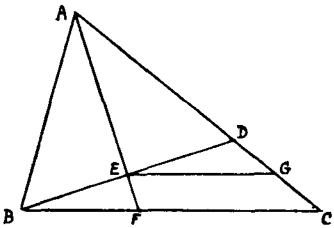
Geometrical Proof of $\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{b-c}{b+c}$.



Consider triangle ABC.

From AC cut off $AD=AB$; join BD; draw AE perpendicular to BD and produce to meet BC in F; draw EG parallel to BC. Then, by Geometry, E and G are mid-points of BD and DC respectively; $A\widehat{B}D=A\widehat{D}B=\frac{1}{2}(B+C)$ since $\widehat{A}$ common to triangles ABD and ABC; $E\widehat{B}F=B-\frac{1}{2}(B+C)=\frac{1}{2}(B-C)$.

$$\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{EF/BE}{EA/BE}=\frac{EF}{EA}=\frac{GC}{GA}=\frac{\frac{1}{2}(b-c)}{\frac{1}{2}(b+c)}=\frac{b-c}{b+c}.$$

ALEX. D. RUSSELL.

Angles between the Medians and Sides of a Triangle.

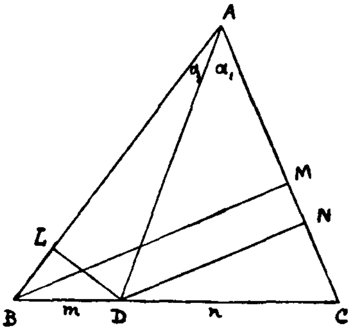


Fig. 1.

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1. Let ABC be a triangle, AD joining A to a point D in BC such that $BD : DC = m : n$. Let $\angle CAD = \alpha_1$ and $\angle BAD = \alpha_2$. Draw BM and $DN \perp^r AC$.

$$\cot\alpha_1 = \frac{AN}{DN} = \frac{AM}{DN} + \frac{MN}{DN} = \frac{(m+n)AM}{n\,BM} + \frac{m\,NC}{n\,DN}$$

$$= \frac{(m+n)}{n}\cot A + \frac{m}{n}\cot C = \frac{m}{n}(\cot A + \cot C) + \cot A \ldots (1)$$

Similarly, $\cot\alpha_2 = \dfrac{n}{m}(\cot A + \cot B) + \cot A$.

$\therefore\ n\cot\alpha_1 - m\cot\alpha_2 = m\cot C - n\cot B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

$$\cot ADC = -\cot(\alpha_1 + C) = \frac{1 - \cot\alpha_1\cot C}{\cot\alpha_1 + \cot C}$$

$$= \frac{1 - \left[\dfrac{m}{n}(\cot A + \cot C) + \cot A\right]\cot C}{\dfrac{m}{n}(\cot A + \cot C) + \cot A + \cot C}$$

$$= \frac{\dfrac{1 - \cot A\cot C}{\cot A + \cot C} - \dfrac{m}{n}\cot C}{\dfrac{m}{n} + 1} = \frac{n\cot B - m\cot C}{m+n} \quad \ldots (3)$$

$\therefore\ (m+n)\cot ADC = m\cot\alpha_2 - n\cot\alpha_1$.

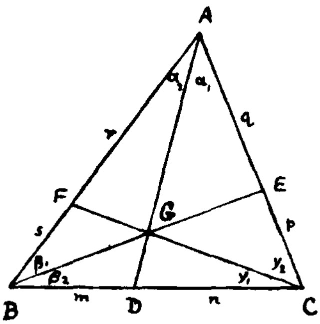

Fig. 2.

2. Suppose three concurrent lines AD, BE, CF be drawn to the sides BC, CA, AB, dividing them in the ratios $m:n$, $p:q$, $r:s$, and intersecting at G, and making angles α_1 and α_2, β_1 and β_2, γ_1 and γ_2 with the sides.

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$$\cot DGC = \cot(\alpha_1+\gamma_2) = \frac{\cot\alpha_1\cot\gamma_2 - 1}{\cot\alpha_1 + \cot\gamma_2}$$

$$= \frac{\left[\frac{m}{n}(\cot A + \cot C) + \cot A\right]\left[\frac{s}{r}(\cot C + \cot A) + \cot C\right] - 1}{\frac{m}{n}(\cot A + \cot C) + \cot A + \frac{s}{r}(\cot C + \cot A) + \cot C}$$

$$= \frac{\frac{m}{n}\cdot\frac{s}{r}(\cot A + \cot C) + \frac{s}{r}\cot A + \frac{m}{n}\cot C - \cot B}{\frac{m}{n} + \frac{s}{r} + 1} \quad \ldots\ldots\ldots\ldots(4)$$

3. If AD, BE, CF are the medians of $\triangle ABC$,

(1) becomes $\cot\alpha_1 = 2\cot A + \cot C$.

$\therefore \cot\alpha_1 + \cot\beta_1 + \cot\gamma_1 = \cot\alpha_2 + \cot\beta_2 + \cot\gamma_2$

$= 3(\cot A + \cot B + \cot C) = 3\cot\omega$.

(2) becomes $\cot\alpha_1 - \cot\alpha_2 = \cot C - \cot B$.

This may be proved as follows: Let DN and DL (Fig. 1) be $\perp^r$ to CA and AB.

Then $AD^2 - DC^2 = AD^2 - DB^2 \quad \therefore AN^2 - NC^2 = AL^2 - LB^2$.

$\therefore (AN - NC)AC = (AL - LB)AB$. But $\triangle ADC = \triangle ADB$.

$\therefore AB \,.\, DL = AC \,.\, DN$. Hence $\dfrac{AN - NC}{DN} = \dfrac{AL - LB}{DL}$.

$\therefore \cot\alpha_1 - \cot C = \cot\alpha_2 - \cot B$.

(3) becomes $\cot ADC = \dfrac{\cot B - \cot C}{2} = \dfrac{\cot\alpha_2 - \cot\alpha_1}{2}$.

$\therefore \cot ADC + \cot BEA + \cot CFB = 0$.

(4) becomes $\cot DGC = \dfrac{2\cot A + 2\cot C - \cot B}{3}$.

$\therefore \cot DGC + \cot EGA + \cot FGB = \cot DGB + \cot EGC$

$+ \cot FGA = \cot A + \cot B + \cot C = \cot\omega$,

and $\cot DGC - \cot DGB = \cot C - \cot B = \cot\alpha_1 - \cot\alpha_2 = 2\cot ADB$.

Several of these results will be found in Hobson's "Trigonometry," Chap. XII.

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An Area Proof of the Proposition

"If AB is divided equally at C and unequally at D, then $AD^2 + DB^2 = 2AC^2 + 2CD^2$."

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