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ON A GAUGE-INVARIANT FUNCTIONAL

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Abstract We define a new functional which is gauge invariant on the space of all smooth connections of a vector bundle over a compact Riemannian manifold. This functional is a generalization of the classical Yang–Mills functional. We derive its first variation formula and prove the existence of critical points. We also obtain the second variation formula.

Keywords: curvature; vector bundle; Yang–Mills connections

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1. Introduction

From the variational point of view, there are a lot of similarities between the theory of harmonic maps and the theory of Yang–Mills connections.

Let (M, g) and (N, h) be two compact Riemannian manifolds and let $F : M \to N$ be a smooth map. Harmonic maps are extremal of the energy functional

$$E(F) = \int_M e(F)\vartheta_g,$$

where $e(F) = \frac{1}{2}\|\mathrm{d}F\|^2$ is the energy density and ϑ_g is the canonical volume element [**2**].

M. Ara introduced and studied another problem of the calculus of variations. He defined the so-called f-energy functional of F as

$$E_f(F) = \int_M f(\tfrac{1}{2}\|\mathrm{d}F\|^2)\vartheta_g,$$

where f is a certain real smooth function, and he called a smooth extremal of E_f an f-harmonic map [**5, 6**].

We introduce and study a problem of the calculus of variations in an analogous way to f-harmonic maps in [**5**]. Namely, we define the f-Yang–Mills functional $\mathcal{YM}_f$, which is gauge invariant on the space of all smooth connections D of a vector bundle E over a compact Riemannian manifold (M, g). The f-Yang–Mills functional is defined by

$$\mathcal{YM}_f(D) = \int_M f(\tfrac{1}{2}\|R^D\|^2)\vartheta_g,$$

where $\|R^D\|$ is the norm of the curvature tensor of a connection D and $f : [0,\infty) \to [0,\infty)$ is a function of class C^2 such that $f'(t) > 0$ for any $t \geqslant 0$. A critical point of $\mathcal{YM}_f$ will be called an f-Yang–Mills connection. We note that if $f(t) = t$, we obtain the classical Yang–Mills functional [**1**] and if $f(t) = \exp(t)$ we obtain the exponential Yang–Mills functional [**4**].

Using a similar method to that in [**1**], we calculate the first and the second variation formulae of the functional $\mathcal{YM}_f$. Once we have obtained the first variation formula of the functional $\mathcal{YM}_f$, the main result of the paper is the following existence theorem.

Theorem 1.1. *Let (M, g) be an n-dimensional compact Riemannian manifold, let G be a compact Lie group and let E be a G-vector bundle over M. Assume that $n \geqslant 5$ and $f''(0) \neq 0$. Then there exists a Riemannian metric $\tilde{g}$ on M which is conformal to g and a G-connection on E such that D is an f-Yang–Mills connection with respect to $\tilde{g}$.*

2. Preliminaries

Let P be a principal G-bundle over a compact Riemannian manifold (M, g), where G is a compact Lie group. We denote by E the associated vector bundle to P by a faithful representation $\rho : G \to O(r)$.

For any vector bundle F over M we denote by $\Gamma(F)$ the space of smooth cross-sections of F and for each $p \geqslant 0$ we denote by $\Omega^p(F) = \Gamma(\Lambda^p T^*M \otimes F)$ the space of all smooth p-forms on M with values in F. Note that $\Omega^0(F) = \Gamma(E)$.

A connection D on the vector bundle E is defined by specifying a covariant derivative, that is, a linear map

$$D : \Omega^0(E) \to \Omega^1(E),$$

such that $D(fs) = \mathrm{d}f \otimes s + fDs$, for any section $s \in \Omega^0(E)$ and any smooth function $f \in C^\infty(M)$.

A connection D is said to be a G-connection if the natural extension of D to tensor bundles of E annihilates the tensors that define the G-structure. We denote by $\mathcal{C}(E)$ the space of all smooth G-connections D on E.

Now let $G(E)$ be the gauge group of the vector bundle E, that is, the group of all automorphisms of E inducing the identity map of M. The gauge group can easily be identified with the space of smooth sections of the bundle of groups $P \times_{\mathrm{Ad}} G$ associated to the adjoint representation, Ad, of G, which is the group of all automorphisms φ of P satisfying $\varphi(ua) = \varphi(u)a$ for any $u \in P$ and $a \in G$. We note that there is a natural action of the gauge group $G(E)$ on the space of G-connections $\mathcal{C}(E)$ given by

$$D^\varphi = \varphi^{-1} \circ D \circ \varphi, \qquad D^\varphi s := \varphi^{-1}(D(\varphi s))$$

for any $s \in \Omega^0(E)$, $\varphi \in G(E)$ and $D \in \mathcal{C}(E)$. We denote by g the Lie algebra of the Lie group G. Related to $G(E)$ is the infinitesimal gauge group or gauge algebra. This can be regarded as the space $\Omega^0(P \times_{\mathrm{Ad}} g)$ of smooth sections of the vector bundle $P \times_{\mathrm{Ad}} g$ which is identified with a subbundle of the bundle $\mathrm{End}(E)$ via the representation ρ, denoted by g_E. The identification is given by

$$P \times_{\mathrm{Ad}} g \ni [(u, A)] \to u \circ \rho(A) \circ u^{-1} \in \mathrm{End}(E).$$

Given a connection on E, the map $D : \Omega^0(E) \to \Omega^1(E)$ can be extended to a generalized de Rham sequence

$$\Omega^0(E) \xrightarrow{d^D = D} \Omega^1(E) \xrightarrow{d^D} \Omega^2(E) \xrightarrow{d^D} \cdots .$$

For each G-connection D of the vector bundle E, the curvature tensor of D, denoted by R^D, is determined by $(d^D)^2 : \Omega^0(E) \to \Omega^2(E)$. It is easy to see that $R^D \in \Omega^2(g_E)$. On the other hand, it holds that

$$R^{D^\varphi} = \varphi^{-1} \circ R^D \circ \varphi$$

for any $\varphi \in \mathcal{C}(E)$.

Let $\langle \cdot\,, \cdot \rangle$ be the inner product on g defined by

$$\langle A, B \rangle = -\tfrac{1}{2}\operatorname{tr}(\rho(A)\rho(B)) = \tfrac{1}{2}\operatorname{tr}(\rho(A)^t \circ \rho(B))$$

for any $A, B \in g$, which induces a fibre metric on $P \times_{\mathrm{Ad}} g$ and thus a fibre metric on $\mathrm{End}(E)$ by

$$\langle C, D \rangle = \tfrac{1}{2}\operatorname{tr}(C^t \circ D)$$

for any $C, D \in \mathrm{End}(E_x)$ and $x \in M$.

If a vector bundle F over M admits a fibre metric $\langle \cdot\,, \cdot \rangle$, we can define an inner product on $\Lambda^p T_x^* M \otimes F_x$ by

$$\langle \psi, \varphi \rangle = \sum_{i_1 < \cdots < i_p} \langle \psi(e_{i_1}, \dots, e_{i_p}), \varphi(e_{i_1}, \dots, e_{i_p}) \rangle,$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $T_x M$ with respect to the metric g. We denote its norm by $\|\cdot\|$. Integrating the above pointwise, the inner product over M gives an inner product in $\Omega^p(F)$. Integration on M shall always be with respect to the Riemannian volume measure. We then define the operator $\delta^D : \Omega^{p+1}(F) \to \Omega^p(F)$, $p \geqslant 0$, to be the formal adjoint of the operator d^D.

3. The first variation formula

Now let $f : [0, \infty) \to [0, \infty)$ be a function of class C^2 such that $f'(t) > 0$ for any $t \geqslant 0$. We define the functional $YM_f : \mathcal{C}(E) \to \mathbb{R}$ by

$$\mathcal{YM}_f(D) = \int_M f(\tfrac{1}{2}\|R^D\|^2)\vartheta_g.$$

We note that if $f(t) = t$, the functional above is the classical Yang–Mills functional and if $f(t) = \exp(t)$, the functional is the exponential Yang–Mills functional [**4**].

It is not difficult to see that

$$\|R^{D^\varphi}\| = \|R^D\|$$

for any $\varphi \in \mathcal{C}(E)$. Thus, the functional $\mathcal{YM}_f$ is invariant under the action of the gauge group $G(E)$ on $\mathcal{C}(E)$.

In the following we shall calculate the first variation of the functional $\mathcal{YM}_f$.

Theorem 3.1. *The first variation of the functional YM_f is given by the formula*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathcal{YM}_f(D^t)=\int_M\langle B,\delta^D(f'(\tfrac{1}{2}\|R^D\|^2)R^D)\rangle\vartheta_g,$$

where $B=\mathrm{d}/\mathrm{d}t|_{t=0}D^t$. Consequently, D is a critical point of $\mathcal{YM}_f$ if and only if

$$\delta^D(f'(\tfrac{1}{2}\|R^D\|^2)R^D)=0.$$

Proof. Let D be a G-connection $D\in\mathcal{C}(E)$ and consider a smooth curve $D^t=D+\alpha^t$ on $\mathcal{C}(E)$, $t\in(-\epsilon,\epsilon)$, such that $\alpha^0=0$, where $\alpha^t\in\Omega^1(g_E)$. The corresponding curvature is given by

$$R^{D^t}=R^D+d^D\alpha^t+\tfrac{1}{2}[\alpha^t\wedge\alpha^t],$$

where we define the bracket of g_E-valued 1-forms φ and ψ by the formula $[\varphi\wedge\psi](X,Y)=[\varphi(X),\psi(Y)]-[\varphi(Y),\psi(X)]$ for any vector fields $X,Y\in\Gamma(TM)$. Indeed, for any vector fields $X,Y\in\Gamma(TM)$ and $u\in\Gamma(E)$, we have

$$\begin{aligned}
R^{D^t}(X,Y)(u)&=D^t_X(D^t_Yu)-D^t_Y(D^t_Xu)-D^t_{[X,Y]}u\\
&=D^t_X(D_Yu+\alpha^t(Y)(u))-D^t_Y(D_Xu+\alpha^t(X)(u))\\
&\quad-D^t_X(D_{[X,Y]}u+\alpha^t([X,Y])(u))\\
&=D_X(D_Yu+\alpha^t(Y)(u))+\alpha^t(X)(D_Yu+\alpha^t(Y)(u))\\
&\quad-D_Y(D_Xu+\alpha^t(X)(u))-\alpha^t(Y)(D_Xu+\alpha^t(X)(u))\\
&\quad-D_{[X,Y]}u-\alpha([X,Y])(u)\\
&=R^D(X,Y)(u)+D_X(\alpha^t(Y)(u))-\alpha^t(Y)(D_Xu)\\
&\quad-(D_Y(\alpha^t(X)(u))-\alpha^t(X)(D_Yu))-\alpha^t([X,Y])(u)\\
&\quad+\alpha^t(X)(\alpha^t(Y)(u))-\alpha^t(Y)(\alpha^t(X)(u))\\
&=R^D(X,Y)(u)+(D_X(\alpha^t(Y))(u)-(D_Y(\alpha^t(X))(u)))\\
&\quad-\alpha^t([X,Y])(u)+\tfrac{1}{2}[\alpha^t\wedge\alpha^t](X,Y)(u)\\
&=R^D(X,Y)(u)+(d^D\alpha^t)(X,Y)(u)+\tfrac{1}{2}[\alpha^t\wedge\alpha^t](X,Y)(u).
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}f(\tfrac{1}{2}\|R^{D^t}\|^2)&=f'(\tfrac{1}{2}\|R^D\|^2)\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\tfrac{1}{2}\|R^{D^t}\|^2\\
&=f'(\tfrac{1}{2}\|R^D\|^2)\bigg\langle\frac{\mathrm{d}}{\mathrm{d}t}R^{D^t},R^D\bigg\rangle\bigg|_{t=0}\\
&=f'(\tfrac{1}{2}\|R^D\|^2)\langle d^DB,R^D\rangle,
\end{aligned}$$

where

$$B=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}D^t\in\Omega^1(g_E).$$

Thus, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{YM}_f(D^t) = \int_M f'(\tfrac{1}{2}\|R^D\|^2)\langle d^D B, R^D\rangle \vartheta_g$$
$$= \int_M \langle B, \delta^D(f'(\tfrac{1}{2}\|R^D\|^2)R^D)\rangle \vartheta_g.$$

□

Example 3.2. If we take $f(t) = at + b$ with $a > 0$, then a G-connection D is a critical point of the functional $\mathcal{YM}_f$ if and only if $\delta^D R^D = 0$. On the other hand, $d^D R^D = 0$ and thus D is a critical point if and only if the curvature tensor R^D is harmonic. For the case when $a = 1$ and $b = 0$, such a connection is called a Yang–Mills connection [**1**].

Example 3.3. If we take $f(t) = \exp t$, then a G-connection D is a critical point of the functional $\mathcal{YM}_f$ if and only if $\delta^D(\exp(\frac{1}{2}\|R^R\|^2)R^D) = 0$. Such a connection is called an exponential Yang–Mills connection [**4**].

For the case of the existence of Yang–Mills connections we have the following result of Katagiri [**3**].

Theorem 3.4. *Let (M, g) be a compact Riemannian manifold of dimension $n \geqslant 5$, let G be a compact Lie group and let E be a smooth G-vector bundle over M. Then there exist a Riemannian metric $\tilde{g}$ on M which is conformally equivalent to the original metric g and a connection D_0 on E such that D_0 is a Yang–Mills connection with respect to $\tilde{g}$.*

In the following we shall prove an existence theorem for critical points of the functional YM_f.

Theorem 3.5. *Let (M, g) be an n-dimensional compact Riemannian manifold, let G be a compact Lie group and let E be a smooth G-vector bundle over M. Assume that $n \geqslant 5$ and $f''(0) \neq 0$. Then there exist a Riemannian metric $\tilde{g}$ on M conformally equivalent to g and a G-connection D on E such that D is a critical point of the functional YM_f.*

Theorem 3.5 follows from Theorem 3.4 and the following result.

Theorem 3.6. *Let (M, g) be an n-dimensional compact Riemannian manifold, let G be a compact Lie group and let E be a smooth G-vector bundle over M. Assume that $n \geqslant 5$ and $f''(0) \neq 0$ and let D be a Yang–Mills connection. Then there exists a Riemannian metric $\tilde{g}$ on M conformally equivalent to g such that D is a critical point of the functional YM_f.*

Proof. First, we note that, due to Theorem 3.4, we can suppose that D is a Yang–Mills connection with respect to the metric g. For a positive C^∞ function σ on M we define $\tilde{g} = \sigma^{-1}g$. If D is a Yang–Mills connection on the vector bundle E, then

$$\delta_g^D R^D = 0 \quad\iff\quad \delta_{\tilde{g}}^D(\sigma^{(n-4)/2}R^D) = 0. \tag{3.1}$$

We suppose that $f''(0) > 0$; the case when $f''(0) < 0$ is similar. Now, as $f''(0) > 0$ and $f \in C^2$, there exists a positive number ϵ such that $f''(t) > 0$ for any $t \in [0, \epsilon)$, and thus

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f' is invertible on the interval $[0,\epsilon)$, with the smooth inverse $H:[f'(0),f'(\epsilon))\to[0,\epsilon)$. Thus, we have the following relations:

$$H(f'(t))=t, \tag{3.2}$$
$$H'(f'(t))f''(t)=1, \tag{3.3}$$

for any $t\in[0,\epsilon)$.

We define now the smooth function

$$F:[(f'(0))^{2/(n-4)},(f'(\epsilon))^{2/(n-4)})\to[0,\epsilon')$$

by

$$F(y)=\frac{H(y^{(n-4)/2})}{y^2}.$$

We shall prove that F is invertible on a certain interval. It is easy to see that

$$F'(y)=\frac{(n-4)H'(y^{(n-4)/2})y^{(n-4)/2}-4H(y^{(n-4)/2})}{2y^3},$$

and thus we let $y=(f'(t))^{2/(n-4)}$. Using the relations (3.2) and (3.3) we get

$$F'((f'(t))^{2/(n-4)})=\frac{(n-4)f'(t)-4tf''(t)}{2f''(t)(f'(t))^{6/(n-4)}}$$

for any $t\in[0,\epsilon)$. If we evaluate the above relation at 0, as $n\geqslant 5$ and $f'>0$, then there exists a positive number $\epsilon''\leqslant\epsilon$ such that $F'((f'(t))^{2/(n-4)})>0$ for any $t\in[0,\epsilon'')$, and thus

$$F:[(f'(0))^{2/(n-4)},(f'(\epsilon''))^{2/(n-4)})\to[0,\epsilon''')$$

is invertible.

We remark that the metric g can be chosen such that $\|R^D\|_g^2<\epsilon'''$. Indeed, for a positive constant C we define the Riemannian metric g' by $g'=Cg$. Then the Yang–Mills equation with respect to g' is the same as that for g. Moreover, since $\|R^D\|_{g'}^2=C^{-2}\|R^D\|_g^2$ and M is compact, we get $\|R^D\|_g^2<\epsilon'''$ for C sufficiently large. Now, if we denote by Φ the smooth inverse of F, we define the positive smooth function σ by

$$\sigma=\Phi(\tfrac{1}{2}\|R^D\|_g^2).$$

Finally, from equation (3.1) we have

$$\begin{aligned}0&=\delta_{\tilde g}^D(\sigma^{(n-4)/2}R^D)\\&=\delta_{\tilde g}^D((\Phi(\tfrac{1}{2}\|R^D\|_g^2))^{(n-4)/2}R^D)\\&=\delta_{\tilde g}^D(f'(\tfrac{1}{2}\sigma^2\|R^D\|_g^2)R^D)\\&=\delta_{\tilde g}^D(f'(\tfrac{1}{2}\|R^D\|_{\tilde g}^2)R^D),\end{aligned}$$

which proves that the Yang–Mills connection D is also a critical point of the functional YM_f with respect to the metric $\tilde g$. □

4. The second variation formula

In this section we obtain the second variation formula of the functional $\mathcal{YM}_f$. Let (M,g) be an n-dimensional compact Riemannian manifold, let G be a compact Lie group and let E be a G-vector bundle over M. Let D be a critical point of the functional $\mathcal{YM}_f$ and D^t be a smooth curve on $\mathcal{C}(E)$ such that $D^t = D + \alpha^t$, where $\alpha^t \in \Omega^1(g_E)$ for all $t \in (-\varepsilon, \varepsilon)$ and $\alpha^0 = 0$. The infinitesimal variation of the connection associated to D^t at $t = 0$ is

$$B := \frac{\mathrm{d}\alpha^t}{\mathrm{d}t}\bigg|_{t=0} \in \Omega(g_E).$$

Following [**1**], define an endomorphism $\mathcal{R}^D$ of $\Omega^1(g_E)$ by

$$\mathcal{R}^D(\varphi)(X) := \sum_{i=1}^{n} [R^D(e_i, X), \varphi(e_i)]$$

for $\varphi \in \Omega(g_E)$ and $X \in \Gamma(TM)$, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on (M,g). Then we obtain the following result.

Theorem 4.1. *Let (M,g) be an n-dimensional compact Riemannian manifold, G a compact Lie group and E a G-vector bundle over M. Let D be an f-Yang–Mills connection on E. Then the second variation of the functional $\mathcal{YM}_f$ is given by*

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathcal{YM}_f(D^t) &= \int_M f''(\tfrac{1}{2}\|R^D\|^2)\langle d^D B, R^D\rangle^2 \vartheta_g \\ &\quad + \int_M f'(\tfrac{1}{2}\|R^D\|^2)(\langle d^D B, d^D B\rangle + \langle B, \mathcal{R}^D(B)\rangle)\vartheta_g \\ &= \int_M \langle B, \mathcal{S}^D(B)\rangle \vartheta_g,\end{aligned}$$

where $\mathcal{S}^D$ is a differential operator acting on $\Omega(g_E)$ defined by

$$\mathcal{S}^D(B) = \delta^D(f''(\tfrac{1}{2}\|R^D\|^2)\langle d^D B, R^D\rangle^2) + \delta^D(f'(\tfrac{1}{2}\|R^D\|^2) d^D B) + f'(\tfrac{1}{2}\|R^D\|^2)\mathcal{R}^D(B).$$

Proof. As $R^{D^t} = R^D + d^D\alpha^t + \frac{1}{2}[\alpha^t \wedge \alpha^t]$ and $\alpha^0 = 0$, we obtain that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} (\tfrac{1}{2}\|R^{D^t}\|^2) = \langle d^D C + [B,B], R^D\rangle + \langle d^D B, d^D B\rangle,$$

where $C := \mathrm{d}^2/\mathrm{d}t^2|_{t=0}\alpha^t$. Thus, we obtain

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} YM_f(D^t) \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_M \tfrac{1}{2} f'(\tfrac{1}{2}\|R^{D^t}\|^2) \frac{\mathrm{d}}{\mathrm{d}t}\|R^{D^t}\|^2 \vartheta_g \\ &\quad = \frac{1}{4}\int_M f''(\tfrac{1}{2}\|R^D\|^2)\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\|R^{D^t}\|^2\bigg)^2 \vartheta_g + \frac{1}{2}\int_M f'(\tfrac{1}{2}\|R^D\|^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0}\|R^{D^t}\|^2 \vartheta_g\end{aligned}$$

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$$= \int_M f''(\tfrac{1}{2}\|R^D\|^2)\langle d^D B, R^D\rangle^2 \vartheta_g + \int_M f'(\tfrac{1}{2}\|R^D\|^2)(\langle d^D C + [B,B], R^D\rangle + \langle d^D B, d^D B\rangle)\vartheta_g.$$

On the other hand, since D is an f-Yang–Mills connection, we have

$$\int_M f'(\tfrac{1}{2}\|R^D\|^2)\langle d^D C, R^D\rangle\vartheta_g = \int_M \langle C, \delta^D(f'(\tfrac{1}{2}\|R^D\|^2)R^D)\rangle\vartheta_g = 0.$$

Finally, one can prove that

$$\langle [B\wedge B], R^D\rangle = \langle B, \mathcal{R}^D(B)\rangle.$$

Indeed,

$$\begin{aligned}
\langle [B\wedge B], R^D\rangle &= \sum_{i<j}\langle [B\wedge B](e_i,e_j), R^D(e_i,e_j)\rangle \\
&= \sum_{i<j}\langle [B(e_i),B(e_j)] - [B(e_j),B(e_i)], R^D(e_i,e_j)\rangle \\
&= 2\sum_{i<j}\langle [B(e_i),B(e_j)], R^D(e_i,e_j)\rangle \\
&= \sum_{i,j=1}^{n}\langle B(e_i), [B(e_j), R^D(e_i,e_j)]\rangle \\
&= \sum_{i=1}^{n}\langle B(e_i), \mathcal{R}^D(e_i)\rangle \\
&= \langle B, \mathcal{R}^D(B)\rangle.
\end{aligned}$$

and thus we obtain the second variation formula. □

The index, nullity and stability of an f-Yang–Mills connection D can be defined in the same way as in the case of the Yang–Mills connection [**1**].

Corollary 4.2. *Let D be an f-Yang–Mills connection for which $\|R^D\|$ is constant and such that $f'' = f'$. Then the stability of a Yang–Mills connection implies the stability of an f-Yang–Mills connection.*

Example 4.3. Let $f(t) = \exp t$ and suppose that D is a G-connection such that $\|R^D\|$ is constant. Then if D is a stable Yang–Mills connection, D is a stable exponential connection.

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