The solution of Cauchy's Problem for linear partial differential equations, with constant coefficients, by means of integrals involving complex variables.

By C. A. STEWART.

(*Received 5th February* 1927. *Read 4th March* 1927.)

The object of this paper is to show how the theory of integrals involving complex variables may be applied to the integration of linear partial differential equations, possessing real, distinct characteristics and constant coefficients. The problem considered is a Cauchy problem (with analytic data)—typical of the equation of real characteristics and the method taken is that of Riemann.[1] For simplicity of exposition, the second order hyperbolic equation is considered, but the results are given in such a form as to indicate an obvious generalisation to equations of higher order.[2]

The problem is to find that solution (known to be unique by Cauchy's existence theorem) of

$$1. \qquad \left(\frac{\partial}{\partial x} + k_1 \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y}\right) V + A \frac{\partial V}{\partial x} + B \frac{\partial V}{\partial y} + CV = \phi(x, y)$$

{where k_1, k_2, A, B, C are real constants, $(k_1 \neq k_2)$}, which is such that on the boundary specified by $x = \psi(y)$, V reduces to a given function $E_0(y)$, and $\dfrac{\partial V}{\partial x}$ to a given function $E_1(y)$.

It is sufficient for our present purpose to assume that ϕ, ψ, E_0, E_1 are analytic in a region ω, containing a portion σ of the boundary; and it is necessary that σ should not be tangential anywhere to a characteristic.

---

[1] Darboux : *Théorie générale des surfaces*, II, pp. 75 *et seq.*

[2] The Riemannian method of integration has been extended by the writer to equations of higher order : *Proc. Lond. Math. Soc.*, 26 (1927). pp. 81–94.

Let $P(x_0, y_0)$ be a point in ω near σ. Draw through P lines of gradient k_1, k_2 to meet σ in R, Q respectively. (Fig. 1.)

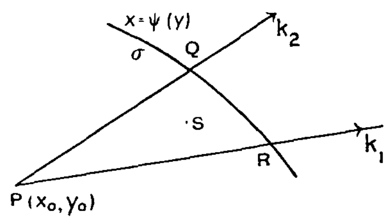

Fig. 1

P can be taken sufficiently near the boundary to ensure that QR is a part of σ; and owing to the conditions we have imposed, no characteristic can meet σ in more than one point. Any point S in the area PQR may be taken as[1]

$$x_0 + u + v,\ y_0 + k_1 u + k_2 v.$$

The differential equation becomes

2. $$V_{uv} + aV_u + bV_v + cV = \phi(x_0 + u + v,\ y_0 + k_1 u + k_2 v)$$

where $a = \dfrac{Ak_2 - B}{k_2 - k_1}$, $b = \dfrac{Ak_1 - B}{k_1 - k_2}$, $c = C$.

The boundary becomes $x_0 + u + v = \psi(y_0 + k_1 u + k_2 v)$, and there are similar changes in the forms of the boundary conditions.

Riemann's method consists substantially in integrating the expression

$$\frac{\partial^2(\lambda V)}{\partial u\,\partial v} - \frac{\partial}{\partial u}\{(\lambda_v - a\lambda)V\} - \frac{\partial}{\partial v}\{(\lambda_u - b\lambda)V\} - \lambda\phi$$

throughout the area P, Q, R of the u, v plane (Fig. 2), this expression being zero if λ is a solution of the adjoint equation:

3. $$\lambda_{uv} - a\lambda_u - b\lambda_v + c\lambda = 0$$

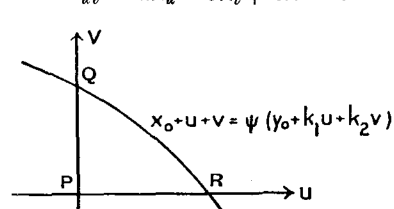

Fig. 2

[1] If there is no term in V_{xx} in the original equation, take the second order terms as $\dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x} + k_1\dfrac{\partial}{\partial y}\right)V$, *and* $x = x_0, + u$, $y = y_0 + k_1 u + v$.

It is shown that the integration provides the value of V at P, viz., $V(x_0, y_0)$, if λ can be found to satisfy the supplementary conditions:

$$\lambda_v = a\lambda, \text{ when } u = 0; \ \lambda_u = b\lambda, \text{ when } v = 0.$$

This solution is usually given in terms of the Bessel Function of zero order, viz.,

4. $$\lambda = e^{av + bu} J_0\{2i\sqrt{\{uv(ab - c)\}}\},$$

but here we shall obtain λ in the form of a double integral involving two complex variables. It will be seen later that this not only simplifies the subsequent integration but it also provides the obvious generalisation.

Consider the double integral

5. $$I = \frac{1}{(2\pi\iota)^2}\iint \frac{e^{\alpha u + \beta v} d\alpha \, d\beta}{\alpha\beta - a\alpha - b\beta + c},$$

where α, β are complex variables describing circles in their respective planes, given by $|\alpha| = R_1$, $|\beta| = R_2$. R_1, R_2 can clearly be taken so large that the function $\alpha\beta - a\alpha - b\beta + c$ does not vanish on the circles of integration (nor at points outside these circles). It is sufficient to take R_1, $R_2 > R_0$, where R_0 is the positive root of the R-equation:

6. $$R^2 = \{|a| + |b|\}R + |c|.$$

For then

$$|\alpha\beta - a\alpha - b\beta + c| \geqslant (R_1 - R_0)(R_0 - |a|) + (R_2 - R_0)(R_0 - |b|) + (R_1 - R_0)(R_2 - R_0) + \{R_0^2 - (|a| + |b|)R_0 - |c|\}.$$
$$i.e. \quad > 0.$$

I is an example of that class of double integrals, where the integrations with respect to α, β are independent of one another. The integrand is analytic and we can differentiate with regard to u, v under the integral sign.

This gives

$$\frac{\partial^2 I}{\partial u \, \partial v} - a\frac{\partial I}{\partial u} - b\frac{\partial I}{\partial v} + cI = \frac{1}{(2\pi\iota)^2}\iint e^{\alpha u + \beta v} \, d\alpha \, d\beta$$
$$= 0$$

since the integrand is everywhere analytic within the circles.

Again $$I(u, 0) = \frac{1}{(2\pi\iota)^2}\iint \frac{e^{\alpha u} d\alpha d\beta}{\alpha\beta - a\alpha - b\beta + c}$$
$$= \frac{1}{2\pi\iota}\int \frac{e^{\alpha u} d\alpha}{\alpha - b} = e^{bu}.$$

https://doi.org/10.1017/S0013091500007410 Published online by Cambridge University Press

Similarly $I(0, v) = e^{\tau v}$; $I(0, 0) = 1$.

$I(u, v)$ is therefore the required solution λ; and we may easily deduce the other form of the result by taking $\alpha = b + t_1$; $\beta = a + t_2$.

Then $$I(u, v) = \frac{e^{bu+av}}{(2\pi\iota)^2}\iint\frac{e^{ut_1+vt_2}dt_1dt_2}{t_1t_2 - ab + c}$$

over $|t_1 + b| = R_1$; $|t_2 + a| = R_2$ and so enclosing the origins and points where $t_1t_2 = ab - c$.

This gives $$\frac{e^{bu+av}}{2\pi\iota}\int\frac{e^{ut_1+v/t_1(ab-c)}dt_1}{t_1}$$
$$= e^{bu+av}J_0(2i\sqrt{\{uv(ab - c)\}}).$$

The application of this result can now be best illustrated by taking the important case of a *linear* boundary; for even in the case of a curved boundary, the solution obtained will give an approximation in the neighbourhood. (*i.e.* when the tangent is taken as the first approximation to the bounding curve).

Take the boundary to be $x = h$,[1] (except when k_1 or k_2 becomes infinite), and for definiteness assume $x_0 < h$, (Fig. 3).

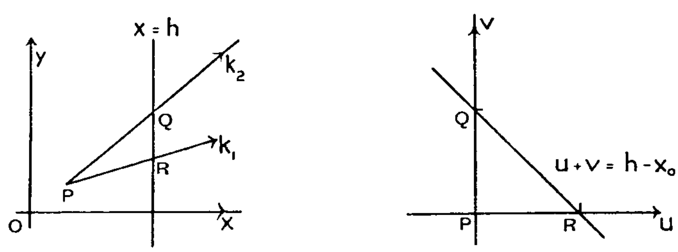

Fig. 3

The solution is provided by the integration of

$$7. \quad \iint\left[\frac{\partial^2(\lambda V)}{\partial u\,\partial v} - \frac{\partial}{\partial u}\{(\lambda_v - a\lambda)V\} - \frac{\partial}{\partial v}\{(\lambda_u - b\lambda)V\} - \lambda\phi\right]dudv = 0$$

where A denotes the area PQR.

The integration of the first term can be effected with regard to either u or v first, giving two forms of the result

$$8. \quad V(x_0, y_0) = \int_Q^R[\lambda(V_u + bV)\,du + (\lambda_v - a\lambda)\,Vdv] - \lambda_R V_R + \iint_A \lambda\phi du\,dv$$
$$= \int_Q^R[-\lambda(V_v + aV)dv - (\lambda_u - b\lambda)\,Vdu] - \lambda_Q V_Q + \iint_A \lambda\phi du\,dv$$

[1] Any linear boundary can be changed to $x = h$, by a suitable linear transformation of the independent variables.

https://doi.org/10.1017/S0013091500007410 Published online by Cambridge University Press

and so we may use them in a convenient combination, determined by the boundary conditions. Suppose we are given that on

$$x = h,\ V = E_0(y),\ V_x = E_1(y).$$

These become:—

On $u + v = h - x_0,\ V = E_0(y_0 + k_1u + k_2v),\ k_2V_u - k_1V_v = (k_2 - k_1)E_1.$
Taking therefore k_2 times the first expression for the solution minus k_1 times the second, we find, (putting $dv = -du$)

$$9.\quad V(x_0y_0) = \iint_A \lambda\phi du dv - \int_Q^R \lambda E_1 du + \int_Q^R \Big(\frac{k_1\lambda_u - k_2\lambda_v - (a+b)\lambda}{k_1 - k_2}\Big)E_0 du$$

$$- \frac{k_2}{k_1 - k_2} e^{b(h-x_0)}E_0\{y_0 + k_1(h - x_0)\} - \frac{k_1}{k_2 - k_1} e^{a(h-x_0)}E_0\{y_0 + k_2(h - x_0)\}.$$

The substitution of our value for λ gives finally

$$10.\quad V(x_0,\ y_0)$$

$$= \frac{1}{(2\pi\iota)^2}\iiiint\frac{e^{\alpha u + \beta v}\phi_0(x_0 + u + v,\ y_0 + k_1u + k_2v)\ du dv d\alpha d\beta}{\alpha\beta - a\alpha - b\beta + c}$$

$$- \frac{1}{(2\pi\iota)^2}\iiint\frac{e^{\alpha u + \beta v}E_1(y_0 + k_1u + k_2v)du d\alpha d\beta}{\alpha\beta - a\alpha - b\beta + c}$$

$$+ \frac{1}{(2\pi\iota)^2}\iiint\frac{e^{\alpha u + \beta v}\{k_1\alpha - k_2\beta - (k_1 - k_2)(a+b)\}E_0(y_0 + k_1u + k_2v)du d\alpha d\beta}{(k_1 - k_2)(\alpha\beta - a\alpha - b\beta + c)}$$

$$- \frac{k_2}{k_1 - k_2} e^{b(h-x_0)}E_0\{y_0 + k_1(h - x_0)\} - \frac{k_1}{k_2 - k_1} e^{a(h-x_0)}E_0\{y_0 + k_2(h - x_0)\}.$$

The scope of the integrals has already been specified, and owing to the nature of the integrands, the integrations may be performed in any order. It will usually be simpler to integrate with respect to the real variables first, these being of the type

$$\iint_A e^{\alpha_1 u + \beta_1 v}\, F(u,\ v)du\, dv,\quad \int_0^k e^{\alpha_1 v}G(u)\,.\,du$$

and so, easily integrated if, for example, F, G were exponential functions or polynomials. Take, therefore,

$$\phi(x,\ y) = e^{\rho x + \sigma y},\ E_0 = e^{\theta_0 y},\ E_1 = e^{\theta_1 y}.$$

Then from the solution obtained, we can deduce the corresponding solutions for polynomials.

We shall require certain results in integration which will be given a general form.

11. (i)

$$\iint \cdots \int e^{a_1u_1 + a_2u_2 + \cdots + a_nu_n}\, du_1 du_2 \cdots du_n = \frac{1}{2\pi i}\int_\gamma \frac{e^{kt}\, dt}{t\,(t-a_1)(t-a_2)\cdots(t-a_n)}$$

where on the left hand side, the integration extends to all positive values of $u_1, u_2, \cdots u_n$, satisfying the relation

$$0 \leqslant u_1 + u_2 + \cdots + u_n \leqslant k \quad (k \text{ positive})$$

and on the right hand side, the contour γ is the circle $|t| = R'$, $(R' > max\ a_r)$.

12. (ii)

$$\iint \cdots \int e^{a_1u_1 + a_2u_2 + \cdots + a_nu_n}\, du_1 du_2 \cdots du_{n-1} = \frac{1}{2\pi i}\int_\gamma \frac{e^{kt} dt}{(t-a_1)(t-a_2)\cdots(t-a_n)}$$

where on the left hand side, the integration extends to all positive values of $u_1, u_2, \cdots u_{n-1}$, satisfying the relation

$$0 \leqslant u_1 + u_2 + \cdots + u_{n-1} \leqslant k, \text{ and } u_n = k - u_1 - u_2 - \cdots - u_{n-1}.$$

Denote the first integral on the left by $V(1, 2, \cdots n)$ and the second by $S(1, 2, \cdots n)$.

(*a*) The substitution $u_r = kt_r$ $(r = 1, \cdots n)$ shows that V and its first $(n-1)$ differential coefficients with regard to k, vanish for $k = 0$.

(*b*) The change of variables from

$$u_1, u_2, \cdots, u_n \text{ to}$$

$$u_1, u_2, \cdots u_{n-1}, \theta\,(= u_1 + u_2 + \cdots + u_n)$$

gives

$$V(1, 2, \cdots n) = \int_0^k F(\theta)\, d\theta$$

where

$$F(\theta) = \iint \cdots \int e^{a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1} + a_n(\theta - u_1 - u_2 - \cdots - u_{n-1})} du_1 du_2 \cdots du_{n-1}$$

taken over all positive values of $u_1, u_2, \cdots u_{n-1}$ satisfying

$$0 \leqslant u_1 + u_2 + \cdots + u_{n-1} \leqslant \theta \quad (0 \leqslant \theta \leqslant k)$$

$$\therefore \quad \frac{dV(1, 2, \cdots n)}{dk} = F(k) = S(1, 2, \cdots n).$$

(c) Integration of $V(1, 2, \cdots n)$, in the original form, with respect to u_n gives

$$V(1, 2, \cdots n) = \frac{S(1, 2, \cdots n)}{a_n} - \frac{V(1, 2, \cdots n-1)}{a_n}$$

$$\therefore \left(\frac{d}{dk} - a_n\right) V(1, 2, \cdots n) = V(1, 2, \cdots n-1).$$

We note that $V_1 = \frac{e^{a_1 k}}{a_1} - \frac{1}{a_1}$ and we therefore interpret S_1 to mean $e^{a_1 k}$, $V_0 = 1$, $S_0 = 0$, and deduce by continued application of the above result that

13. $$\left(\frac{d}{dk} - a_1\right)\left(\frac{d}{dk} - a_2\right) \cdots \left(\frac{d}{dk} - a_n\right) V(1, 2, \cdots n) = 1.$$

$V(1, 2, \cdots n)$ is therefore that solution of equation 13 which satisfies the initial conditions.

$$V = 0, \ \frac{dV}{dk} = 0, \cdots \frac{d^{n-1}V}{dk^{n-1}} = 0, \text{ when } k = 0.$$

The contour integral

$$\frac{1}{2\pi\iota}\int_\gamma \frac{e^{kt}\,dt}{t(t-a_1)\cdots(t-a_n)}$$

gives this solution, for the result of substituting this on the left hand side of equation 13. Gives

$$\frac{1}{2\pi\iota}\int_\gamma e^{kt}\frac{dt}{t} = 1.$$

Also putting $k = 0$ in the contour integral above and in those obtained by differentiating once, twice up to $(n-1)$ times will give integrands of the type

$$\frac{t^m}{t(t-a_1)(t-a_2)\cdots(t-a_n)} \qquad (m = 0, 1, \cdots n-1)$$

These do not possess singularities at infinity and therefore vanish.

By differentiation we get

$$S(1, 2, \cdots n) = \frac{1}{2\pi\iota}\int_\gamma \frac{e^{kt}\,dt}{(t-a_1)(t-a_2)\cdots(t-a_n)}.$$

In the solution for $V(x_0, y_0)$, the part depending on ϕ is

$$\frac{1}{(2\pi\iota)^2}\iiiint \frac{e^{\alpha u + \beta v + \rho(x_0+u+v) + \sigma(y_0 + k_1 u + k_2 v)}du\,dv\,d\alpha\,d\beta}{\alpha\beta - a\alpha - b\beta + c}$$

over $\quad 0 \leqslant u + v \leqslant h - x_0;\ |\alpha| = R_1;\ |\beta| = R_2$

$$= \frac{1}{(2\pi\iota)^3}\iiint \frac{e^{\rho x_0 + \sigma y_0 + t(h - x_0)}d\alpha\,d\beta\,dt}{t(t-\alpha-\rho-\sigma k_1)(t-\beta-\rho-\sigma k_2)(\alpha\beta - a\alpha - b\beta + c)}$$

for $\ |t| = R',\ |\alpha| = R_1,\ |\beta| = R_2.$

Now take R' so large that $|t-\rho-\sigma k_1| > R_1, R_2$

and $|t-\rho-\sigma k_2| > R_1, R_2$.

Then, in integrating with respect to α, β, we note that the integrand possesses a singularity only at *one* place *outside* the circles

$|\alpha| = R_1$, $|\beta| = R_2$, viz., where $\alpha = t-\rho-\sigma k_1$, $\beta = t-\rho-\sigma k_2$

and so the above integral becomes

$$\frac{1}{2\pi\iota}\int\frac{e^{\rho x_0+\sigma y_0+t(h-x_0)}dt}{t\{(t-\rho-\sigma k_1)(t-\rho-\sigma k_2)-a(t-\rho-\sigma k_1)-b(t-\rho-\sigma k_2)+c\}}$$

$$=\frac{1}{2\pi\iota}\int\frac{e^{\rho r_0+\sigma y_0+t(h-x_0)}dt}{tf(\rho-t,\ \sigma)}$$

where $f(D_1, D_2)V=\phi$ is the *original* differential equation 1, D_1, D_2 denoting the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$.

The part of the solution depending on E_1 is proved similarly to be

$$-\frac{1}{2\pi\iota}\int\frac{e^{\theta_1 y_0+t(h-x_0)}dt}{f(-t,\ \theta_1)}$$

The part of the solution depending on E_0 is

$$\frac{1}{(2\pi\iota)^2}\iiint\frac{e^{\alpha u+\beta v}\{k_1\alpha-k_2\beta-(k_1-k_2)(a+b)\}e^{\theta_0(y_0+k_1u+k_2v)}dud\alpha d\beta}{(k_1-k_2)(\alpha\beta-a\alpha-b\beta+c)}$$

$$-\frac{k_2}{k_1-k_2}e^{b(h-x_0)+\theta_0 y_0+\theta_0(h-x_0)k_1}-\frac{k_1}{k_2-k_1}e^{a(h-x_0)+\theta_0 y_0+\theta_0(h-x_0)k_2}$$

The part of this involving the integral is

$$\frac{1}{(2\pi\iota)^3}\iiint\frac{\{k_1\alpha-k_2\beta-(k_1-k_2)(a+b)\}e^{\theta_0 y_0+t(h-x_0)}dtd\alpha d\beta}{(k_1-k_2)(t-\alpha-\theta_0 k_1)(t-\beta-\theta_0 k_2)(\alpha\beta-a\alpha-b\beta+c)}$$

But in this integral, there are 3 places outside $|\alpha|=R_1$, $|\beta|=R_2$ where the integrand has a singularity

(i) $\alpha=t-\theta_0 k_1$, $\beta=t-\theta_0 k_2$ (ii) $\alpha=\infty$, $\beta=t-\theta_0 k_2$ (iii) $\alpha=t-\theta_0 k_1$, $\beta=\infty$ ($\alpha=\infty$, $\beta=\infty$ is not a singularity).

The first gives $\frac{1}{2\pi\iota}\int\frac{\{t-(k_1+k_2)\theta_0-a-b\}\,e^{\theta_0 y_0+t(h-x_0)}\,dt}{f(-t,\ \theta_0)}$.

The second gives $-\frac{k_1}{k_1-k_2}e^{\theta_0 y_0+(a+\theta_0 k_2)(h-x_0)}$.

The third gives $\frac{k_2}{k_1-k_2}e^{\theta_0 y_0+(b+\theta_0 k_1)(h-x_0)}$.

The part, therefore, depending on E_0 is simply

$$\frac{1}{2\pi\iota}\int\frac{\{t-(k_1+k_2)\theta_0-a-b\}\,e^{\theta_0 y_0+t(h-x_0)}\,dt}{f(-t_1\,\theta_0)}\,.$$

The solution[1] is therefore given by

14. $V(x_0 y_0)$

$$=\frac{1}{2\pi\iota}\int\left[\frac{e^{\rho x_0+\sigma y_0}}{tf(\rho-t,\,\sigma)}-\frac{e^{\theta_1 y_0}}{f(-t,\,\theta_1)}+\frac{\{t-(k_1+k_2)\theta_0-A\}}{f(-t,\,\theta_0)}e^{\theta_0 y_0}\right]e^{t(h-x_0)}dt.$$

This solution obviously applies to equations of elliptic or parabolic type, although the *method* of obtaining it is peculiar to the hyperbolic type.

Examples:

(i) Hyperbolic type:

Find the solution of

$$V_{xx}-3V_{xy}+2V_{yy}+2V_x+4V_y+V=1$$

which is such that $V_x=y$, $V=0$ when $x=0$.
The first part is

$$\frac{1}{2\pi\iota}\int\frac{e^{-tx_0}\,dt}{t(t-1)^2}=1+\text{coefficient of }\frac{1}{T}\text{ in }\frac{e^{-x_0(1+T)}}{T^2}$$

$$=1-x_0e^{-x_0}.$$

The second part is obtained from

$$-\frac{1}{(2\pi\iota)}\int\frac{e^{\theta_1 y+t(h-x_0)}dt}{t^2+t(3\theta_1-2)+(2\theta_1{}^2+4\theta_1+1)}\quad\text{which gives}$$

$$\frac{2e^{\theta_1 y_0-x_0(1-\frac{3}{2}\theta_1)}\sinh\frac{x_0}{2}\sqrt{(\theta_1{}^2-28\theta_1)}}{\sqrt{(\theta_1{}^2-28\theta_1)}}\,.$$

The coefficient of θ_1 in this is $e^{-x_0}(x_0y_0+\frac{3}{2}x_0{}^2-\frac{7}{6}x_0{}^3)$.

The third part is zero.

The required solution is $1+e^{-x}(xy+\frac{3}{2}x^2-\frac{7}{6}x^3-x)$.

[1] *Cf.* Zeilon: Arkev für Matematik, Astronomi och Fysik, 6 (1910).

(ii) Parabolic type:

Find the solution of $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial y}$, satisfying

$$V = A, \quad V_x = By \text{ on } x = 0$$

1st part is zero:

2nd part is the coefficient of θ_1 in the expansion of

$$-\frac{B}{2\pi\iota}\int \frac{e^{\theta_1 y_0 - tx_0}\, dt}{t^2 - \theta_1}$$

that is in $\frac{Be^{\theta_1 y_0} \sinh x_0\sqrt{\theta_1}}{\sqrt{\theta_1}}$ and therefore is $Bx_0(y_0 + x_0^2/6)$.

3rd part is $\frac{A}{2\pi\iota}\int \frac{te^{-tx_0}\, dt}{t^2} = A$.

Solution is $A + Bx(y + x^2/6)$.

(iii) Elliptic type:

Find the solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = cV$ satisfying $V = A \sin py$, $V_x = 0$, when $x = 0$.

Corresponding to $E_0 = e^{\iota py}$, $E_1 = 0$, the solution is

$$\frac{1}{2\pi\iota}\int \frac{te^{\iota py_0 - tx_0}\, dt}{t^2 - p^2 - c}$$

$$= Ae^{\iota py_0} \cosh x_0\sqrt{(p^2 + c)}.$$

Therefore the solution is

$$A \sin py \cosh x\sqrt{(p^2 + c)}.$$

https://doi.org/10.1017/S0013091500007410 Published online by Cambridge University Press