BULL. AUSTRAL. MATH. SOC. 40C05, 46J05
VOL. 23 (1981), 191-194.

ON A THEOREM OF BRUDNO OVER NON-ARCHIMEDIAN FIELDS

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A classical theorem of Brudno, dealing with the consistency of summability with regular matrices is shown by example not to hold over a non-archimedian field.

1.

Following Monna [1], attempts have been made in recent times to study different summability methods over non-archimedian fields which are complete in the metric of valuation. In all such attempts, as in [3], [4], significant differences in contrast to the classical case have been obtained. The object of the present short note is to prove by an example that the classical theorem of Brudno [2] dealing with the consistency of regular matrices is not true in general in the non-archimedian case. In §2, we shall describe the necessary preliminaries, where as in §3, we shall establish our claim.

2.

Let K be a non-archimedian field which is complete under the metric of valuation denoted by $|\ |$. We note that the valuation $|\ |$ is non-archimedian if and only if $|n| < 1$ for every integer n considered as an element of K . Thus, in a field with non-trivial non-archimedian valuation, the sequence $\{1, 2, 3, \ldots\} = \{n\}$ is a bounded sequence in the metric of valuation.

Received 18 September 1980.

https://doi.org/10.1017/S0004972700007036 Published online by Cambridge University Press

Let $A = (a_{np})$, $n, p = 1, 2, 3, \ldots$, be a matrix defined over such a field. For $n = 1, 2, 3, \ldots$, let us write

$$y_n = \sum_{p=1}^{\infty} a_{np}x_p .$$

For every sequence $x = \{x_n\}$ defined over K , let $\{y_n\}$ be convergent for each n . y_n is called the A-transform of x . If $y_n \to y$ as $n \to \infty$ in the metric of valuation, then x is said to be A-summable to y . A is said to be convergence preserving if $\lim_{n\to\infty} y_n$ exists for every convergent sequence x . A is called regular if in addition $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$. Such regular matrices are also known as Toeplitz matrices. The theorem given below is practically contained in [1].

THEOREM (Monna). *A matrix* $A = (a_{np})$ *is a regular matrix defined over* K *if and only if* $\sup_{n,p} |a_{np}| \le M$ *where* M *is a constant,*

$\lim_{n\to\infty} a_{np} = 0$ *for every fixed* p , $\sum_{p=1}^{\infty} a_{np} = A_n \to 1$ *as* $n \to \infty$.

The following is the classical Brudno theorem on a regular matrix for which a simple proof was given by Petersen [2].

THEOREM (Petersen). *Let every bounded sequence summable by a Toeplitz matrix* A *also be summable by a Toeplitz matrix* B . *Then it is summable to the same value by* B *as by* A .

Petersen [2] established this theorem by showing that if two regular matrix methods $A = (a_{mn})$ and $B = (b_{mn})$ sum bounded sequence $\{s_n\}$ to different sums, then there exists a bounded sequence which is summed by A but not by B .

3.

In this section we shall give examples of two regular matrices A and B over K such that every bounded sequence summed by A is also summed by B and show that there exists a bounded sequence summable by these two regular matrices to two different sums.

Let $A = (a_{np})$ and $B = (b_{np})$ be defined as follows:

$$a_{np} = \begin{cases} n+1 & \text{when } p = n , \\ -n & \text{when } p = n+1 , \\ 0 & \text{for all other values of } n \text{ and } p ; \end{cases}$$

$$b_{np} = \begin{cases} n+2 & \text{when } p = n , \\ -(n+1) & \text{when } p = n+1 , \\ 0 & \text{for all other values of } n \text{ and } p . \end{cases}$$

The matrix A satisfies the conditions of the theorem of Monna given in §2 as seen below.

(i) Since $|n+1| = \text{Max}\{|n|, 1\}$ and $|n| < 1$, we have $|n+1| = 1$. Hence we have from this $\sup_{n,p} |a_{np}| \le \sup\{|n+1|, |n|\} = 1$.

(ii) Since each column of A contains infinitely many zeros and $|n+1| = 1$ and $|n| < 1$, $a_{np} \to 0$ as $n \to \infty$.

(iii) $\sum_{p=1}^{\infty} a_{np} = n + 1 - n = 1 \to 1$ as $n \to \infty$.

Hence $A = (a_{np})$ is a regular matrix. In a similar manner, we can verify that B is also a regular matrix over K.

As a next step, we shall show that every bounded sequence summed by A is also summed by B. For this let $\{x_n\}$ be any bounded sequence. If y_n is the A-transform of x_n, then we have $y_n = (n+1)x_n - nx_{n+1}$. If y'_n is the B-transform of x_n, then

$$y'_n = (n+2)x_n - (n+1)x_{n+1} .$$

The relation between y_n and y'_n is easily seen to be

$$y'_n = y_n + (x_n - x_{n+1}) .$$

Hence $|y_n|' \le \text{Max}\{|y_n|, |x_n - x_{n+1}|\} \le |y_n| + \lambda$ where $|x_n| \le \lambda$ for all n, λ being a constant. Thus we have $|y'_n| \le |y_n| + \lambda$.

If $\{y_n\}$ is convergent, then $\{y'_n\}$ is also convergent. Thus if

https://doi.org/10.1017/S0004972700007036 Published online by Cambridge University Press

$\{x_n\}$ is summable by A, then it is summable by B also. This shows that the bounded convergence field of A is contained in the bounded convergence field of B.

We establish our claim by showing that there exists a bounded sequence summable by these two regular matrices to two different sums. For this consider the bounded sequence $N = \{n\} = \{1, 2, 3, \ldots\}$ in K. The A-transform of the sequence N gives rise to the sequence $\{y_n\} = \{0, 0, 0, 0, \ldots\}$. So N is A-summable to 0. The B-transform of the sequence N gives rise to the sequence $\{y_n'\} = \{-1, -1, -1, \ldots\}$. So N is B-summable to -1. Hence given two regular methods A and B defined over K such that every bounded sequence summed by A is also summed by B, there exists a bounded sequence $N = \{n\}$ summable by A and B to two different sums which cannot happen in the case of the classical Brudno's theorem. This establishes our claim.

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