Can. J. Math., Vol. XXVI, No. 1, 1974, pp. 185-189

SOME C^*-ALGEBRAS WITH OUTER DERIVATIONS, II

GEORGE A. ELLIOTT

1. In this paper we shall consider the class of C^*-algebras which are inductive limits of sequences of finite-dimensional C^*-algebras. We shall give a complete description of those C^*-algebras in this class every derivation of which is inner.

THEOREM. *Let A be a C^*-algebra. Suppose that A is the inductive limit of a sequence of finite-dimensional C^*-algebras. Then the following statements are equivalent:*

(i) *every derivation of A is inner;*

(ii) *A is the direct sum of a finite number of algebras each of which is either commutative, the tensor product of a finite-dimensional and a commutative with unit, or simple with unit.*

2. *Remark.* There are two consequences of the above result which may hold more generally. Let A be a C^*-algebra which is the inductive limit of a sequence of finite-dimensional C^*-algebras, and suppose that every derivation of A is inner. Then

(1) every derivation of each quotient of A by a closed two-sided ideal is inner, and

(2) A is the direct sum of a commutative algebra and an algebra with unit. It is known (see [**9**]) that (2) need not hold without some additional restriction for a C^*-algebra A having only inner derivations, but it is conceivable that the restriction of separability is strong enough (see [**4**]). *Added in proof.* This is the case; see Akemann, Elliott, Pedersen, Tomiyama, Amer. J. Math. (to appear).

3. LEMMA. *Let A be a C^*-algebra which is the inductive limit of a sequence of finite-dimensional C^*-algebras, and suppose that every derivation of A is inner. Then every primitive quotient of A is simple with unit.*

Proof. By [**4**, 2], every primitive quotient of A has a unit. This is also established in the course of the following proof that every primitive quotient of A is simple.

Let P be a primitive ideal of A, and suppose that I is a closed two-sided ideal of A containing P. We must show that I is equal to P or to A.

There exists an increasing sequence $A_1 \subseteq A_2 \subseteq \ldots$ of finite-dimensional sub-C^*-algebras of A with union dense in A. By [**1**, 3.1], $\bigcup_n I \cap A_n$ is dense in I. Denote the unit of $I \cap A_n$ by e_n, $n = 1, 2, \ldots$, and set $e_{n+1} - e_n = f_n$; f_n is a projection permutable with each element of A_n.

Received August 28, 1972 and in revised form, October 31, 1972.

If all except finitely many f_n are in P then $\bigcup_n I \cap A_n$ has a unit modulo P. By continuity this is also a unit for I modulo P. Hence by primitivity of P, $I = P$ or $I = A$.

Suppose that infinitely many f_n are not in P. We shall arrive at a contradiction. Passing to a subsequence of $A_1 \subseteq A_2 \subseteq \ldots$, we may equally well suppose that no f_n is in P. If $\lambda = (\lambda_n)$ is a bounded sequence in $\mathbf{C}$, then for each $x \in A$ the series

$$\sum_{n=1}^{\infty} \lambda_n(f_n x - x f_n)$$

is convergent. Indeed, if k is large enough that $||x - x_\epsilon|| < \epsilon(2 \sup_n|\lambda_n|)^{-1}$ for some $x_\epsilon \in A_k$, then

$$\left\|\left[\sum_{n=k+1}^{k+p} \lambda_n f_n, x\right]\right\| = \left\|\left[\sum_{n=k+1}^{k+p} \lambda_n f_n, x - x_\epsilon\right]\right\| < \epsilon,$$

where $[a, b] = ab - ba$. Therefore by the hypothesis that every derivation of A is inner, for every bounded sequence $\lambda = (\lambda_n)$ in $\mathbf{C}$ there exists $y_\lambda \in A$ such that

$$\sum_{n=1}^{\infty} \lambda_n(f_n x - x f_n) = y_\lambda x - x y_\lambda, \quad x \in A.$$

Representing A/P faithfully as an irreducible C^*-algebra of operators, we find that for each bounded sequence $\lambda = (\lambda_n)$ in $\mathbf{C}$, the image $\dot{y}_\lambda$ of y_λ in A/P differs from the operator $\sum_{n=1}^{\infty} \lambda_n \dot{f}_n$ by a scalar. Since for each bounded sequence $\lambda = (\lambda_n)$ in $\mathbf{C}$ we have

$$\left\|\sum_{n=1}^{\infty} \lambda_n \dot{f}_n\right\| = \sup_n|\lambda_n|,$$

it follows that the set of operators

$$\{\dot{y}_\lambda | \lambda = (\lambda_n) \text{ a bounded sequence in } \mathbf{C}\}$$

is not norm separable. This contradicts the separability of A.

4. Lemma. *Let A be a C^*-algebra which is the inductive limit of a sequence of finite-dimensional C^*-algebras, and suppose that every derivation of A is inner. Then every primitive ideal of A of infinite codimension is a direct summand.*

Proof. Let P be a primitive ideal of A of infinite codimension. We must show that $P + P' = A$, where P' is the annihilator of P. If $P + P' \neq A$, then, since by 3, A/P is simple, $P' = 0$. Therefore it is enough to show that $P' \neq 0$.

There exists an increasing sequence $A_1 \subseteq A_2 \subseteq \ldots$ of finite-dimensional sub-C^*-algebras of A with union dense in A. Denote by e_n the unit of $P \cap A_n$, and by g_n the complement of e_n in A_n. The increasing sequence $(A_1 + P)/P \subseteq (A_2 + P)/P \subseteq \ldots$ of sub-C^*-algebras of A/P has union dense in A/P. Since A/P is of infinite dimension it follows that the dimension of $(A_n + P)/P$ tends to infinity. Since $(A_n + P)/P$ is isomorphic to $A_n/A_n \cap P$ which in turn is isomorphic to g_nA_n, $n = 1, 2, \ldots$, we have dim $g_nA_n \to \infty$. Hence, passing to a subsequence of $A_1 \subseteq A_2 \subseteq \ldots$ we may suppose that dim g_nA_n is strictly increasing.

Suppose that $P' = 0$. Then for each $n = 1, 2, \ldots$ there is an $m = m(n) \geqq n + 1$ such that $e_mg_nA_n$ is isomorphic to g_nA_n. Passing to a subsequence of $A_1 \subseteq A_2 \subseteq \ldots$ (e.g., $A_1 \subseteq A_{m(1)} \subseteq A_{m(m(1))} \subseteq \ldots$) we may suppose that $m(n) = n + 1$; that is, that $e_{n+1}g_nA_n$ is isomorphic to g_nA_n, $n = 1, 2, \ldots$.

Fix $n = 1, 2, \ldots$. Since dim $g_nA_n >$ dim $g_{n-1}A_{n-1}$ there is a projection $f_n' \in g_nA_n$, $0 \neq f_n' \neq g_n$, such that f_n' is permutable with each element of $g_{n-1}A_{n-1}$. Since the quotient map $A \to A/P$ is injective on g_nA_n, the image of f_n' in A/P is a nonscalar projection. By 3, A/P is simple; the image of f_n' in A/P is therefore noncentral. Hence, for some $m' = m'(n) \geqq n, f_n'$ is noncentral in $g_{m'}A_{m'}$. Replacing f_n' by $g_{m'}f_n' \in g_{m'}A_{m'}$ and passing to a subsequence of $A_1 \subseteq A_2 \subseteq \ldots$ (e.g., $A_1 \subseteq A_{m'(1)} \subseteq A_{m'(m'(1))} \subseteq \ldots$; here we are assuming that $m'(k)$ has been defined for all $k = 1, 2, \ldots$), we may suppose that $m' = n$; that is, that f_n' is noncentral in g_nA_n. This entails that $||f_n'x_n - x_nf_n'|| = 1$ for some $x_n \in g_nA_n$ with $||x_n|| = 1$. Since $g_ne_{n-1} = g_ne_n = 0$, we have $f_n'e_{n-1}A_{n-1} = 0$. Therefore f_n' is permutable with each element of A_{n-1}. Set $e_{n+1}f_n' = f_n$. Then $f_n \in A_{n+1}$ and f_n is permutable with each element of A_{n-1}. Since $e_{n+1}g_nA_n$ is isomorphic to g_nA_n, we have $||f_nx_n - x_nf_n|| = 1$. Moreover,

$$f_n = f_ng_ne_{n+1} = f_ng_ne_{n+1}(e_{n+1} - e_n) = f_n(e_{n+1} - e_n).$$

Let $\lambda = (\lambda_n)$ be a bounded sequence in $\mathbf{C}$. Then the same remark as in 3 shows that for each $x \in A$ the series

$$\sum_{n=1}^{\infty} \lambda_n(f_nx - xf_n)$$

is convergent. Hence there exists $y_\lambda \in A$ such that

$$\sum_{n=1}^{\infty} \lambda_n(f_nx - xf_n) = y_\lambda x - xy_\lambda, \quad x \in A.$$

With $\lambda = (\lambda_n)$ a bounded sequence in $\mathbf{C}$ denote by δ_λ the derivation of A: $x \mapsto y_\lambda x - xy_\lambda$. Then for $n = 1, 2, \ldots$,

$$||\delta_\lambda|| \geqq ||\delta_\lambda(x_n)|| \geqq ||(e_{n+1} - e_n)\delta_\lambda(x_n)(e_{n+1} - e_n)|| = ||\lambda_n(f_nx_n - x_nf_n)|| = |\lambda_n|.$$

Moreover, $||\delta_\lambda|| \leqq 2||y_\lambda||$. Thus,

$$\sup_n|\lambda_n| \leqq 2||y_\lambda||.$$

Since A is separable the set

$$\{y_\lambda | \lambda = (\lambda_n) \text{ a bounded sequence in } \mathbf{C}\}$$

is norm separable. From the preceding inequality it follows that the linear space

$$\{\lambda = (\lambda_n) \text{ a bounded sequence in } \mathbf{C}\}$$

https://doi.org/10.4153/CJM-1974-018-6 Published online by Cambridge University Press

is separable in the norm $||\lambda|| = \sup_n|\lambda_n|$. This is known not to be true, and the supposition $P' = 0$ is therefore inconsistent with the hypotheses.

5. LEMMA. *Let A be a C^*-algebra which is the inductive limit of a sequence of finite-dimensional C^*-algebras, and suppose that every derivation of A is inner. Then there are only finitely many primitive ideals of A of infinite codimension.*

Proof. Suppose that infinitely many primitive ideals $P_1, P_2, \ldots$ of A have infinite codimension, and denote by $I_1, I_2, \ldots$ the annihilators of $P_1, P_2, \ldots$. Then by 4, for each $n = 1, 2, \ldots$ $A = P_n + I_n$. Since by 3 each I_n is simple, if $n \neq n'$ then $I_nI_{n'} = 0$.

There exists an increasing sequence $A_1 \subseteq A_2 \subseteq \ldots$ of finite-dimensional sub-C^*-algebras of A with union dense in A. For each $n = 1, 2, \ldots, I_n$ has infinite dimension, so $I_n \cap A_n \neq I_n$. Since I_n is a direct summand of A, there exists a noncentral projection f_n in I_n which is not in $I_n \cap A_n$, and which is permutable with each element of A_n.

Claim. The sequence of inner derivations of A determined by $\sum_{n=1}^k f_n$, $k = 1, 2, \ldots$, converges simply to an outer derivation of A. Convergence follows from the fact that f_n is permutable with each element of A_n, $n = 1, 2, \ldots$, and that the f_n are mutually orthogonal projections (they belong to orthogonal ideals). Suppose that the limit, δ, clearly a derivation of A, is inner, determined by $y \in A$. For each $n = 1, 2, \ldots$, since I_n is simple and f_n is a noncentral projection in I_n, there exists $x_n \in I_n$ of unit norm such that

$$||\delta(x_n)|| = ||f_nx_n - x_nf_n|| > 1/2.$$

Since f_n is permutable with each element of A_n, x_n also may be chosen to be permutable with each element of A_n. On the other hand, there exists y_0 in some A_{n0} such that $||y - y_0|| < 1/4$. Then we have

$$||\delta(x_{n0})|| = ||yx_{n0} - x_{n0}y|| = ||(y - y_0)x_{n0} - x_{n0}(y - y_0)|| < 2(1/4) = 1/2.$$

This is a contradiction, whence δ must be outer.

6. LEMMA. *Let A be a C^*-algebra which is generated by its projections. Suppose that* Prim A *is separated, and that A has a unit. Then the centre of A is generated by its projections.*

Proof. By [**6**, Theorem 4.1], the functions $t \mapsto ||x + t||$ on Prim A with $x \in A$ are continuous. Since A is generated by its projections the functions $t \mapsto ||e + t||$ with e a projection in A separate points of Prim A. By [**2**, 8.16], for every projection e in A there exists a central projection e' such that $e' + t = ||e + t||$, $t \in$ Prim A. It follows that the centre of A is generated by its projections.

7. *Remark.* In 6 it is not necessary to assume that A has a unit. The assumption that Prim A is separated, however, cannot be omitted (W. Green, private communication).

8. *Proof of Theorem* 1. (ii) $\Rightarrow$ (i). Since a derivation is zero on central

idempotents it is enough to consider the cases that A is either commutative, the tensor product of a finite-dimensional algebra and a commutative algebra with unit, or simple with unit. The first case is covered by [**11**, Corollary 2.2], the second by [**3**, 1] (for example), and the third by [**8**].

(i) $\Rightarrow$ (ii). By 3, 4 and 5, A is the direct sum of finitely many simple algebras with unit together with an algebra having only finite-dimensional primitive quotients. By [**4**, 3], the direct summand of A with only finite-dimensional primitive quotients is a finite direct sum of homogeneous algebras of finite order each of which is either commutative or with unit. By [**6**, Theorems 4.2, 4.1, 3.3 and Lemma 4.3], together with [**5**, Theorem 3.1] and 6 above, each direct summand of A which is homogeneous of finite order is either commutative or the tensor product of a finite-dimensional algebra and a commutative algebra with unit.

9. *Application.* Let G be a countable, locally finite discrete group. Suppose that every derivation of the C^*-algebra of G is inner. Then the commutator subgroup of G is finite.

To see this it is enough by [**7**, Theorem 1] to show that the left regular representation of G is not of type II. Since G is locally finite, the left regular representation of G determines a faithful representation of $C^*(G)$, the C^*-algebra of G (this can be seen directly or by using the fact that G is amenable). Again because G is locally finite, $C^*(G)$ satisfies the hypothesis of 1. Since $C^*(G)$ has a one-dimensional quotient (corresponding to the trivial representation of G), by 1, $C^*(G)$ has a nonzero commutative direct summand. So, therefore, also does the von Neumann algebra generated by the left regular representation of G.

This answers negatively a question of Sakai (see [**10**, Problem 3]).

REFERENCES

1. O. Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. *171* (1972), 195–234.

2. J. Dauns and K. H. Hofmann, *Representations of rings by sections*, Mem. Amer. Math. Soc. *83* (1968).

3. G. A. Elliott, *Derivations of matroid C*-algebras*, Invent. Math. *9* (1970), 253–269.

4. ——— *Some C*-algebras with outer derivations*, Rocky Mountain J. Math. *3* (1973), 501–506.

5. J. M. G. Fell, *The structure of algebras of operator fields*, Acta Math. *106* (1961), 233–280.

6. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. *70* (1951), 219–255.

7. ——— *Group algebras in the large*, Tôhoku Math. J. *3* (1951), 249–256.

8. S. Sakai, *Derivations of simple C*-algebras*, J. Functional Analysis *2* (1968), 202–206.

9. ——— *Derivations of simple C*-algebras, III*, Tôhoku Math. J. *23* (1971), 559–564.

10. ——— *Derived C*-algebras of primitive C*-algebras* (to appear).

11. I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. *129* (1955), 260–264.

The Institute for Advanced Study,
Princeton, New Jersey

https://doi.org/10.4153/CJM-1974-018-6 Published online by Cambridge University Press