J. Austral. Math. Soc. 19 (Series A) (1975), 290–292.

ON A CLASS OF FINITE GROUPS

J. W. WAMSLEY

(Received 27 November 1972, revised 10 April 1973)

Communicated by G. E. Wall

1.

Let G be a finite P-group. Denote $\dim H^1(G, Z_p)$ by $d(G)$ and $\dim H^2(G, Z_p)$ by $r(G)$, then $d(G)$ is the minimal number of generators of G and G has a presentation

$$G = F/R = \{x_1, \cdots, x_{d(G)} \mid R_1, \cdots, R_m\},$$

where F is free on $x_1, \cdots, x_{d(G)}$ and R is the normal closure in F of $R_1, \cdots, R_m$. We have always that $m \geqq r(G) = d(R/[F,R])$ and we say that G belongs to a class, $\mathscr{G}p$, of the finite p-groups if $m = r(G)$. It is well known (see for example Johnson and Wamsley (1970)) that if G and H are finite p-groups then $r(G \times H) = r(G) + r(H) + d(G)d(H)$ and hence $G,\ H \in \mathscr{G}p$ implies $G \times H \in \mathscr{G}p$, also it is shown in Wamsley (1972) that if G is any finite p-group then there exists an $H \in \mathscr{G}p$ such that $G \times H$ belongs to $\mathscr{G}p$. Let $G^1 = G$ and $G^k = G^{k-1} \times G$ then we show in this note that if G is any finite p-group, there exists an integer $n(G)$, such that $G^k \in \mathscr{G}p$ for alal $k \geqq n(G)$.

2.

Let G be a finite p-group of nilpotency class c, then G has a presentation $G = F/R = \{x_1, \cdots, x_{d(G)} \mid R_1, \cdots, R_{r(G)},\ S_1, \cdots, S_t\}$ where $S_1, \cdots, S_t$ are commutators of weight $c + 2$. Define $b(G)$ to be the minimal t such that G has a presentation of the above form, then $G \in \omega p$ if and only if $b(G) = 0$.

Lemma. *Suppose H and G are finite p-groups then*

$$b(G \times H) \leqq \max(b(G) - r(H), 0) + \max(b(H) - r(G), 0).$$

Proof.

$$G = \{x_1, \cdots, x_{d(G)} \mid R_1, \cdots, R_{r(G)},\ S_j,\ 1 \leqq i \leqq d(G),\ 1 \leqq j \leqq b(G)\},$$

$$H = \{x'_1, \cdots, x'_{d(H)} \mid R'_1, \cdots, R'_{r(H)},\ S'_j,\ 1 \leqq i \leqq d(H),\ 1 \leqq j \leqq b(H)\}.$$

Let K be presented on generators $x_1, \cdots, x_{d(G)}$, $x'_1, \cdots, x'_{d(H)}$ with relators, $[x_i, x'_j]$, $R_m S'^{-1}_m$, $R'_n S_n^{-1}$ S'_u/S_v, where $1 \leqq i \leqq d(G)$, $1 \leqq j \leqq d(H)$, $1 \leqq m \leqq r(G)$, $1 \leqq n \leqq r(H)$, $r(G)+1 \leqq u \leqq b(H)$, $r(H)+1 \leqq v \leqq b(G)$. Then S_n is in the centre of K and hence K is of class $c+1$ and therefore class c. Therefore $K = G \times H$ and $b(K) \leqq \max(b(G)-r(H),0)+\max(b(H)-r(G),0)$.

We have inductively that

$$d(G^k) = kd(G) \text{ and } r(G^k) = kr(G) + (k(k-1)/2)d(G)^2.$$

Also the lemma states that $b(G^2) \leqq 2b(G)$ and hence $b(G^{2^k}) \leqq 2^k b(G)$. Choose a k such that $b(G) \leqq 2^{k-3}d(G)^2$ and consider $G^{2^k} \times G^{2^k}$. We have,

$$b(G^{2^{k+1}}) \leqq 2\max(b(G^{2^k}) - r(G^{2^k}), 0),$$

and since

$$b(G^{2^k}) \leqq 2^k b(G) \leqq 2^{2k-3}d(G)^2 \leqq r(G^{2^k}),$$

then $b(G^{2^{k+1}}) = 0$.

Let α be such that $b(G^\alpha) = 0$ where $\alpha \geqq 2^{k+1}$ then we will show that $b(G^{\alpha+1}) = 0$. We have $r(G^\alpha) \geqq 2^{2k-3}d(G)^2$ and $b(G) \leqq 2^{k-3}d(G)^2$ whence by the lemma

$$\begin{aligned} b(G^{\alpha+1}) &\leqq \max(-r(G),0) + \max(b(G) - r(G^\alpha), 0) \\ &\leqq \max(2^{k-3}d(G)^2 - 2^{2k-3}d(G)^2, 0) \\ &\leqq 0, \text{ and we have proved the following:} \end{aligned}$$

THEOREM. *Let G be a finite p-group. Then there exists an integer $n(G) > 0$ such that $G^k \in \mathscr{G}p$ for all $k \geqq n(G)$.*

References

D. L. Johnson and J. W. Wamsley (1970), 'Minimal relations for certain finite p-groups,' *Israel. J. Math.* **8**, 349–356.

J. W. Wamsley (1972), 'On a class of groups of prime-power order,' *Israel J. Math.* **11**, 297–298.

School of Mathematics
Flinders University
South Australia

https://doi.org/10.1017/S1446788700031475 Published online by Cambridge University Press