c=625U_0+5{,}000\Delta^1+17{,}500\Delta^2+35{,}000\Delta^3+, \&c.$

$$\therefore\ 625U_c=S-2{,}500\Delta^2-17{,}500\Delta^3.$$

In the same manner we find

$$1{,}000U_c=\Sigma-3{,}750\Delta^2-26{,}250\Delta^3$$

and from these two equations

$$U_c=\frac{2\,\Sigma-3\,S}{125}$$

which is more intelligible when written thus:

$$U_c=\frac{2\,S_{5.5.5.4.2}-3\,S_{5.5.5.5}}{125}$$

The shortened working, which saves several columns, will be presently seen. (For explanation see *J.I.A.*, xxv, 23.)

https://doi.org/10.1017/S0020268100005321 Published online by Cambridge University Press

$l_x - l^1_x$ (Dr. Sprague)	Fives: Three middle terms	Fives: Two outside terms	Fives: Three less Two	Fives	Fives	Fives	Last ÷ 125 or × ·008
64	...	...	...	...	...	...	...
73	...	...	...	...	...	...	...
80	241	157	84	...	...	...	...
88	261	170	91	...	...	...	...
93	278	178	100	488	...	...	...
97	288	184	104	512	...	...	...
98	291	182	109	526	2,539	...	...
96	283	175	108	520	2,488	...	...
89	263	158	105	493	2,329	10,999	88
78	227	133	94	437	2,039	9,474	76
60	175	98	77	353	1,604	7,259	58
37	106	53	53	236	1,014	4,323	35
9	21	− 3	24	85	273	686	5
− 25	− 79	− 67	− 12	− 97	− 607	− 3,570	− 29
− 63	−192	−135	− 57	− 304	−1,598	− 8,289	− 66
−104	−311	−206	−105	− 527	−2,652	−13,241	−106
−144	−429	−275	−154	− 755	−3,705	−18,122	−145
−181	−537	−338	−199	− 969	−4,679	−22,568	−181
−212	−627	−387	−240	−1,150	−5,488	−26,185	−209
−234	−689	−418	−271	−1,278	−6,044	−28,591	−229
−243	−714	−428	−286	−1,336	−6,269	−29,465	−236
−237	−696	−414	−282	−1,311	−6,111	−28,598	−229
−216	−633	−376	−257	−1,194	−5,553	−25,936	−207
−180	−529	−314	−215	− 992	−4,621	−21,619	−173
−133	−390	−236	−154	− 720	−3,382	−15,978	−128
− 77	−230	−146	− 84	− 404	−1,952	− 9,507	− 76
− 20	− 63	− 53	− 10	− 72	− 470	...	...
34	94	35	59	236	918	...	...
80	226	109	117	490	...	...	...
112	321	167	154	668	...	...	...
129	374	204	170	...	...	...	...
133	386	218	168	...	...	...	...
124	...	...	...	...	...	...	...
106	...	...	...	...	...	...	...

It will be seen that the quantities to be added to l^1_x agree very closely with those which Dr. Sprague shows to be wanting; and in any case where it would be of advantage to graduate the differences still remaining, the work could be carried to the utmost degree of exactness.

The test, therefore, establishes the applicability of formulas of this kind when the distortion is cured which Dr. Sprague has pointed out.

It remains to consider what effect this procedure has upon the formula in regard to adjustment of irregularities. The first application makes U_c, which I will now call U_0, equal to

$$\cdot200U_0 + \cdot192(U_{-1}+U_{+1}) + \cdot144(U_{-2}+U_{+2}) + \cdot080(U_{-3}+U_{+3}) + \cdot024(U_{-4}+U_{+4}) - \cdot016(U_{-6}+U_{+6}) - \cdot016(U_{-7}+U_{+7}) - \cdot008(U_{-8}+U_{+8})$$

obtained as follows, the terms of the numerator being differenced at commencement.

https://doi.org/10.1017/S0020268100005321 Published online by Cambridge University Press

U_n	S_4	$S_{4,2}$	$2S_{4,2}$	S_5	$3S_5$	$2S_{4,2}-3S_5$*	Fives	Fives	Fives	Last $\div125$ or $\times\cdot008$	
...	...	...	...	...	...	...	...	...	−1	−·008	U_{-8}
...	...	...	...	...	...	...	...	..	−2	−·016	U_{-7}
...	...	...	...	...	...	...	...	−1	−2	−·016	U_{-6}
...	...	...	...	...	...	...	...	−1	0	0	U_{-5}
...	...	...	...	...	...	...	−1	0	3	·024	U_{-4}
...	...	...	...	...	...	...	0	2	10	·080	U_{-3}
...	1	1	2	1	3	−1	1	3	18	·144	U_{-2}
...	1	2	4	1	3	1	2	6	24	·192	U_{-1}
1	1	2	4	1	3	1	1	7	25	·200	U_0
...	1	2	4	1	3	1	2	6	24	·192	U_{+1}
...	...	1	2	1	3	−1	1	3	18	·144	U_{+2}
...	...	...	...	...	...	...	0	2	10	·080	U_{+3}
...	...	..	...	...	...	...	−1	0	3	·024	U_{+4}
...	...	...	...	...	...	...	...	−1	0	0	U_{+5}
...	...	...	...	...	...	...	...	−1	−2	−·016	U_{+6}
...	...	...	...	...	...	...	...	...	−2	−·016	U_{+7}
...	...	...	...	...	...	...	...	...	−1	−·008	U_{+8}

When, in like manner, the differences between the first and last columns have been expanded and the two sets of results combined, the completed operation makes U_0 equal to

$\cdot229696\ U_0$	$\Delta-\cdot005440$
$+\cdot224256(U_{-1}+U_{+1})$	$-\cdot067840$
$+\cdot156416(U_{-2}+U_{+2})$	$-\cdot089600$
$+\cdot066816(U_{-3}+U_{+3})$	$-\cdot072320$
$-\cdot005504(U_{-4}+U_{+4})$	$-\cdot014720$
$-\cdot020224(U_{-5}+U_{+5})$	$-\cdot010240$
$-\cdot030464(U_{-6}+U_{+6})$	$+\cdot010240$
$-\cdot020224(U_{-7}+U_{+7})$	$+\cdot017600$
$-\cdot002624(U_{-8}+U_{+8})$	$+\cdot012864$
$+\cdot010240(U_{-9}+U_{+9})$	$-\cdot004608$
$+\cdot005632(U_{-10}+U_{+10})$	$-\cdot003584$
$+\cdot002048(U_{-11}+U_{+11})$	$-\cdot001920$
$+\cdot000128(U_{-12}+U_{+12})$	$-\cdot000640$
$-\cdot000512(U_{-13}+U_{+13})$	...
$-\cdot000512(U_{-14}+U_{+14})$	$+\cdot000256$
$-\cdot000256(U_{-15}+U_{+15})$	$+\cdot000192$
$-\cdot000064(U_{-16}+U_{+16})$	

This is correct to seventh differences, and the coefficients show the distribution of an irregularity occurring at U_0 and amounting to unity.

In constructing a formula to include in summation 15 terms only, we note that U_c is now U_7, and find that

$$375U_c=S_{5.5.5.3}-1{,}250\Delta^2-7{,}500\Delta^3.$$

Also, bearing in mind that we are using two summations of different scope (U_0 in the shorter being U_1 in the longer when they are referred to the same centre),

$$125U_c=S_{5.5.5}-375\Delta^2-2{,}250\Delta^3.$$

* In working, begin with this column (three middle terms of five, less two outside terms).

From these equations $U_c = \dfrac{10S_{5.5.5} - 3S_{5.5.5.3}}{125}$, which expands as follows:

U_n	S_3	$3S_3$	$10U_n - 3S_3$	Fives	Fives	Fives	Last ÷125 or ×·008	
...	...	...	...	...	...	− 3	−·024	U_{-7}
...	...	...	...	...	...	− 2	−·016	U_{-6}
...	...	...	...	...	− 3	0	0	U_{-5}
...	...	...	...	...	1	3	·024	U_{-4}
...	...	...	...	−3	2	7	·056	U_{-3}
...	...	...	...	+4	3	21	·168	U_{-2}
...	1	3	−3	+1	4	24	·192	U_{-1}
1	1	3	+7	+1	11	25	·200	U_0
...	1	3	−3	+1	4	24	·192	U_{+1}
...	...	...	...	+4	3	21	·168	U_{+2}
...	...	...	...	−3	2	7	·056	U_{+3}
...	...	...	...	...	1	3	·024	U_{+4}
...	...	...	...	...	−3	0	0	U_{+5}
...	...	...	...	...	...	− 2	−·016	U_{+6}
...	...	...	...	...	...	− 3	−·024	U_{+7}

This formula is the exact equivalent of Woolhouse's. Indeed, it should claim no more than to be a ready means of obtaining Mr. Woolhouse's results; for it is he who has laid down the lines on which arithmetical graduation should proceed, and whose work I have imitated in the desire "to devise a method of adjustment as even and correct as that of Mr. Woolhouse and more easy in application" (*J.I.A.*, xxiv, 44). Certainly, the arrangement in black and white with which he left the practical part of his subject (*J.I.A.*, xxi, 45) was troublesome; and it did not admit of the check by addition which can be applied to the columnar arrangement. These objections were afterwards met by Mr. Ackland (*J.I.A.*, xxiii, 355), and the formula now given merely does his work by a shortened process, thus:

Age	d_x (H^M)	S_3	$3S_3$	$10d_x - 3S_3$	Fives	Fives	Fives	÷125 or ×·008
66	220	...	...	...	...	...	...	...
67	220	677	2,031	169	...	...	...	...
68	237	703	2,109	261	...	...	...	...
69	246	696	2,088	372	1,000	...	...	...
70	213	681	2,043	87	1,312	...	...	...
71	222	703	2,109	111	1,048	5,973	...	...
72	268	733	2,199	481	1,324	6,471	...	...
73	243	811	2,433	− 3	1,289	6,331	31,920	255·36
74	300	784	2,352	648	1,498	6,711	...	...
75	241	786	2,358	52	1,172	6,434	...	...
76	245	710	2,130	320	1,428	...	...	...
77	224	695	2,085	155	1,047	...	...	...
78	226	609	2,007	253	...	...	...	..
79	219	641	1,923	267	...	...	...	...
80	196	...	...	...	...	...	..	...

The two formulas are now equally short in lateral working (compare *J.I.A.*, xxv, 23), and the saving of labour is a material set-off against the added work of graduating the primary errors or differences.

When graduation by this formula is completed in the manner hereinbefore proposed, U_0 becomes equal to

$\cdot 220736\ U_0$	$\Delta - \cdot 000320$
$+\cdot 220416(U_{-1}+U_{+1})$	$-\cdot 017280$
$+\cdot 203136(U_{-2}+U_{+2})$	$-\cdot 184320$
$+\cdot 018816(U_{-3}+U_{+3})$	$-\cdot 022080$
$-\cdot 003264(U_{-4}+U_{+4})$	$-\cdot 010560$
$-\cdot 013824(U_{-5}+U_{+5})$	$-\cdot 013760$
$-\cdot 027584(U_{-6}+U_{+6})$	$-\cdot 007360$
$-\cdot 034944(U_{-7}+U_{+7})$	$+\cdot 048960$
$+\cdot 014016(U_{-8}+U_{+8})$	$-\cdot 004160$
$+\cdot 009856(U_{-9}+U_{+9})$	$-\cdot 006400$
$+\cdot 003456(U_{-10}+U_{+10})$	$-\cdot 002304$
$+\cdot 001152(U_{-11}+U_{+11})$	$-\cdot 001408$
$-\cdot 000256(U_{-12}+U_{+12})$	$-\cdot 000512$
$-\cdot 000768(U_{-13}+U_{+13})$	$+\cdot 000192$
$-\cdot 000576(U_{-14}+U_{+14})$	

This also is correct to seventh differences, but the formula is evidently less suited than the other to even graduation. The base of this is $-3U_{-1}+7U_0-3U_{+1}$; the base of the other is $-U_{-2}+U_{-1}+U_0+U_{+1}-U_{+2}$.

A section from a completed graduation of $H^M\ d_x$ by each of the two formulas will be as follows:—

First Formula (Higham): $U_0 = \dfrac{2S_{5.5.5.4.2}-3S_{5.5.5.5}}{125}$.

Age	d_x Ungraduated	First Graduation	Difference	Difference Graduated	Completed Graduation	Δ′	Δ″
60	1,840	1,747	93	− 5	1,742	+ 85	+ 5
61	1,860	1,831	29	− 4	1,827	+ 90	+ 1
62	1,910	1,916	− 6	1	1,917	+ 91	− 16
63	2,000	2,001	− 1	7	2,008	+ 75	− 9
64	2,060	2,074	− 14	9	2,083	+ 66	− 11
65	2,150	2,139	11	10	2,149	+ 55	− 17
66	2,200	2,195	5	9	2,204	+ 38	− 21
67	2,200	2,240	− 40	2	2,242	+ 17	+ 22
68	2,370	2,274	96	−15	2,259	+ 39	+ 13
69	2,460	2,320	140	−22	2,298	+ 52	+ 24
70	2,130	2,374	−244	−24	2,350	+ 76	− 1
71	2,220	2,439	−219	−13	2,426	+ 75	+ 1
72	2,680	2,500	180	1	2,501	+ 76	− 60
73	2,430	2,553	−123	24	2,577	+ 16	− 50
74	3,000	2,564	436	29	2,593	− 34	− 53
75	2,410	2,535	−125	24	2,559	− 87	− 20
76	2,450	2,464	− 14	8	2,472	−107	− 19

Second Formula (Woolhouse): $U_c = \frac{10S_{5.5.5} - 3S_{5.5.5.3}}{125}$.

Age	d_x Ungraduated	First Graduation	Difference	Difference Graduated	Completed Graduation	Δ^1	Δ^2
60	1,840	1,747	93	− 4	1,743	+ 80	+ 14
61	1,860	1,828	32	− 5	1,823	+ 94	− 4
62	1,910	1,917	− 7	0	1,917	+ 90	− 7
63	2,000	2,001	− 1	6	2,007	+ 83	− 34
64	2,060	2,079	− 19	11	2,090	+ 49	+ 23
65	2,150	2,135	15	4	2,139	+ 72	− 33
66	2,200	2,199	1	12	2,211	+ 39	− 29
67	2,200	2,243	− 43	7	2,250	+ 10	+ 5
68	2,370	2,274	96	−14	2,260	+ 15	+ 82
69	2,460	2,307	153	−32	2,275	+ 97	− 66
70	2,130	2,383	−253	−11	2,372	+ 31	+ 71
71	2,220	2,427	−207	−24	2,403	+102	− 31
72	2,680	2,503	177	2	2,505	+ 71	− 30
73	2,430	2,554	−124	22	2,576	+ 41	−119
74	3,000	2,578	422	39	2,617	− 78	+ 28
75	2,410	2,527	−117	12	2,539	− 50	− 86
76	2,450	2,474	− 24	15	2,489	−136	+ 22

In graduating the differences it is convenient, for avoidance of negative signs, to add a constant at commencement and take it off afterwards. For instance: add 500 at outset, drop 2,000 in the first fives, and take off 100 at the end. And when there is much irregularity, it is well to postpone till after the first fives the differencing of the terms of the numerator.

I am not without hope that the foregoing may be of service to those whose skill qualifies them to use the graphic method. A clear and undistorted presentation of what a record of mortality does say must afford some assistance in the determination of what it meant to say.

I am, Sir,
Your obedient servant,
J. A. HIGHAM.

https://doi.org/10.1017/S0020268100005321 Published online by Cambridge University Press