CRITERIA FOR FOURIER TRANSFORMS

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1. A criterion due to Salem

Salem [1] gave the following criterion for Fourier-Lebesgue sequences $(c_n)_{n=-\infty}^{\infty}$.

Denote by E the set of continuously differentiable periodic functions u such that u' has an absolutely convergent Fourier series, and let E_1 denote the set of $u \in E$ satisfying $||u||_\infty \leqq 1$. In order that a sequence $(c_n)_{n=-\infty}^{\infty}$ be the sequence of Fourier coefficients of an integrable periodic function, it is necessary and sufficient that the following two conditions be satisfied:

(S_1) The formally integrated series $\sum_{n\neq 0} c_n e^{inx}/(in)$ converges to a continuous function.

(S_2) If $(u_k)_{k=1}^{\infty}$ is a sequence extracted from E_1 such that

$$\lim_{k\to\infty} ||u_k||_2 = 0,$$

then

$$\lim_{k\to\infty} \sum_{n=-\infty}^{\infty} c_n \hat{u}_k(n) = 0.$$

In the above, $||\cdot||_p$ denotes the usual norm (or quasinorm, if $0 < p < 1$) in the L^p-space formed relative to Haar measure on the circle group, and

$$\hat{u}(n) = (1/2\pi) \int_{-\pi}^{\pi} u(x) e^{-inx} dx.$$

In §§ 1—5 we record an analogue of Salem's criterion, applicable to an arbitrary compact Hausdorff Abelian group G, in which no condition of the type (S_1) appears; an analogue for the case in which G is locally compact but not compact; and a related comment regarding the Lebesgue-Radon-Nikodým theorem. Thereafter we discuss some other somewhat similar criteria on the basis of a general theorem about Banach spaces. Most of the results we formulate could be extended to non-Abelian compact groups, and indeed to fairly general orthogonal expansions on finite measure spaces.

NOTATION. Throughout the paper X will denote the character group of G. $L^p(G)$ denotes the usual Lebesgue space formed relative to Haar measure on G; $C(G)$ the space of continuous functions on G; and, if G is noncompact, $C_0(G)$ denotes the space of continuous functions on G which tend to zero at infinity. For $1 \leqq p \leqq \infty$ when G is compact, and for $p = 1$ when G is noncompact, $\mathscr{F}L^p(G)$ will denote the set of functions on X which are Fourier transforms of elements of $L^p(G)$. For compact G, $\mathscr{F}C(G)$ is defined analogously. $M(G)$ denotes the space of bounded Radon measures on G, identifiable with the topological dual of $C_0(G)$ (or of $C(G)$ when G is compact).

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In place of the sequence (c_n), we consider a bounded measurable function F on X. We select any exponent p satisfying $0 < p < \infty$. The choice of E is left free, save for the following assumptions:

(a) E is a linear subspace of $C_0(G) \cap L^1(G)$ such that each $u \in E$ has a Fourier transform $\hat{u}$ which is integrable over X; the $\hat{u}(u \in E)$ form a dense subspace of $L^1(X)$.

(b) There exists a number $c \geqq 0$ such that each continuous function v on G having a compact support is the pointwise limit of a sequence (or the uniform limit of a net) (u_k) of functions in E satisfying $||u_k||_\infty \leqq c||v||_\infty$ and $||u_k||_p \leqq c||v||_p$.

One might, for example, take E to consist of all finite linear combinations of continuous positive definite functions in $L^1(G)$, i.e., of all continuous functions $u \in L^1(G)$ such that $\hat{u} \in L^1(X)$.

As before, E_1 will denote the set of $u \in E$ satisfying $||u||_\infty \leqq 1$.

We consider the following hypothesis on F:

$(S)_p$ If $(u_k)_{k=1}^\infty$ is a sequence extracted from E_1 such that

$$\lim_{k\to\infty} ||u_k||_p = 0,$$

then

$$\lim_{k\to\infty} \int_X F \cdot \hat{u}_k d\xi = 0.$$

In what follows we shall use the fact that, whether or not G is compact, $(S)_p$ signifies exactly that to each $\varepsilon > 0$ corresponds a number $c(\varepsilon) = c(\varepsilon, F, p) \geqq 0$ for which

$$\left|\int_X F \cdot \hat{u} d\xi\right| \leqq \varepsilon \cdot ||u||_\infty + c(\varepsilon) \cdot ||u||_p \tag{1}$$

for each $u \in E$. The verification is simple and is left to the reader.

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THEOREM 1. *Assume that G is compact. With the notation and assumptions of § 2, in order that F shall belong to $\mathscr{F}L^1(G)$, it is necessary and sufficient that it satisfy condition $(S)_p$ for some (and so for all) p.*

PROOF. Suppose first that $F = \hat{f}$ for some $f \in L^1(G)$. Then, thanks to (a) of § 2 and the Fubini-Tonelli theorem, we have

$$\int_X F \cdot \hat{u} d\xi = \int_G f(-x)u(x)dx \tag{2}$$

for $u \in E$. That $(S)_p$ is satisfied, follows from (2) and Lebesgue's theorem [in the form that $\int g_k \to 0$ whenever the sequence (g_k) is dominated and converges to zero in measure]. Alternatively, it may be shown in the following manner that (1) is fulfilled. Suppose $u \in E$ and define

$$S_\lambda = \{x \in G : |u(x)| > \lambda\} \quad \text{for} \quad \lambda > 0,$$

noting that $m(S_\lambda) \leqq \lambda^{-p}||u||_p^p$. We have from (2)

$$\left|\int_X F \cdot \hat{u} d\xi\right| \leqq ||u||_\infty \cdot \int_{-S_\lambda} |f| dx + \lambda \cdot ||f||_1. \tag{3}$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_S |f| dx \leqq \varepsilon$ whenever $m(S) \leqq \delta$. Taking $\lambda = \delta^{-1/p}||u||_p$, (3) accordingly gives

$$\left|\int_X F \cdot \hat{u} d\xi\right| \leqq \varepsilon \cdot ||u||_\infty + \delta^{-1/p}||f||_1 \cdot ||u||_p,$$

which is (1) with $c(\varepsilon) = \delta^{-1/p}||f||_1$. In this way we see that $(S)_p$ is necessary.

To prove the sufficiency of $(S)_p$, we begin by noting that when G is compact the inequality (1) entails that

$$\left|\int_X F \cdot \hat{u} d\xi\right| \leqq c' \cdot ||u||_\infty \tag{4}$$

for $u \in E$. This, combined with (a) of § 2 and the Hahn-Banach theorem, entails that there exists a bounded Radon measure μ on G such that

$$\int_X F \cdot \hat{u} d\xi = \int_G u(-x) d\mu(x) \tag{5}$$

for $u \in E$. The Fubini-Tonelli theorem and (a) of § 2 combine with (5) to show that $F = \hat{\mu}$. It therefore remains to show that μ is absolutely continuous (relative to m).

Now (b) of § 2, taken together with (1) and (5), shows that

$$\left|\int_G v d\mu\right| \leqq \varepsilon \cdot ||v||_\infty + c'(\varepsilon) \cdot ||v||_p \tag{6}$$

for any continuous v. If h is a positive continuous function, (6) shows that

$$\begin{aligned} \int_G h d|\mu| &= \operatorname{Sup}\left\{\left|\int_G hv d\mu\right| : ||v||_\infty \leqq 1\right\} \\ &\leqq \varepsilon \cdot ||h||_\infty + c'(\varepsilon) \cdot ||h||_p. \end{aligned} \tag{7}$$

By [3], pp. 184, 188, (7) yields $|\mu|(U) \leqq \varepsilon + c'(\varepsilon) \cdot m(U)^{1/p}$ for any relatively compact open subset U of G, which is enough to show that μ is absolutely continuous.

4. The case in which G is noncompact

Scrutiny of the preceding proof shows that in this case (4) no longer follows from (1). But in any case (4) is equivalent to the demand that F be equal l.a.e. on X to the transform of some bounded Radon measure on G.

It therefore follows that necessary and sufficient conditions in order that F be equal l.a.e. on X to the transform of a function in $L^1(G)$ are:

$(S)_p$ As before.

(S') Any condition known to be necessary and sufficient to ensure that F be equal l.a.e. on X to a Fourier-Stieltjes transform.

A possible condition of type (S') is

(S'_1) F is continuous and

$$|\sum c_r F(\xi_r)| \leqq \text{const.} \|\sum c_r \xi_r\|_\infty$$

for each trigonometric polynomial $\sum c_r \xi_r$ on G.

For this condition, due to Eberlein, see [2], p. 32. Another possibility is

$$(S'_2) \qquad \limsup_\alpha \int_G \left| \int_X \hat{r}_\alpha(\xi) \hat{F}(\xi) \xi(x) d\xi \right| dx < \infty,$$

where (r_α) is a suitable approximate identity in $L^1(G)$.

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It is perhaps worth pointing out that part of the arguments in § 3 suffice to establish the following variant of the Lebesgue-Radon-Nikodým theorem (cf. [3], Theorem 4.15.1).

Let X be any Hausdorff locally compact space and m a positive Radon measure on X. Let p be any exponent satisfying $0 < p < \infty$. A Radon measure μ on X has the form $\mu = f \cdot m$ for some locally m-integrable function f, if and only if to each $\varepsilon > 0$ and each compact subset K of X corresponds a number $c_K(\varepsilon) = c_K(\varepsilon, p, \mu) \geqq 0$ such that

$$\left| \int_X u d\mu \right| \leqq \varepsilon \cdot \|u\|_\infty + c_K(\varepsilon) \cdot \|u\|_p$$

for each continuous function u on X having its support contained in K. [The norm $\|\cdot\|_p$ is constructed relative to the measure m.]

If X is a C^∞ manifold, the same system of inequalities applied to the indefinitely differentiable functions u with supports contained in K, is necessary and sufficient in order that a distribution μ on X shall be of the form $f \cdot m$ with f a locally m-integrable function.

In the preceding statements one may replace the set of all compact subsets K of X by any chosen open covering of X.

6. A result about Banach spaces

Suppose that B is a Banach space with topological dual B'. Assume also that conditions (i) and (ii) immediately below are satisfied.

(i) V is a linear subspace of B' with the property that there exists a number $m \geqq 0$ such that each $f' \in B'$ is the weak (i.e., $\sigma(B', B)-$) limit of a net (f'_i) extracted from V and satisfying $\|f'_i\| \leqq m \cdot \|f'\|$.

(ii) A is a subset of B, the linear combinations of elements of which are everywhere dense in B.

We then have the following theorem, which will be seen in § 7 to have some interesting concrete applications.

THEOREM 2. *The notations and assumptions being as immediately above, suppose further that λ is a linear functional defined on V. In order that λ be generated by an element of B, i.e., that there should exist $f \in B$ such that*

$$\lambda(f') = f'(f) \qquad (f' \in V), \tag{9}$$

it is necessary and sufficient that to each $\varepsilon > 0$ shall correspond a number $c(\varepsilon) \geqq 0$ and a finite subset S_ε of A such that

$$|\lambda(f')| \leqq \varepsilon \cdot \|f'\| + c(\varepsilon) \cdot \mathrm{Sup}_{g \in S_\varepsilon} |f'(g)| \qquad (f' \in V). \tag{10}$$

PROOF. *Necessity.* If λ has the form (9), and if $\varepsilon > 0$ is assigned, choose (as is possible on account of (ii)) $f_1, \cdots, f_n \in A$ and scalars $\alpha_1, \cdots, \alpha_n$ so that

$$\|f - \textstyle\sum_{i=1}^n \alpha_i f_i\| \leqq \varepsilon.$$

Then one has for $f \in V'$

$$|\lambda(f')| = |f'(f)| \leqq |f'(f - \textstyle\sum_{i=1}^n \alpha_i f_i)| + |\sum_{i=1}^n \alpha_i \cdot f'(f_i)|$$
$$\leqq \varepsilon \cdot \|f'\| + [\textstyle\sum_{i=1}^n |\alpha_i|] \cdot \mathrm{Sup}_{1 \leqq i \leqq n} |f'(f_i)|,$$

which shows that (10) holds for the choice

$$c(\varepsilon) = \textstyle\sum_{i=1}^n |\alpha_i|, \qquad S_\varepsilon = \{f_1, \cdots, f_n\} \subset A.$$

Sufficiency. The first step is to show that, if λ satisfies the stated condition, then it can be extended into a linear functional $\bar{\lambda}$ on B' which satisfies the same type of condition with B' in place of V.

To do this, suppose $f' \in B'$ and choose (as is possible on account of (i)) a net (f'_i) of elements of V converging weakly to f' and such that $\|f'_i\| \leqq m \cdot \|f'\|$. Given any $\delta > 0$, apply (10) with $\varepsilon = \delta/4m\|f'\|$ to derive

$$|\lambda(f'_i) - \lambda(f'_j)| = |\lambda(f'_i - f'_j)|$$
$$\leqq \varepsilon \cdot \|f'_i - f'_j\| + c(\varepsilon) \cdot \mathrm{Sup}_{g \in S_\varepsilon} |(f'_i - f'_j)(g)|$$
$$\leqq \tfrac{1}{2}\delta + c(\varepsilon) \cdot \mathrm{Sup}_{g \in S_\varepsilon} |(f'_i - f'_j)(g)|.$$

From this it appears that the net $(\lambda(f'_\iota))$ is Cauchy, so that $\lim_\iota \lambda(f'_\iota)$ exists finitely. Similar use of (10) shows also that the value of this limit is independent of the chosen net (f'_ι): this value may therefore be taken as the value assigned to $\bar{\lambda}(f')$. Yet a third use of (10) shows that

$$|\bar{\lambda}(f')| \leqq m\varepsilon \cdot \|f'\| + c(\varepsilon) \cdot \operatorname{Sup}_{g \in S_\varepsilon} |f'(g)|$$

for $f' \in B'$, thereby verifying that $\bar{\lambda}$ satisfies the same type of condition as does λ. Thus we may as well assume from the outset that $V = B'$.

On making the assumption $V = B'$, it is evident from (10) that the restriction of λ to each ball in B' is weakly continuous. The alleged result therefore follows from the Banach-Grothendieck theorem (see, for example, [3], Theorem 8.5.1).

REMARKS. (1) If B is separable, and if one chooses a sequence $(f_n)_{n=1}^\infty$ everywhere dense in the unit ball of B, one may replace (10) by

$$(11) \qquad |\lambda(f')| \leqq \varepsilon \cdot \|f'\| + c(\varepsilon) \cdot \sum_{n=1}^\infty 2^{-n} |f'(f_n)|,$$

with a possibly different value for $c(\varepsilon)$.

(2) There is no difficulty in principle (merely some complication in detail) in formulating Theorem 2 for the case in which B is any complete locally convex space.

Supposing (p_α) to be a family of seminorms defining the topology of B, let p'_α be the norm on B' dual to p_α. In place of (i) one would assume that to each index α corresponds a number $m_\alpha \geqq 0$ such that each $f' \in B'$ is the weak limit of a net (f'_ι) extracted from V such that $p'_\alpha(f'_\iota) \leqq m_\alpha \cdot p'_\alpha(f')$. Then the place of (10) would be taken by the demand that to each index α and each number $\varepsilon > 0$ shall correspond a number $c_\alpha(\varepsilon) \geqq 0$ and a finite set $S_{\alpha,\varepsilon} \subset A$ such that

$$(12) \qquad |\lambda(f')| \leqq \varepsilon \cdot p'_\alpha(f') + c_\alpha(\varepsilon) \cdot \operatorname{Sup}_{g \in S_{\alpha,\varepsilon}} |f'(g)| \qquad (f' \in V).$$

In addition, the conclusion remains valid whenever the set V_1 (hitherto assumed to coincide with B'), formed of weak limits f' in B' of nets (f'_ι) extracted from V and satisfying the preceding conditions, is weakly closed in B'. For in this case V_1 is identifiable with the topological dual of $B/V_1^0 = B_1$ (see, for example, [3], Proposition 8.1.2 and Theorem 8.1.5) and we may argue with B_1 and V_1 in place of B and V, respectively, at the same time replacing A by its natural image in the quotient space B_1.

7. Applications of Theorem 2

THEOREM 3. *If G is compact Abelian, a function F on X belongs to $\mathscr{F}L^1(G)$ if and only if to each $\varepsilon > 0$ corresponds a number $c(\varepsilon) \geqq 0$ and a finite subset S_ε of X such that*

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(13) $$\left|\int_X F\cdot\hat{u}d\xi\right| \leqq \varepsilon\cdot\|u\|_\infty + c(\varepsilon)\cdot \text{Sup}_{\xi\in S_\varepsilon}|\hat{u}(\xi)|$$

for all trigonometric polynomials u on G.

PROOF. This is a direct application of Theorem 2 if we take $B = L^1(G)$, $B' = L^\infty(G)$, $A =$ the set of all character functions, $V =$ the set of all trigonometric polynomials, and

(14) $$\lambda(u) = \int_X F\cdot\hat{u}d\xi = \sum_{\xi\in X} F(\xi)\hat{u}(\xi)$$

for trigonometric polynomials u.

THEOREM 4. *If G is compact Abelian, a function F on X belongs to* $\mathscr{F}C(G)$ *if and only if to each* $\varepsilon > 0$ *corresponds a number* $c(\varepsilon) \geqq 0$ *and a finite subset* S_ε *of X such that*

(15) $$\left|\int_X F\cdot\hat{u}d\xi\right| \leqq \varepsilon\cdot\|u\|_1 + c(\varepsilon)\cdot \text{Sup}_{\xi\in S_\varepsilon}|\hat{u}(\xi)|$$

for all trigonometric polynomials u on G.

PROOF. This again is a direct application of Theorem 2. This time we choose $B = C(G)$, $B' = M(G)$, A as before, $V =$ the set of all measures of the form udx where u is a trigonometric polynomial, and $\lambda(udx) = \lambda(u)$ as in (14).

REMARKS.

(1) If it be assumed *a priori* that F is bounded, one may in (13) and (15) allow u to vary over any superspace of the trigonometric polynomials which comprises only functions with absolutely convergent Fourier series.

(2) Whilst it is immediately obvious that any F, which satisfies (13) for each $\varepsilon > 0$ and a suitable $c(\varepsilon)$, also satisfies (1) for any $p \geqq 1$, any $\varepsilon > 0$, and the same $c(\varepsilon)$, the converse implication is not obviously valid (even if different $c(\varepsilon)$'s are allowed). Thus neither Theorem 1 nor Theorem 3 is immediately derivable from the other.

(3) Theorem 3 may also be compared with the wellknown assertion that $F \in \mathscr{F}L^p(G)$ $(1 < p \leqq \infty)$ if and only if

$$\left|\int_X F\cdot\hat{u}d\xi\right| \leqq \text{const.}\ \|u\|_{p'}$$

for each trigonometric polynomial u on G, p' being the conjugate exponent defined by $1/p + 1/p' = 1$.

8. Applications to biorthogonal systems

For simplicity we consider only the case of Banach spaces B. Suppose that the families $(e_k)_{k\in K}$ and $(e'_k)_{k\in K}$ in B and B', respectively, are biorthogo-

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nal, so that $e'_{k'}(e_k) = \delta_{k,k'}$, and that conditions (i) and (ii) are satisfied when we take V [resp. A] to be the set of finite linear combinations of the e'_k [resp. the e_k]. This will certainly be the case if, for example, there exists a net (σ_i) of "summability factors" such that

$$f = \lim_i \sum_{k \in K} \sigma_i(k) \cdot e'_k(f) e_k \qquad (f \in B),$$

$$\text{Sup}_i \| \sum_{k \in K} \sigma_i(k) \cdot e'_k(f) e_k \| \leqq m \cdot \|f\| \qquad (f \in B).$$

Then it follows from Theorem 2 that a given scalar-valued function F on K has the form $F(k) = e'_k(f)$ for some $f \in B$ if and only if to each $\varepsilon > 0$ corresponds a number $c(\varepsilon) \geqq 0$ and a finite set $S_\varepsilon \subset K$ such that

$$| \sum_{k \in K} F(k) f'(e_k) | \leqq \varepsilon \cdot \|f'\| + c(\varepsilon) \cdot \text{Sup}_{k \in S_\varepsilon} |f'(e_k)|$$

for each $f' \in V$.

The proof is exactly like those of Theorem 2 and Theorem 3, and the result is an extension of these theorems to sufficiently regular biorthogonal systems.

References

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