Can. J. Math. Vol. **46** (3), 1994 pp. 648–661

CONJUGACY CLASSES IN ALGEBRAIC MONOIDS II

MOHAN S. PUTCHA

ABSTRACT. Let M be a connected linear algebraic monoid with zero and a reductive unit group. We show that there exist reductive groups $G_1, \ldots, G_t$, each with an automorphism, such that the conjugacy classes of M are in a natural bijective correspondence with the twisted conjugacy classes of G_i, $i = 1, \ldots, t$.

Introduction. The objects of study in this paper are connected linear algebraic monoids M with zero. This means by definition that the underlying set of M is an irreducible affine variety and that the product map is a morphism (*i.e.* a polynomial map). We will assume further that the unit group G is a reductive group. In an earlier paper [6], the author found affine subsets $M_1, \ldots, M_k$, reductive groups $G_1, \ldots, G_k$ with respective automorphisms $\sigma_1, \ldots, \sigma_k$, and surjective morphisms $\theta_i: M_i \longrightarrow G_i$ such that: (1) Every element of M is conjugate to an element of some M_i, and (2) If $a, b \in M_i$, then a is conjugate to b in M if and only if there exists $x \in G_i$ such that $x\theta_i(a)\,\sigma_i(x)^{-1} = \theta_i(b)$. However it can happen that an element in M_i is conjugate to an element in M_j with $i \neq j$. We were not at that time able to handle this situation. Indeed the problem has baffled us since then. Finally we are able to give a complete solution. We show that in the above situation, every element of M_i is conjugate to an element of M_j, and every element of M_j is conjugate to an element of M_i. We also find necessary and sufficient conditons within the Weyl group or the Renner monoid, for this to happen. As an application we show that if $e = e^2 \in M$ and $a, b \in eMe$, then a is conjugate to b in M if and only if a is conjugate to b in eMe.

1. **Preliminaries.** Throughout this paper $\mathbb{Z}^+$ will denote the set of all positive integers. Let G be a connected linear algebraic group defined over an algebraically closed field. The *radical* $R(G)$ is the maximal closed connected normal solvable subgroup of G and the *unipotent radical* $R_u(G)$ is the group of unipotent elements of $R(G)$. We will assume that G is a *reductive group*, *i.e.* $R_u(G) = 1$. Then $R(G) \subseteq C(G)$, the center of G. Moreover $G = R(G)G_0$ where $G_0 = (G, G)$ is a semisimple group, *i.e.* $R(G_0) = 1$. Also G_0 is a product of simple closed normal subgroups of G. We refer to [1], [2] for details. If σ is an automorphism of G, then we say that $a, b \in G$ are *σ-conjugate* if $b = xa\sigma(x)^{-1}$ for some $x \in G$.

Research partially supported by NSF Grant DMS-9200077.

Received by the editors November 16, 1992.

AMS subject classification: 20G99, 20M10.

Fix a pair of opposite Borel subgroups B, B^- of G so that $T = B \cap B^-$ is a maximal torus. Let $W = W(G) = N_G(T)/T$ denote the *Weyl group* of G. Let S denote the fundamental generating set of reflections of W. Then the following axioms of Tits are valid [2; Section 29.1]:

$$\text{(T1)} \qquad \theta B\sigma \subseteq B\sigma B \cup B\theta\sigma B \quad \text{for all } \sigma \in W,\ \theta \in S$$

$$\text{(T2)} \qquad \theta B\theta \neq B \quad \text{for all } \theta \in S.$$

For $I \subseteq S$, $P_I = BW_IB$ and $P_I^- = B^-W_IB^-$ are a pair of *standard opposite parabolic subgroups*, where W_I is the subgroup of W generated by I. $L_I = P_I \cap P_I^-$ is a reductive group, called a *standard Levi subgroup* of G. We have, $W(P_I) = W(P_I^-) = W(L_I) = W_I$. Subgroups of G containing a Borel subgroup, *i.e.* a conjugate of B, are called *parabolic subgroups*. If P is a parabolic subgroup of G containing T, then there is a unique *opposite parabolic subgroup* P^- of G containing T such that $L = P \cap P^-$ is a reductive group. Then L is a *Levi factor* of P and $P = LR_u(P)$, $L \cap R_u(P) = 1$, where $R_u(P)$ is the unipotent radical of P. This is called a *Levi decomposition* of P. If B_1, B_2 are Borel subgroups of G containing T, then G is expressible as the following disjoint union:

$$G = \bigsqcup_{\sigma \in W} B_1\sigma B_2.$$

This is called the *Bruhat decomposition* of G.

LEMMA 1.1. *Let P_1, P_2 be parabolic subgroups of G with Levi decompositions $P_1 = L_1U_1$, $P_2 = L_2U_2$ such that $T \subseteq L_1 \cap L_2$. Suppose $a \in U_1$, $b \in L_1$, $\sigma \in W$ such that $ab \in P_2\sigma$. Then $a \in P_2$.*

PROOF. Let $\sigma = nT$. Then $n \in P_2P_1$. There exist $\theta_1, \theta_2 \in W$, $I, J \subseteq S$, such that $P_1 = \theta_1^{-1}P_I\theta_1$ and $P_2 = \theta_2^{-1}P_J\theta_2$. Then

$$\begin{aligned} P_2P_1 &= \theta_2^{-1}(BW_JB\theta_2\theta_1^{-1}B)W_IB\theta_1 \\ &= \theta_2^{-1}BW_J\theta_2\theta_1^{-1}BW_IB\theta_1, \quad \text{by } (T1) \\ &= \theta_2^{-1}BW_J\theta_2\theta_1^{-1}W_IB\theta_1, \quad \text{by } (T1). \end{aligned}$$

Since $n \in P_2P_1$, we see by the Bruhat decomposition that $\theta_2\sigma\theta_1^{-1} \in W_J\theta_2\theta_1^{-1}W_I$. So

$$\sigma \in \theta_2^{-1}W_J\theta_2 \cdot \theta_1^{-1}W_I\theta_1 = W(L_2) \cdot W(L_1).$$

Hence there exists $m \in N_G(T) \cap L_1$ such that $abm \in P_2$. Since $a \in U_1$ and $bm \in L_1$, we see by [6; Fact 1.3] that $a, bm \in P_2$. ■

LEMMA 1.2. *Let $I \subseteq S$, $L = L_I$. Let $\sigma_1, \ldots, \sigma_t, \theta_1, \ldots, \theta_t \in W$ such that $\bigcap_{i=1}^t \sigma_iL\theta_i \neq \emptyset$. Then $\bigcap_{i=1}^t \sigma_iW_I\theta_i \neq \emptyset$.*

PROOF. Let $B_i = \sigma_i^{-1}B\sigma_i \cap L$, $B_i' = \theta_iB\theta_i^{-1} \cap L$, $i = 1, \ldots, t$. All of these are Borel subgroups of L containing T. By the Bruhat decomposition for L,

$$L = B_iW_IB_i' \subseteq \sigma_i^{-1}B\sigma_iW_I\theta_iB\theta_i^{-1}, \quad i = 1, \ldots, t.$$

Hence

$$\sigma_i L\theta_i \subseteq B\sigma_i W_I \theta_i B, \quad i = 1, \dots, t.$$

Thus

$$\emptyset \neq \bigcap_{i=1}^{t} \sigma_i L\theta_i \subseteq \bigcap_{i=1}^{t} B\sigma_i W_I \theta_i B.$$

By the Bruhat decomposition for G, $\bigcap_{i=1}^{t} \sigma_i W_I \theta_i \neq \emptyset$. ∎

Now for monoids. By a (linear) *algebraic monoid*, we mean a monoid M such that the underlying set is an affine variety and the product map is a morphism. The identity component of M will be denoted by M^c. We will use the same notation for an algebraic group. We will assume that M is connected (*i.e.* $M = M^c$) and that M has a zero. We will further assume that the unit group G is reductive. We call such a monoid a *reductive monoid*. Typically such monoids arise by taking lined Zariski closures of linear representations of reductive groups. We refer to [5] for the general theory of algebraic monoids. We will let $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$ denote the usual Green's relations on M. If $a, b \in M$, then $a\mathcal{R}b$ if $aM = bM$, $a\mathcal{L}b$ if $Ma = Mb$, $a\mathcal{J}b$ if $MaM = MbM$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. If $X \subseteq M$, then

$$E(X) = \{e \in X \mid e^2 = e\}$$

will denote the set of idempotents in X. If $e \in E(M)$, then by the author [3], [4],

$$C_G^r(e) = \{g \in G \mid ge = ege\}$$
$$C_G^l(e) = \{g \in G \mid eg = ege\}$$

are opposite parabolic subgroups of G with common Levi factor $C_G(e)$. We will let

$$\tilde{G}_e^r = R_u\big(C_G^r(e)\big), \quad \tilde{G}_e^l = R_u\big(C_G^l(e)\big)$$

denote the unipotent radicals of $C_G^r(e)$ and $C_G^l(e)$ respectively. Then

$$\tilde{G}_e^r e = \{e\}, \quad e\tilde{G}_e^l = \{e\}$$
$$C_G^r(e) = C_G(e) \cdot \tilde{G}_e^r, C_G^l(e) = C_G(e) \cdot \tilde{G}_e^l.$$

Let

$$\hat{G}_e = \{g \in G \mid ge = e = eg\} \triangleleft C_G(e), \quad G_e = \hat{G}_e^c.$$

By [6; Fact 1.1], [5; Corollary 4.34] we have,

$$C_G(e) = G_e \cdot C_G(G_e)$$
$$\hat{G}_e \subseteq G_e \cdot C\big(C_G(e)\big), \quad C_G(\hat{G}_e) = C_G(G_e).$$

By [6; Fact 1.3], we have,

LEMMA 1.3. *Let* $e, f \in E(\bar{T})$. *Then*

$$C^r_G(e) \cap C^l_G(f) = [\tilde{G}^l_f \cap C_G(e)][C_G(e,f)][\tilde{G}^r_e \cap C_G(f)][\tilde{G}^r_e \cap \tilde{G}^l_f].$$

For $e \in E(\bar{T})$, $\sigma = nT \in W$, let $e^\sigma = n^{-1}en$. This is clearly independent of the choice of n. Let

$$W(e) = W\big(C_G(e)\big) = C_W(e) = \{\sigma \in W \mid e^\sigma = e\}.$$

We also let

$$\begin{aligned} W_e &= \{\sigma \in W \mid f^\sigma = f \text{ for all } f \in E(\bar{T}) \text{ with } f \leq e\} \\ &= \{nT \mid n \in N_G(T) \cap G_e\} \cong W(G_e). \end{aligned}$$

Here $f \leq e$ means $ef = fe = f$. Note that T_e, rather than T, is a maximal torus of G_e. By [6; Facts 1.1, 1.2, 1.3, Lemma 1.6], we have

LEMMA 1.4. *Let* $e_1, \ldots, e_t \in E(\bar{T})$, $V = C_G(e_1, \ldots, e_t)$. *Then*

$$V = C_G(G_{e_1}, \ldots, G_{e_t}) \cdot V_{e_1} \cdots V_{e_t}$$
$$C_G(T_{e_1}, \ldots, T_{e_t}) = C_G(G_{e_1}, \ldots, G_{e_t}) \cdot T.$$

For $e_1, \ldots, e_t \in E(\bar{T})$, *we let*

$$W(e_1, \ldots, e_t) = W(e_1) \cap \cdots \cap W(e_t) = W\big(C_G(e_1, \ldots, e_t)\big).$$

By the author [3], the semigroup way of viewing the Borel subgroup B is via the *cross-section lattice*:

$$\Lambda = \Lambda(B) = \{e \in E(\bar{T}) \mid B \subseteq C^r_G(e)\}.$$

Then $|\Lambda \cap J| = 1$ for each $\mathcal{J}$-class ($= G \times G$ orbit) J and for all $e, f \in \Lambda$, $f \in MeM$ if and only if $e \geq f$.

The monoid analogue of the Weyl group $W(G)$ is the *Renner monoid*,

$$\mathrm{Ren}(M) = \overline{N_G(T)}/T.$$

$\mathrm{Ren}(M)$ is a finite fundamental inverse monoid with idempotent set $E(\bar{T})$ and unit group W. By Renner [7], M is the disjoint union:

$$M = \bigsqcup_{r \in \mathrm{Ren}(M)} BrB.$$

For more recent advances in this direction, we refer to Renner [9], where in particular an exciting new $\mathcal{H}$-cross-section submonoid O is found. This new monoid is related to the minimum length right and left coset representatives of W_I in W.

2. **Main section.** Let M be a reductive monoid with unit group G. Call two elements $a, b \in M$ *conjugate* if $b = a^x = x^{-1}ax$ for some $x \in G$. We are interested in the conjugacy classes in M. Renner [8] has shown that the conjugacy class of an element is closed if and only if the element lies in the closure of a torus. In general the conjugacy classes in M (as opposed to the full matrix monoid) can be very complicated. For example in general the number of conjugacy classes of nilpotent elements in M is infinite. None the less, major progress was made by the author [6]. The story begins with the following affine subset of M, for $e \in E(\bar{T})$, $\sigma \in W$:

$$M_{e,\sigma} = eC_G(e^\delta \mid \delta \in \langle\sigma\rangle)\sigma$$

where $\langle\sigma\rangle$ denotes the cyclic group generated by σ. In general $e\sigma = e\tau$ does not imply $M_{e,\sigma} = M_{e,\tau}$. See Example 2.2. Clearly

$$M^\pi_{e,\sigma} = \pi^{-1}M_{e,\sigma}\pi = M_{e,\sigma^\pi} \quad \text{for all } \pi \in W(e).$$

Now $V = C_G(e^\delta \mid \delta \in \langle\sigma\rangle)$ is a reductive group with a closed normal subgroup

$$V' = \prod_{\delta\in\langle\sigma\rangle} \hat{V}_{e^\delta}$$

where as usual $\hat{V}_f = \{x \in V \mid xf = fx = f\}$. Then $G_{e,\sigma} = V/V'$ is a reductive group and σ induces an automorphism $\bar{\sigma}$ of $G_{e,\sigma}$. Clearly there is a natural surjective morphism $\xi: M_{e,\sigma} \longrightarrow G_{e,\sigma}$ given by $\xi(exn) = xV'$ for $x \in V, \sigma = nT$. Following is the main result of [6].

THEOREM 2.1. *Every element of M is conjugate to an element of some $M_{e,\sigma}$, $e \in \Lambda$, $\sigma \in W$. If $a, b \in M_{e,\sigma}$, then a is conjugate to b in M if and only if a is conjugate to b by an element of V if and only if $\xi(a)$ and $\xi(b)$ are $\bar{\sigma}$-conjugate in $G_{e,\sigma}$.*

If $a \in M_{e,\sigma}$, $b \in M_{f,\theta}$, $e,f \in \Lambda$, and if a is conjugate to b in M, then clearly $e = f$. However it need not be that $\sigma = \theta$. So the main question left open in [6] was the consideration of the situation when $M_{e,\sigma}$ and $M_{e,\theta}$ have conjugate elements. Complicated by the fact that unequal $M_{e,\sigma}$'s can have non-empty intersection, the solution evaded us for five years. Finally we are able to give a complete solution. We begin by introducing a new closed subset $N_{e,\sigma}$ of $M_{e,\sigma}$ (see Lemma 1.4):

$$\begin{aligned} N_{e,\sigma} &= eC_G(T_{e^\delta} \mid \delta \in \langle\sigma\rangle)\sigma \\ &= eC_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)T\sigma \\ &= eC_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)\sigma. \end{aligned}$$

Clearly

$$N^\pi_{e,\sigma} = \pi^{-1}N_{e,\sigma}\pi = N_{e,\sigma^\pi} \text{ for all } \pi \in W(e).$$

Let $\pi \in W_e$. Then $\pi = mT$ for some $m \in G_e \cap N_G(T)$. Let $a \in N_{e,\sigma}$. Then $a = egn$ for some $g \in C_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)$, $n \in N_G(T)$ with $\sigma = nT$. Then for all $i \geq 0$, $n^ign^{-i} \in C_G(G_e)$ and hence is centralized by m. Thus we see by induction on i that

$$(mn)^ig(mn)^{-i} = mn^ign^{-i}m^{-1} = n^ign^{-i} \in C_G(G_e).$$

Hence $g \in C_G(G_{e^\delta} \mid \delta \in \langle \pi\sigma \rangle)$. So

$$egn = emgn = egmn \in N_{e,\pi\sigma}.$$

So $N_{e,\sigma} \subseteq N_{e,\pi\sigma}$. Similarly $N_{e,\pi\sigma} \subseteq N_{e,\sigma}$. Hence

$$N_{e,\sigma} = N_{e,\pi\sigma} \quad \text{for all } \pi \in W_e.$$

Thus $N_{e,\sigma}$ depends only on the element $e\sigma$ in Ren(M). For this reason we write $N_{e\sigma}$ for $N_{e,\sigma}$.

EXAMPLE 2.2. Let M denote the multiplicative monoid of all 5×5 matrices over an algebraically closed field. Let

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0. \end{bmatrix}$$

Then $M_{e,\sigma}$ consists of matrices of the form

$$\begin{bmatrix} 0 & 0 & a & b & 0 \\ 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad ad \neq bc.$$

On the other hand $e\sigma = e\theta$ and $M_{e,\theta} = N_{e\sigma} = N_{e\theta}$ consists of matrices of the form

$$\begin{bmatrix} 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad a \neq 0,\ b \neq 0.$$

THEOREM 2.3. *(i) If $r, s \in$ Ren(M) with $N_r \cap N_s \neq \emptyset$, then $N_r = N_s$.*

(ii) If $\theta \in W(e^\delta \mid \delta \in \langle \sigma \rangle)$, then $N_{e\theta\sigma} \subseteq M_{e,\sigma}$ and $N_{e\theta\sigma} = N^{\pi}_{e\sigma}$ for some $\pi \in W(e^\delta \mid \delta \in \langle \sigma \rangle)$.

(iii) Any element of $M_{e,\sigma}$ is conjugate to some element of $N_{e\sigma}$.

(iv) Any element of M is conjugate to an element of $N_{e\sigma}$ for some $e \in \Lambda$, $\sigma \in W$.

(v) The map $\xi : M_{e,\sigma} \longrightarrow G_{e,\sigma}$ remains surjective when restricted to $N_{e\sigma}$. Hence the conjugacy classes in $N_{e\sigma}$ are in a natural bijective correspondence with the $\bar{\sigma}$-conjugacy classes of $G_{e,\sigma}$.

PROOF. (i) Let $r = e\sigma$, $e \in E(\bar{T})$, $\sigma \in W$. Then $e\mathcal{R}s$ and hence $s = e\theta$ for some $\theta \in W$. Let $a \in N_r \cap N_s$. Then there exist $g \in C_G(G_{e^\delta} \mid \delta \in \langle \sigma \rangle)$, $h \in C_G(G_{e^\delta} \mid \delta \in \langle \theta \rangle)$, $m, n \in N_G(T)$, such that $\sigma = nT$, $\theta = mT$ and $a = egn = ehm$. Then $a\mathcal{L}n^{-1}en$ and $a\mathcal{L}m^{-1}em$. Hence $n^{-1}en = m^{-1}em$. So $nm^{-1} \in C_G(e)$. Thus $gn = zhm$ for some $z \in \hat{G}_e$.

Let $x \in C_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)$. Since n normalizes $C_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)$, so does $gn = zhm$. Hence for all $i \geq 0$, $(zhm)^i x (zhm)^{-i} \in C_G(G_e)$. Since $z \in \hat{G}_e$ and $C_G(G_e) = C_G(\hat{G}_e)$, we see by induction that for all $i > 0$,

$$(hm)^i x (hm)^{-i} = (zhm)^i x (zhm)^{-i} \in C_G(G_e).$$

Now

$$(hm)^i = h(mhm^{-1})(m^2hm^{-2})\cdots(m^{i-1}hm^{1-i})m^i$$

and $m^jhm^{-j} \in C_G(G_e)$ for all $j \geq 0$. It follows that $m^ixm^{-i} \in C_G(G_e)$ for all $i \geq 0$. Hence $x \in C_G(G_{e^\delta} \mid \delta \in \langle\theta\rangle)$. Thus $C_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle) \subseteq C_G(G_{e^\delta} \mid \delta \in \langle\theta\rangle)$. So

$$exn = egn \cdot (n^{-1}g^{-1}xn) = ehm \cdot (n^{-1}g^{-1}xn) = eh \cdot m(n^{-1}g^{-1}xn)m^{-1}.m$$

and $m(n^{-1}g^{-1}xn)m^{-1} \in C_G(e^\delta \mid \delta \in \langle\theta\rangle)$. Thus $exn \in N_{e\theta}$. So $N_{e\sigma} \subseteq N_{e\theta}$. Similarly $N_{e\theta} \subseteq N_{e\sigma}$ and $N_{e\sigma} = N_{e\theta}$.

(ii) By Lemma 1.4, $\theta = pT$, $p = p_0 \cdots p_s q$ with $p_i \in V_{e^{\sigma^i}} \cap N_G(T)$, where $V = C_G(e^\delta \mid \delta \in \langle\sigma\rangle)$ and $q \in V_0^c \cap N_G(T)$, where $V_0 = C_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)$. Let $\theta_i = p_iT \in W_{e^{\sigma^i}} \cap W(e^\delta \mid \delta \in \langle\sigma\rangle)$, $\theta' = qT$. Then θ' commutes with each element of $W_{e^{\sigma^j}}$ for all j. By (i), $N_{e\sigma} = N_{e\theta'\sigma}$. Now $\theta_1 \cdots \theta_s \in W(e^\delta \mid \delta \in \langle\sigma\rangle)$,

$$\begin{aligned}(\theta_1\cdots\theta_s)^{-1}(e\theta_0\cdots\theta_s\theta'\sigma)(\theta_1\cdots\theta_s) &= (\theta_1\cdots\theta_s)^{-1}(e\theta_1\cdots\theta_s\theta'\sigma)(\theta_1\cdots\theta_s)\\ &= (\theta_1\cdots\theta_s)^{-1}(\theta_1\cdots\theta_s)e\theta'\sigma\theta_1\cdots\theta_s\\ &= e\theta'\sigma\theta_1\cdots\theta_s\\ &= e\theta_1'\cdots\theta_s'\theta'\sigma\end{aligned}$$

where $\theta_i' = \sigma\theta_i\sigma^{-1} \in W_{e^{\sigma^{i-1}}} \cap W(e^\delta \mid \delta \in \langle\sigma\rangle)$, $i = 1,\ldots,s$. Inductively we see that $\pi(e\theta\sigma)\pi^{-1} = e\theta'\sigma$ for some $\pi \in W(e^\delta \mid \delta \in \langle\sigma\rangle)$. Hence

$$N_{e\theta\sigma} = N^\pi_{e\theta'\sigma} = N^\pi_{e\sigma} \subseteq M^\pi_{e,\sigma} = M_{e,\sigma}$$

(v) follows from Lemma 1.4 and then (iii), (iv) follow from Theorem 2.1. ∎

Let $a \in M_{e,\sigma}$, $b \in M_{e,\theta}$, $e^\sigma = f_1$, $e^\theta = f_2$. Then $e\mathcal{R}a\mathcal{L}f_1$, $e\mathcal{R}b\mathcal{L}f_2$.

LEMMA 2.4. *Let $e, f_1, f_2 \in E(\bar{T})$, $a, b \in M$ such that $e\mathcal{R}a\mathcal{L}f_1$, $e\mathcal{R}b\mathcal{L}f_2$. If a and b are conjugate in M, then there exists $\pi \in W(e)$ such that $f_1^\pi = f_2$.*

PROOF. There exists $x \in G$ such that $xax^{-1} = b$. Then

$$xex^{-1}\mathcal{R}xax^{-1} = b\mathcal{R}e.$$

So $x \in C_G^r(e)$. Now

$$xf_1x^{-1}\mathcal{L}xax^{-1} = b\mathcal{L}f_2.$$

Hence by [5; Chapter 6], f_1 and f_2 are conjugate in $\overline{C_G^r(e)}$. Hence there exists $m \in N_G(T) \cap C_G^r(e) = N_G(T) \cap C_G(e)$ such that $m^{-1}f_1m = f_2$. So $\pi = mT \in W(e)$ and $f_1^\pi = f_2$. ∎

In preparation for our main theorem, we prove the following technical lemma.

LEMMA 2.5. *Let $e,f \in E(\bar{T})$. Define a relation $\equiv$ on G as: $g_1 \equiv g_2$ if there exist $x \in C_G(e,f)$, $a \in \tilde{G}^l_f \cap C_G(e)$, $b \in \tilde{G}^r_e \cap C_G(f)$ such that $axg_1 = g_2xb$. Then*

(i) $\equiv$ is an equivalence relation on G.

(ii) If $\sigma = nT \in W$, $e^\sigma = f$, $k \in \mathbb{Z}^+$, $x, y \in C_G(e^{\sigma^j} \mid j = 0, \dots, k-1)$, $x \in \tilde{G}^l_{e^{\sigma^k}}$, then $xyn \equiv yn$.

(iii) Let $\theta = mT \in W$, $e^\theta = f$, $u \in C_G(G_{e^\delta} \mid \delta \in \langle\theta\rangle)$, $z \in G_e$. Then there exists $\sigma = nT \in W$, $v \in C_G(e^\delta \mid \delta \in \langle\sigma\rangle)$, such that $zum \equiv vn$ and $\theta = \pi_0 \cdots \pi_t \sigma$ for some $\pi_i \in W_{e^{\sigma^i}} \cap W(e^{\sigma^j} \mid 0 \leq j \leq i)$, $i = 0, \dots, t$.

PROOF. (i) Suppose $g_1, g_2 \in G$ with $g_1 \equiv g_2$. Then there exist $a \in \tilde{G}^l_f \cap C_G(e)$, $x \in C_G(e,f)$, $b \in \tilde{G}^r_e \cap C_G(f)$ such that $axg_1 = g_2xb$. Then

$$(x^{-1}a^{-1}x)x^{-1}g_2 = g_1x^{-1}(xb^{-1}x^{-1})$$

with $x^{-1}a^{-1}x \in \tilde{G}^l_f \cap C_G(e)$, $x \in C_G(e,f)$, $xb^{-1}x^{-1} \in \tilde{G}^r_e \cap C_G(f)$. Thus $\equiv$ is symmetric. Clearly $\equiv$ is reflexive. Next let $g_1, g_2, g_3 \in G$ such that $g_1 \equiv g_2 \equiv g_3$. Then there exist $a, c \in \tilde{G}^l_f \cap C_G(e)$, $x, y \in C_G(e,f)$, $b, d \in \tilde{G}^r_e \cap C_G(f)$ such that

$$axg_1 = g_2xb, \quad cyg_2 = g_3yd.$$

Then

$$c(yay^{-1})(yx)g_1 = g_3(yx)(x^{-1}dx)b$$

with $c(yay^{-1}) \in \tilde{G}^l_f \cap C_G(e)$, $yx \in C_G(e,f)$, $(x^{-1}dx)b \in \tilde{G}^r_e \cap C_G(f)$. Thus $g_1 \equiv g_3$ and $\equiv$ is an equivalence relation on G.

(ii) We prove by induction on k. If $k = 1$, then $x \in C_G(e) \cap \tilde{G}^l_f$ and the result is clear. So let $k > 1$. Then $x \in C_G(e,f)$, $nxn^{-1} \in C_G(e^{\sigma^j} \mid j = 0, \dots, k-2) \cap \tilde{G}^l_{e^{\sigma^{k-1}}}$. Hence $y(nxn^{-1})y^{-1} \in C_G(e^{\sigma^j} \mid j = 0, \dots, k-2) \cap \tilde{G}^l_{e^{\sigma^{k-1}}}$. Thus by the induction hypothesis,

$$xyn \equiv ynx = y(nxn^{-1})y^{-1}.yn \equiv yn.$$

(iii) Suppose inductively that

$$y \in H = \prod_{i=0}^{k} [C_G(e^{\theta^j} \mid j = 0, \dots, k) \cap G_{e^{\theta^i}}].$$

Then by [6; Facts 1.1, 1.2, 1.3], H is a reductive group and $P \cap H$ is a parabolic subgroup of H for all parabolic subgroups P of G with $T \subseteq P$. Further, $T_o = T_e \cdots T_{e^{\theta^k}}$ is a maximal torus of H. Now $P_1 = C^l_G(e^{\theta^{k+1}})$ and $P_2 = C^r_G(\theta e \theta^{-1})$ are parabolic subgroups of G containing T. Hence $P_1 \cap H$ and $P_2 \cap H$ are parabolic subgroups of H containing T_0. By the Bruhat decomposition for H, there exists $p \in N_G(T) \cap H$ such that $y \in (P_1 \cap H)p(P_2 \cap H)$. So there exist $y_1 \in P_1 \cap H$, $y_2 \in P_2 \cap H$ such that $y = y_1py_2$. By [6; Fact 1.3], $y_2 = y_3y_4$ for some $y_3 \in H \cap C_G(\theta e \theta^{-1})$, $y_4 \in H \cap \tilde{G}^r_{\theta e \theta^{-1}}$. So by [6; Facts 1.1, 1.2, 1.3],

$$m^{-1}y_3m \in \prod_{i=1}^{k+1} [C_G(e^{\theta^j} \mid j = 0, \dots, k+1) \cap G_{e^{\theta^i}}]$$

$$m^{-1}y_4m \in C_G(f) \cap \tilde{G}^r_e.$$

Hence

$$\begin{aligned} yum &= y_1py_2um \\ &= y_1upy_2m \\ &= y_1upm(m^{-1}y_3m)(m^{-1}y_4m) \\ &\equiv (m^{-1}y_3m)y_1upm. \end{aligned}$$

Now $y_1 = y_5y_6$ for some $y_5 \in H \cap \tilde{G}^l_{e^{\theta^{k+1}}}$, $y_6 \in H \cap C_G(e^{\theta^{k+1}})$. Hence by [6; Facts 1.1, 1.2, 1.3], $(m^{-1}y_3m)y_1 = y_7y_8$, where

$$y_7 = (m^{-1}y_3m)y_5(m^{-1}y_3m)^{-1} \in \tilde{G}^l_{e^{\theta^{k+1}}}$$

$$y_8 = (m^{-1}y_3m)y_6 \in \prod_{i=0}^{k+1}[C_G(e^{\theta^j} \mid j = 0,\dots,k+1) \cap G_{e^{\theta^i}}].$$

Let $\sigma = pm$. We see by induction that for all $i \geq 0$,

$$(pm)^{-i}u(pm)^i = m^{-i}um^i \in C_G(G_e).$$

Hence $u \in C_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)$. We claim that $e^{\sigma^j} = e^{\theta^j}$ for $j = 0,\dots,k+1$. We prove this by induction. For $j = 0$, this is obvious. So assume $e^{\theta^j} = e^{\sigma^j}$, $j \leq k$. Now $\pi = pT \in C_W(e^{\theta^j})$ and $\sigma = \pi\theta$. So

$$e^{\sigma^{j+1}} = (e^{\theta^j})^\sigma = (e^{\theta^j})^{\pi\theta} = (e^{\theta^j})^\theta = e^{\theta^{j+1}}.$$

Now by (ii),

$$\begin{aligned} yum &\equiv (m^{-1}y_3m)y_1upm \\ &= y_7(y_8u)pm \\ &\equiv y_8upm. \end{aligned}$$

Now $\pi = \pi_0 \cdots \pi_k$, with $\pi_i \in W_{e^{\sigma^i}} \cap W(e,\dots,e^{\sigma^k})$, $i = 0,\dots,k$.

Thus starting with $y = z$ and $k = 0$, and proceeding inductively to $k = |W|$, we find $\sigma = nT \in W$, $y \in C_G(e^\delta \mid \delta \in \langle\sigma\rangle$ such that $u \in C_G(G_{e^\delta} \mid \delta \in \langle\sigma\rangle)$, $\theta = \pi_0 \cdots \pi_t\sigma$ with $\pi_i \in W_{e^{\sigma^i}} \cap W(e,\dots,e^{\sigma^i})$, $i = 0,\dots,t$, and $zum \equiv yun$. This completes the proof. ∎

We are now ready to prove our main theorem.

THEOREM 2.6. *The following conditions are equivalent for $e \in \Lambda$ and $\sigma, \theta \in W$:*

(i) There exists an element of $M_{e,\sigma}$ that is conjugate to an element of $M_{e,\theta}$.

(ii) Every element of $M_{e,\sigma}$ is conjugate to an element of $M_{e,\theta}$ and every element of $M_{e,\theta}$ is conjugate to an element of $M_{e,\sigma}$.

(iii) There exists $\gamma \in W$ with $\theta = \pi_0 \cdots \pi_t\gamma$ and $\pi_i \in W_{e^{\gamma^i}} \cap W(e,\dots,e^{\gamma^i})$, $i = 0,\dots,t$, such that

$$\bigcap_{i\geq 0} \gamma^i W(e)\sigma^{-i} \neq \emptyset.$$

(iv) There exists $\gamma \in W$ with $e\theta$ conjugate to $e\gamma$ in Ren(M)*, such that*

$$\bigcap_{i\geq 0} \gamma^i W(e)\sigma^{-i} \neq \emptyset.$$

(v) $N^{\pi}_{e\sigma} = N_{e\theta}$ *for some* $\pi \in W(e)$.

PROOF. (i) $\Rightarrow$ (iii) Let $f = e^{\sigma}$. By Lemma 2.4 there exists $\eta \in W(e)$ such that $f^{\eta} = e^{\theta}$. We can replace θ by $\eta\theta\eta^{-1}$. Then having found the appropriate $\pi_0, \ldots, \pi_t, \gamma$ with respect to $\eta\theta\eta^{-1}$, we can replace them by $\eta^{-1}\pi_0\eta, \ldots, \eta^{-1}\pi_t\eta, \eta^{-1}\gamma\eta$, respectively. Thus without loss of generality, we can assume that $e^{\theta} = f$.

There exists $A_1 \in M_{e,\sigma}$ that is conjugate to some $A_2 \in M_{e,\theta}$. By Theorem 2.3, we can assume that $A_2 \in N_{e\theta}$. So there exist $u \in C_G(e^{\delta} \mid \delta \in \langle\sigma\rangle)$, $v \in C_G(G_{e^{\delta}} \mid \delta \in \langle\theta\rangle)$ such that $A_1 = eun$, $A_2 = evm$, $\sigma = nT$, $\theta = mT$. There exists $X \in G$ such that $XA_1X^{-1} = A_2$. Since $A_1, A_2, \in eMf$, $X \in C^r_G(e) \cap C^l_G(f)$. By Lemma 1.3,

$$C^r_G(e) \cap C^l_G(f) = [C_G(e) \cap \tilde{G}^l_f][C_G(e,f)][C_G(f) \cap \tilde{G}^r_e][\tilde{G}^r_e \cap \tilde{G}^l_f].$$

Since $A_1, A_2 \in eMf$, we can assume without loss of generality that

$$X \in [C_G(e) \cap \tilde{G}^l_f][C_G(e,f)][C_G(f) \cap \tilde{G}^r_e].$$

So there exist $a \in C_G(e) \cap \tilde{G}^l_f$, $x \in C_G(e,f)$, $b \in C_G(f) \cap \tilde{G}^r_e$ such that $X = axb$. From $XA_1 = A_2X$, we get

$$eaxun = evmxb.$$

Since $e^{\sigma} = e^{\theta}$, $nm^{-1} \in C_G(e)$. Hence

$$(axun)(vmxb)^{-1} = axu(nb^{-1}x^{-1}n^{-1})nm^{-1}v \in C_G(e).$$

Hence

$$axun = zvmxb \tag{1}$$

for some $z \in \hat{G}_e$. Since $\hat{G}_e \subseteq C\big(C_G(e)\big) \cdot G_e$, we can assume without loss of generality (by changing u appropriately), that $z \in G_e$. In the notation of Lemma 2.5, $un \equiv zvm$. By Lemma 2.5 (iii), we can change θ, m, v appropriately, so that $un \equiv vm$ with $v \in C_G(e^{\delta} \mid \delta \in \langle\theta\rangle)$. Let us therefore assume that

$$axun = vmxb. \tag{2}$$

Note that now $A_2 \in M_{e,\theta}$ and not $N_{e\theta}$. By (2),

$$\begin{aligned} ax &= vmxbn^{-1}u^{-1} \\ &= vmxbm^{-1}(mn^{-1}u^{-1}nm^{-1})mn^{-1} \in C^r_G(\theta e\theta^{-1})\theta\sigma^{-1}. \end{aligned}$$

Since $a \in \tilde{G}^l_f$ and $x \in C_G(f)$, we see by Lemma 1.1 that $a, xnm^{-1} \in C^r_G(\theta e\theta^{-1})$. By [6; Fact 1.3], we can factor

$$\begin{aligned} a = c_1a_1 \quad &\text{for some } c_1 \in \tilde{G}^r_{\theta e\theta^{-1}} \text{ and} \\ &a_1 \in C_G(e, \theta e\theta^{-1}) \cap \tilde{G}^l_f. \end{aligned} \tag{3}$$

Similarly we can factor

$$(4)\qquad x = y_1x_1 \quad \text{for some } y_1 \in \tilde{G}^r_{\theta e\theta^{-1}},\ x_1 \in C_G(\theta e\theta^{-1})mn^{-1} = \theta C_G(e)\sigma^{-1}.$$

Since $y_1(x_1nm^{-1}) = xnm^{-1} \in C_G(e,f)\sigma\theta^{-1}$, we see by Lemma 1.1 that $y_1 \in C_G(e,f)$. Hence

$$(5)\qquad x_1 \in C_G(e,f).$$

By (2)

$$c_1a_1y_1x_1un = vmy_1x_1b.$$

Hence

$$(6)\qquad wa_1x_1un = vmy_1x_1$$

where by (3), (4),

$$\begin{aligned} w &= vmy_1x_1n^{-1}u^{-1}x_1^{-1}a_1^{-1} \\ &= c_1a_1y_1x_1unb^{-1}n^{-1}u^{-1}x_1^{-1}a_1^{-1} \\ &= c_1 \cdot a_1 \cdot y_1 \cdot (x_1nm^{-1})[m(n^{-1}un)b^{-1}(n^{-1}un)^{-1}m^{-1}](x_1nm^{-1})^{-1} \cdot a_1^{-1} \in \tilde{G}^r_{\theta e\theta^{-1}}. \end{aligned}$$

Suppose inductively that for $k \in \mathbb{Z}^+$,

$$(7)\qquad x = y_1 \cdots y_kx_k$$

where

$$(8)\qquad \begin{gathered} y_i \in \tilde{G}^r_{\theta^i e\theta^{-i}}, \quad i = 1, \ldots, k \\ x_k \in \bigcap_{j=-1}^{k} \theta^j C_G(e)\sigma^{-j}. \end{gathered}$$

Further assume that there exist

$$(9)\qquad \begin{gathered} w_i \in C_G(\theta^j e\theta^{-j} \mid i+1 \leq j \leq k) \cap \tilde{G}^r_{\theta^i e\theta^{-i}}, \quad i = 1, \ldots, k \\ a_k \in C_G(\theta^j e\theta^{-j} \mid j = 0, \ldots, k) \cap \tilde{G}^l_f \end{gathered}$$

such that

$$(10)\qquad w_k \cdots w_1a_kx_kun = vmy_kx_k.$$

By (3)–(6) we see that (7)–(10) are valid for $k = 1$, since

$$C_G(f) = \theta^{-1}C_G(e)\theta = \theta^{-1}C_G(e)\sigma\theta^{-1} \cdot \theta = \theta^{-1}C_G(e)\sigma.$$

Since $x_k \in \theta^j C_G(e)\sigma^{-j}$ for $-1 \leq j \leq k$, we see that

$$(11)\qquad x_kn^jm^{-j} \in C_G(\theta^j e\theta^{-j}), \quad -1 \leq j \leq k.$$

https://doi.org/10.4153/CJM-1994-035-6 Published online by Cambridge University Press

Hence

$$\begin{aligned} w_k \cdots w_1 a_k x_k &= vmy_k x_k n^{-1}u^{-1} \\ &= v \cdot m(y_k x_k n^k m^{-k})m^{-1} \cdot m^{k+1}(n^{-k-1}u^{-1}n^{k+1})m^{-k-1} \\ &\quad \cdot m^{k+1}n^{-k-1} \in C^r_G(\theta^{k+1}e\theta^{-k-1})\theta^{k+1}\sigma^{-k-1}. \end{aligned}$$

By (9), (11) and the repeated use of Lemma 1.1, we see that

$$w_1, \ldots, w_k, a_k, x_k n^{k+1}m^{-k-1} \in C^r_G(\theta^{k+1}e\theta^{-k-1}). \tag{12}$$

Hence we can factor by [6; Fact 1.3],

$$\begin{gathered} x_k = y_{k+1}x_{k+1} \quad \text{with } y_{k+1} \in \tilde{G}^r_{\theta^{k+1}e\theta^{-k-1}} \text{ and} \\ x_{k+1} \in C_G(\theta^{k+1}e\theta^{-k-1})m^{k+1}n^{-k-1} = \theta^{k+1}C_G(e)\sigma^{-k-1}. \end{gathered} \tag{13}$$

By (11), (13) and Lemma 1.1,

$$y_{k+1} \in C_G(\theta^j e\theta^{-j}), \quad -1 \leq j \leq k.$$

Hence by (11), (13),

$$x_{k+1} = y_{k+1}^{-1}x_k \in C_G(\theta^j e\theta^{-j})\theta^j\sigma^{-j} = \theta^j C_G(e)\sigma^{-j}, \quad -1 \leq j \leq k. \tag{14}$$

By (13), (14),

$$x_{k+1} \in \bigcap_{j=-1}^{k+1} \theta^j C_G(e)\sigma^{-j}. \tag{15}$$

By (9), (12) and [6; Fact 1.3], we can factor

$$\begin{gathered} a_k = c_{k+1}a_{k+1}, \quad c_{k+1} \in \tilde{G}^r_{\theta^{k+1}e\theta^{-k-1}}, \\ a_{k+1} \in C_G(\theta^j e\theta^{-j} \mid 0 \leq j \leq k) \cap \tilde{G}^l_f \end{gathered} \tag{16}$$

and for $i = 1, \ldots, k$,

$$\begin{gathered} w_i = q_i w_i', \quad q_i \in \tilde{G}^r_{\theta^{k+1}e\theta^{-k-1}}, \\ w_i' \in C_G(\theta^j e\theta^{-j} \mid i+1 \leq j \leq k+1) \cap \tilde{G}^r_{\theta^i e\theta^{-i}}. \end{gathered} \tag{17}$$

Now

$$p = vmy_k m^{-1}v^{-1} \in \tilde{G}^r_{\theta^{k+1}e\theta^{-k-1}} \tag{18}$$

and by (10), (13), (16), (17),

$$\begin{aligned} (q_k w_k') \cdots (q_1 w_1')(c_{k+1}a_{k+1})y_{k+1}x_{k+1}un &= vmy_k x_k \\ &= pvmx_k \\ &= pvmy_{k+1}x_{k+1}. \end{aligned}$$

Since $q_1,\ldots,q_k,c_{k+1},y_{k+1},p \in \tilde{G}^r_{\theta^{k+1}e\theta^{-k-1}}$ and since $a_{k+1},w'_1,\ldots,w'_k \in C_G(\theta^{k+1}e\theta^{-k-1})$ we see that

$$w'_{k+1}w'_k\cdots w'_1 a_{k+1}x_{k+1}un = vmy_{k+1}x_{k+1}$$

for some $w'_{k+1} \in \tilde{G}^r_{\theta^{k+1}e\theta^{-k-1}}$. This completes the induction step. In particular (15) is valid for $k = |W|$. Hence by Lemma 1.2,

$$\bigcap_{j\geq 0}\theta^j W(e)\sigma^{-j} = \bigcap_{j=1}^{k}\theta^j W(e)\sigma^{-j} \neq \emptyset.$$

(iii) $\Rightarrow$ (iv) We show that $e\theta$ is conjugate to $e\gamma$ in Ren(M). We do this by induction on t. If $t = 0$, then $e\theta = e\gamma$. So let $t > 0$. Then $e\theta = e\pi_1\cdots\pi_t\gamma = \pi_1\cdots\pi_t e\gamma$ is conjugate in Ren(M) to $e\gamma\pi_1\cdots\pi_t = e\pi'_1\cdots\pi'_t\gamma$, where

$$\pi'_i = \gamma\pi_i\gamma^{-1} \in W_{e^{\gamma^{i-1}}} \cap W(e,\ldots,e^{\gamma^{i-1}}), \quad i = 1,\ldots,t.$$

By the induction hypothesis, $e\theta$ is conjugate to $e\sigma$ in Ren(M).

(iv)$\Rightarrow$ (v) If $e\theta$ and $e\gamma$ are conjugate in Ren(M), then they are conjugate by an element of $W(e)$. Thus without loss of generality we can assume that $\gamma = \theta$. Let

$$\pi \in \bigcap_{i\geq 0}\theta^i W(e)\sigma^{-i}.$$

Then $\pi\sigma^i\theta^{-i} \in W(\theta^i e\theta^{-i})$ for all $i \geq 0$. Now

$$\pi(e\sigma)\pi^{-1} = e\pi\sigma\pi^{-1} = e(\pi\sigma\pi^{-1}\theta^{-1})\theta.$$

Clearly $\pi N_{e\sigma}\pi^{-1} = N_{e\pi\sigma\pi^{-1}}$. Now for all $i \in \mathbb{Z}^+$,

$$\begin{aligned}\pi\sigma\pi^{-1}\theta^{-1} &= (\pi\sigma^i\theta^{-i})\theta(\theta^{i-1}\sigma^{i-1}\pi^{-1})\theta^{-1} \in W(\theta^i e\theta^{-i})\cdot\theta W(\theta^{i-1}e\theta^{1-i})\theta^{-1}\\ &= W(\theta^i e\theta^{-i}).\end{aligned}$$

It follows that $\pi\sigma\pi^{-1}\theta^{-1} \in W(e^\delta \mid \delta \in \langle\theta\rangle)$. By Theorem 2.3(ii), $N_{e\pi\sigma\pi^{-1}}$ is conjugate to $N_{e\theta}$ by an element of $W(e^\delta \mid \delta \in \langle\theta\rangle)$. It follows that $N_{e\theta}$ is conjugate to $N_{e\sigma}$ by an element of $W(e)$.

(v) $\Rightarrow$ (ii) follows from Theorem 2.3, and (ii) $\Rightarrow$ (i) is obvious. ∎

By Theorems 2.3 and 2.6, we have,

COROLLARY 2.7. *There exist reductive groups $G_1,\ldots,G_t$ with respective automorphisms $\sigma_1,\ldots,\sigma_t$, such that the conjugacy classes of M are in a natural bijective correspondence with the σ_i-conjugacy classes of G_i, $i = 1,\ldots,t$.*

COROLLARY 2.8. *Let $\sigma = nT$, $\theta = mT \in W$, $e \in E(\bar{T})$ such that en and em are conjugate in M. Then there exists $\pi = pT \in W$ such that en and ep are conjugate by an element in $C_G(e^\delta \mid \delta \in \langle\sigma\rangle)$, and $e\theta$ and $e\pi$ are conjugate in* Ren(M).

The following answers affirmatively [6; Conjecture 2.7].

COROLLARY 2.9. *Let $e \in E(M)$, $a, b \in eMe$. Then a and b are conjugate in eMe if and only if a and b are conjugate in M.*

PROOF. We can assume that $e \in \Lambda$. Now $eC_G(e)$ is the unit group of eMe. Thus if a and b are conjugate in eMe, then they are conjugate in M by an element in $C_G(e)$. Assume conversely that a and b are conjugate in M. We need to show that they are conjugate by an element in $C_G(e)$. By Theorem 2.1 applied to eMe, we can assume that $a \in M_{h,\sigma}$, $b \in M_{h,\theta}$ for some $h \in e\Lambda$ and $\sigma, \theta \in W(e)$. By Lemma 2.4, h^σ and h^θ are conjugate in $\overline{C_G(h)}$. By [5; Chapter 6], h^σ and h^θ are conjugate in $e\overline{C_G(h)}e$. It follows that h^σ and h^θ are conjugate by an element in $W(e,h)$. Thus without loss of generality we can assume that $h^\sigma = h^\theta = h'$. By Theorem 2.6, a and b are conjugate by an element in $C_G(h,h')$. Now $hC_G(h) = hC_G(e,h)$ and $h'C_G(h') = h'C_G(e,h')$. Hence

$$C_G(h) = G_h \cdot C_G(e,h) = G_h \cdot C_G(e, G_h)$$
$$C_G(h') = G_{h'} \cdot C_G(e,h') = G_{h'} \cdot C_G(e, G_{h'}).$$

By [6; Facts 1.1, 1.2],

$$C_G(h,h') \subseteq C_G(e)[G_h \cap G_{h'}].$$

Since $a, b \in hMh'$, it follows that a and b are conjugate by an element in $C_G(e)$. ∎

REFERENCES

1. R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, Wiley, 1985.
2. J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, 1981.
3. M. S. Putcha, *A semigroup approach to linear algebraic groups*, J. Algebra **80**(1983), 164–185.
4. ———, *Determinant functions on algebraic monoids*, Comm. Algebra **11**(1983), 695–710.
5. ———, *Linear algebraic monoids*, London Math. Soc. Lecture Note Series **133**, Cambridge Univ. Press, 1988.
6. ———, *Conjugacy classes in algebraic monoids*, Trans. Amer. Math. Soc. **303**(1987), 529–540.
7. L. E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids*, J. Algebra **101**(1986), 303–338.
8. ———, *Conjugacy classes of semisimple elements, and irreducible representations of algebraic monoids*, Comm. Algebra **116**(1988), 1933–1943.
9. ———, *Analogue of the Bruhat decomposition for algebraic monoids II. The length function and the trichotomy*, to appear.

Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27695-8205
U.S.A.