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Small sets with large power sets

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One problem in set theory without the axiom of choice is to find a reasonable way of estimating the size of a non-well-orderable set; in this paper we present evidence which suggests that this may be very hard. Given an arbitrary fixed aleph κ we construct a model of set theory which contains a set X such that if $Y \subseteq X$ then either Y or $X - Y$ is finite, but such that κ can be mapped into $S(S(S(X)))$. So in one sense X is large and in another X is one of the smallest possible infinite sets. (Here $S(X)$ is the power set of X.)

1. Preliminaries

We work in Zermelo-Fraenkel (ZF) set theory, without the axiom of choice but with the axiom of foundation.

Notations. If $f : X \to Y$ and $A \subseteq X$ then:

$f''A = \{y : (\exists x \in A)(f(x) = y)\}$;

$X \overset{*}{\geq} Y$ means that X can be mapped onto Y ;

$A \,\Delta\, B = A \cup B - (A \cap B)$.

We write $|X|$ for the cardinal of X , $S(X)$ for the power set of X , $S_\kappa(X)$ for $\{Y \subseteq X : |Y| < \kappa\}$, $X^{[n]}$ for $\{Y \subseteq X : |Y| = n\}$. ${}^A B$ is the set of functions from A into B ; $B^A = |{}^A B|$. 'X is finite' means that X has n elements for some $n < \omega$.

Relative constructibility. We write L for Gödel's constructible universe. If X is a transitive set, $L(X)$ is the smallest transitive

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proper class which contains X and satisfies ZF. It can be shown that, inside $L(X)$, any element of $L(X)$ can be defined from X, an element of L and a finite number of elements of X.

If X is not transitive, by $L(X)$ we mean $L(\mathrm{TC}(X))$, where

$$\mathrm{TC}(X) = \{X\} \cup X \cup (\bigcup X) \cup (\bigcup\bigcup X) \cup \dots .$$

$\mathrm{TC}(X)$ (the transitive closure of X) is the smallest transitive set with X as a member.

Forcing. We follow Shoenfield [3], but adopt a different convention for names in the forcing language: for $x \in M$ we take x as a name for x, and we take $\mathbf{G}$ as a name for G. We adopt from [3] the notation $H_\kappa(A, B)$ for

$$\{f : \mathrm{dom}(f) \in S_\kappa(A),\ \mathrm{ran}(f) \subseteq B\} .$$

We note the following symmetry lemma.

LEMMA. *Let M be a countable transitive model of* ZF, *$P \in M$ a notion of forcing, $\pi \in M$ an automorphism of P and $\varphi(v_0, v_1)$ a* ZF-*formula. Then*

$$p \Vdash \varphi(\mathbf{G}, x) \leftrightarrow \pi^{-1}p \Vdash \varphi(\pi''\mathbf{G}, x)$$

where $x \in M$ and $p \in P$.

2. Dedekind-finite sets

In this section all proofs are carried out in ZF; no use is made of the axiom of choice.

A *Dedekind-finite* (DF) set is defined to be a set not equinumerous with any of its proper subsets; a DF cardinal is the cardinal of a DF set. In the absence of the axiom of choice infinite DF sets may exist.

LEMMA 2.1. *The following are equivalent;*

(i) X *is* DF;

(ii) $|X| \neq |X| - 1$;

(iii) $\omega \nleq |X|$.

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LEMMA 2.2. *For an arbitrary set* X , $\omega \leq^* X$ *iff* $\omega \leq 2^X$.

Proof. This is due to Kuratowski ([4], p. 94-95). We say a set X is *quasi-minimal* (QM) if X is infinite but has only finite and cofinite subsets ($Y \subseteq X$ is cofinite if $X - Y$ is finite). Clearly a QM set is DF; in fact it is obvious that X QM $\to X \not\geq^* \omega$.

The name 'quasi-minimal' (due to Hickman) arises as follows. By Lemma 2.1 *(ii)* the only cardinal minimal among the infinite cardinals is ω . However we put an equivalence relation on infinite cardinals thus: $m \equiv m'$ if there is $n < \omega$ such that either $m + n = m'$ or $m' + n = m$. Write $[m]$ for the equivalence class of m , and set $[m] \leq_1 [m']$ if $m \leq m'$. Then $[m]$ is minimal under the partial order $\leq_1$ iff $m = \omega$ or m is QM.

LEMMA 2.3. *If* $X \not\geq^* \omega$ *and* $n < \omega$ *then* $X^{[n]} \not\geq^* \omega$.

Proof. It is straightforward to prove that if $Y \not\geq^* \omega$ and $Z \not\geq^* \omega$ then $Y \times Z \not\geq^* \omega$. So $X^n \not\geq^* \omega$, and trivially $X^n \geq^* X^{[n]}$.

THEOREM 2.4. *Let* X *be* QM, κ *an aleph.*

(i) $\kappa \leq |X| \to \kappa < \omega$.

(ii) $\kappa \leq |S(X)| \to \kappa < \omega$.

(iii) $\kappa \leq |S(S(X))| \to \kappa \leq 2^\omega$.

Proof. Since $X \not\geq^* \omega$ it follows from Lemma 2.2 that X and $S(X)$ are both DF, which establishes *(i)* and *(ii)*.

We note that $|S(X)| = 2.|S_\omega(X)|$ (this may be seen by associating each infinite subset of X with its complement), and so $|S(S(X))| = |S(S_\omega(X))|^2$. To establish *(iii)* it then suffices to show

$$\kappa \leq |S(S_\omega(X))| \to \kappa \leq 2^\omega .$$

For if λ is an aleph, m any infinite cardinal and $\lambda \leq m^2$, then $\lambda \leq m$ (see [2], Lemma 6.13, p. 55).

Suppose then that $f : \kappa \to S(S_\omega(X))$ is one-to-one. Set

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$f_n(\alpha) = f(\alpha) \cap X^{[n]}$. Now $\{f_n(\alpha) : \alpha \in \kappa\} \subseteq S(X^{[n]})$, and by Lemma 2.3, $X^{[n]} \not\leq^* \omega$, so by Lemma 2.2, $S(X^{[n]})$, and thus $\{f_n(\alpha) : \alpha \in \kappa\}$, is DF. However $\{f_n(\alpha) : \alpha \in \kappa\}$ has a canonical well-order and so is finite, and can be canonically mapped into ω . Combining these canonical maps for each n yields a one-to-one map of $A = \{f_n(\alpha) : \alpha \in \kappa$ and $n < \omega\}$ into $\omega \times \omega$. Now $f(\alpha)$ is determined by $(f_n(\alpha))_{n<\omega}$, which is an ω-sequence of elements of A , and so $f(\alpha)$ can be associated with an element of ${}^{\omega}(\omega\times\omega)$. It follows that κ can be mapped one-to-one into ${}^{\omega}(\omega\times\omega)$, and so $\kappa \leq 2^{\omega}$.

3. A large QM set

In this section we construct the model promised in the abstract. Theorem 2.4 shows why we have to look at $S(S(S(X)))$ rather than some smaller power of X .

Let M be a countable transitive model of ZF + $V = L$, κ a $(\text{successor aleph})^M$. Then $M \models 2^{\lambda} = \kappa$ for some aleph λ of M . We take $\left[H_\kappa\left(({}^{\lambda}2)\times\kappa,\ 2\right)\right]^M$ as our notion of forcing, with the partial order defined by $p \leq q$ iff $p \supseteq q$. Let G be generic over M with respect to this notion.

LEMMA 3.1. *(i)* M *and* $M[G]$ *have the same cofinality* (cf) *function and the same alephs.*

(ii) For $\alpha < \kappa$ *and* $x \in M$, $({}^{\alpha}x)^M = ({}^{\alpha}x)^{M[G]}$.

Proof. We note that κ is $(\text{regular})^M$. We assume the terms 'μ-closed' and 'μ-chain condition' from [3], §10. Our notion of forcing satisfies the κ^+-chain condition (by [3], Lemma 10.3) and is κ-closed, so our results follow from [3], Lemma 10.2, and Lemma 10.6 and Corollary.

We now work in $M[G]$ unless otherwise stated. For $f \in {}^{\lambda}2$ set $G(f) = \bigcup\{p(f) : p \in G\}$. Note that by $p(f)$ we mean

$\{\langle\alpha, \beta\rangle : \langle\langle f, \alpha\rangle, \beta\rangle \in p\}$. Set $G^* = \{G(f) : f \in {}^{\lambda}2\}$. It can be shown by standard arguments that each $G(f)$ is a member of ${}^{\kappa}2$ and that if $f \neq g$ then

$$|\{i < \kappa : (G(f))(i) \neq (G(g))(i)\}| = \kappa .$$

For $r \in {}^{\kappa}2$ set

$$[r] = \{s \in {}^{\kappa}2 : |\{i < \kappa : s(i) \neq r(i)\}| < \kappa\} .$$

Set $X = \{[r] : r \in G^*\}$. Define for $\alpha < \lambda$,

$$Y_\alpha = \{\{[G(f)], [G(f')]\} : f, f' \in {}^{\lambda}2, f'(\alpha) = 1-f(\alpha) \text{ and } f'(\beta) = f(\beta) \text{ for } \beta \neq \alpha\} .$$

Then Y_α is a partition of X into disjoint two-element subsets, and $\alpha \neq \beta \rightarrow Y_\alpha \cap Y_\beta = \emptyset$. Set

$$Y = \{K : K \text{ is a partition of } X \text{ into two-element subsets and } K-Y_\alpha \text{ is finite for some } \alpha < \lambda\} ,$$

$$Z = \{\langle K, \alpha\rangle : K \in Y \text{ and } K-Y_\alpha \text{ is finite}\} .$$

The model which is to contain the large QM set is $N = (L(Z))^{M[G]}$.

The motivation of the construction is as follows. If we set $N' = (L(X))^{M[G]}$ it can be readily shown that $N' \models X$ is QM. In constructing N we add enough sets to N' to make X large in the desired sense, but not enough to destroy the quasi-minimality of X .

LEMMA 3.2. *(i)* M, N *and* $M[G]$ *have the same* cf *function and alephs.*

(ii) For $\alpha < \kappa$ *and* $x \in M$, $({}^{\alpha}x)^M = ({}^{\alpha}x)^N = ({}^{\alpha}x)^{M[G]}$.

(iii) $N \models 2^\lambda = \kappa$.

Proof. *(i)* and *(ii)* are immediate from Lemma 3.1, since $M \subset N \subset M[G]$; *(iii)* is essentially just a special case of *(ii)*.

THEOREM 3.3. $N \models \kappa \leq |S(S(S(X)))|$.

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Proof. Clearly in N, $Y \subseteq S(X^{[2]})$, and also $Z : Y \to \lambda$ is onto. So $N \models S(S(X)) \,{}^{*}\!\geq \lambda$. It follows that

$$N \models 2^{\lambda} \leq |S(S(S(X)))|$$

and the result follows by Lemma 3.2 *(iii)*.

We now look at $\mathrm{TC}(Z)$. If $x \in \mathrm{TC}(Z)$ then either $x = Z$ or $x = \langle K, \beta \rangle$ for some $K \in Y$ and $\beta < \lambda$ or $x \in Y$ or ... or $x \in M$. It can easily be seen that in all cases either $x = Z$ or x is codable by (at worst) some Y_{α}, a finite number of elements of G^* and an element of M. We recall from §1 that inside N every element of N is definable from $\mathrm{TC}(Z)$, an element of M $\left(= (L)^{M[G]}\right)$ and a finite number of elements of $\mathrm{TC}(Z)$. By using the coding just mentioned we have that inside N any element of N is definable from Z, a finite number of Y_{α}'s, a finite number of elements of G^*, and an element of M (as $\mathrm{TC}(Z)$ is definable from Z).

We proceed to a continuity lemma, but first introduce some notation. Suppose A is a set, $s \subseteq A$ and $f : A \to 2$. We define $f^{s} : A \to 2$ thus:

$$f^{s}(a) = f(a) \text{ if } a \notin s \, ; \quad f^{s}(a) = 1 - f(a) \text{ if } a \in s \, .$$

LEMMA 3.4. *Suppose that*

$$N \models \varphi\left(Z, Y_{\alpha_1}, \ldots, Y_{\alpha_n}, G(f_1), \ldots, G(f_m), x, [G(f)]\right),$$

where $x \in M$ *and* $f \neq f_i^{s}$ *for* $1 \leq i \leq m$ *and any* $s \subseteq \{\alpha_1, \ldots, \alpha_n\}$. *Let* $g \in {}^{\lambda}2$ *be any function such that* $g \neq f_i^{s}$ *for* $1 \leq i \leq m$ *and* $s \subseteq \{\alpha_1, \ldots, \alpha_n\}$. *Then*

$$N \models \varphi\left(Z, Y_{\alpha_1}, \ldots, Y_{\alpha_n}, G(f_1), \ldots, G(f_m), x, [G(g)]\right).$$

Proof. Let ψ be a formula such that

$$M[G] \models \psi(G, y) \leftrightarrow N \models \varphi\left(Z, Y_{\alpha_1}, \ldots, Y_{\alpha_n}, G(f_1), \ldots, G(f_m), x, [G(f)]\right).$$

ψ 'describes' the construction of Z, Y_{α_1}, and so on, from G and also relativizes φ to the class N. Here $y \in M$. Take $p \in G$ such that $p \Vdash \psi(G, y)$. Set, for $s \subseteq \{\alpha_1, \ldots, \alpha_n\}$,

$$A_s = \{i \in \operatorname{dom}(p(f^s)) : p(f^s, i) \neq (G(g^s))(i)\},$$

$$B_s = \{i \in \operatorname{dom}(p(g^s)) : p(g^s, i) \neq (G(f^s))(i)\}.$$

Then $A_s, B_s \in S_\kappa(\kappa)$ and so by Lemma 3.2 *(ii)*, $A_s, B_s \in M$.

For $A \in S_\kappa(\kappa)$ we define an automorphism σ_A of $H_\kappa(\kappa, 2)$ thus:

$$\left.\begin{array}{l} (\sigma_A(t))(i) = 1 - t(i) \quad \text{if} \quad i \in \operatorname{dom}(t) \cap A \\ (\sigma_A(t))(i) = t(i) \quad \text{if} \quad i \in \operatorname{dom}(t) - A \end{array}\right\} \quad \text{for} \quad t \in H_\kappa(\kappa, 2)$$

and $\operatorname{dom}(\sigma_A(t)) = \operatorname{dom}(t)$.

We define an automorphism π of $H_\kappa(({}^\lambda 2) \times \kappa, 2)$ thus:

$$\left.\begin{array}{l} (\pi p)(f^s) = \sigma_{A_s}(p(g^s)) \\ (\pi p)(g^s) = \sigma_{B_s}(p(f^s)) \end{array}\right\} \quad \text{for} \quad s \subseteq \{\alpha_1, \ldots, \alpha_n\},$$

$$(\pi p)(h) = p(h) \quad \text{for} \quad h \neq f^s, g^s \quad \text{for any} \quad s \subseteq \{\alpha_1, \ldots, \alpha_n\}.$$

Then $\pi \in M$, and

$$(\pi^{-1}p)(f^s) = \sigma_{B_s}(p(g^s)) \subset G(f^s),$$

$$(\pi^{-1}p)(g^s) = \sigma_{A_s}(p(f^s)) \subset G(g^s),$$

$$(\pi^{-1}p)(h) = p(h) \subset G(h) \quad (\text{for} \quad h \neq f^s, g^s).$$

It follows that $\pi^{-1}p \in G$. Now $p \Vdash \psi(G, y)$, so by the symmetry lemma of §1, $\pi^{-1}p \Vdash \psi(\pi''G, y)$, whence

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$$M[G] \models \psi(\pi''G,\ y)\ .$$

Now

$$(\pi''G)\left(f^s\right) = \sigma_{A_s}\left(G\left(g^s\right)\right)\ ,$$

$$(\pi''G)\left(g^s\right) = \sigma_{B_s}\left(G\left(f^s\right)\right)\ ,$$

$$(\pi''G)(h) = G(h)\ ,\quad h \neq f^s,\ g^s \quad \text{for any} \quad s \subseteq \{\alpha_1,\ \ldots,\ \alpha_n\}\ .$$

So $\left[(\pi''G)\left(f^s\right)\right] = \left[G\left(g^s\right)\right]$, $\left[(\pi''G)\left(g^s\right)\right] = \left[G\left(f^s\right)\right]$, and $[(\pi''G)(h)] = [G(h)]$ for $h \neq f^s,\ g^s$.

Thus the change from G to $\pi''G$ leaves X , and thence Y and Z , unchanged, leaves $G\left(f_1\right),\ \ldots,\ G\left(f_m\right)$ unchanged and carries $[G(f)]$ to $[G(g)]$. Take $\alpha \in \{\alpha_1,\ \ldots,\ \alpha_n\}$; for $s \subseteq \{\alpha_1,\ \ldots,\ \alpha_n\}$ set $s' = s \vartriangle \{\alpha\}$. Then $\left\{\left[G\left(f^s\right)\right],\ \left[G\left(f^{s'}\right)\right]\right\} \in Y_\alpha$ and $\left\{\left[G\left(g^s\right)\right],\ \left[G\left(g^{s'}\right)\right]\right\} \in Y_\alpha$. The change from G to $\pi''G$ carries each of these pairs to the other, so Y_α is carried into itself.

In conclusion

$$M[G] \models \psi(\pi''G,\ y) \leftrightarrow N \models \varphi\left(Z,\ Y_{\alpha_1},\ \ldots,\ Y_{\alpha_n},\ G\left(f_1\right),\ \ldots,\ G\left(f_m\right),\ x,\ [G(g)]\right)\ .$$

Since $M[G] \models \psi(\pi''G,\ y)$, the proof is complete.

THEOREM 3.5. $N \models X$ *is* QM.

Proof. We work in N . Suppose that

$$N \models A \text{ is an infinite subset of } X\ .$$

By the remarks after Theorem 3.3 we may assume that A is defined in terms of $Z,\ Y_{\alpha_1},\ \ldots,\ Y_{\alpha_n},\ G\left(f_1\right),\ \ldots,\ G\left(f_m\right)$ say, and $x \in M$. Take $a \in A - \left\{\left[G\left(f_i^s\right)\right] : 1 \leq i \leq m \text{ and } s \subseteq \{\alpha_1,\ \ldots,\ \alpha_n\}\right\}$. Now $a = [G(f)]$ for some f , so the sentence '$a \in A$' may be written in the form

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$$N \models \varphi\left(Z, Y_{\alpha_1}, \ldots, Y_{\alpha_n}, G(f_1), \ldots, G(f_m), x, [G(f)]\right) .$$

Application of Lemma 3.4 shows that

$$N \models A \supseteq X - \left\{\left[G\left(f_i^s\right)\right] : 1 \le i \le m \text{ and } s \subseteq \{\alpha_1, \ldots, \alpha_n\}\right\} ,$$

so $N \models A$ is cofinite.

In conclusion we have shown that for κ an arbitrary aleph it is possible to have a QM set X such that $\kappa < |S(S(S(X)))|$. This result is one of a series: Hickman [1] and the author (PhD thesis, University of Bristol, 1971) have independently shown that it is possible to have a DF set X such that $\kappa < |S(X)|$ (again for κ an arbitrary aleph); indeed Hickman obtains $\kappa < |X^{[2]}|$. Also the author (unpublished) has shown that it is possible to have a set X such that $X \ {}^*\!\!\not\geq \omega$ (whence by Lemma 2.2, $S(X)$ is DF) but such that $\kappa < |S(S(X))|$. It should be emphasised that none of these results have anything to do with the possibility that 2^{ω} can be large; in all the models concerned if κ, λ are alephs then

$$\kappa \le 2^{\lambda} \rightarrow \kappa \le \lambda^{+} .$$

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