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THE STRUCTURE OF STABLE COMPONENTS

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ABSTRACT. Let A be an artin algebra. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of A. If $\mathfrak{C}$ is periodic, then the structure of $\mathfrak{C}$ is known. Here, we are going to consider the case when $\mathfrak{C}$ is non-periodic: we will show that $\mathfrak{C}$ is isomorphic to $\mathbf{Z}\mathfrak{S}$ with $\mathfrak{S}$ a valued quiver. In particular, there is no cyclic path in $\mathfrak{C}$.

1. **The main result.** For basic concepts and notations in representation theory of algebras, we refer to [R]. Now we recall some of them. A *quiver* $Q = \big(V(Q), A(Q), s_Q, e_Q\big)$ is given by two sets $V(Q)$, $A(Q)$, and two maps $s_Q, e_Q \colon A(Q) \longrightarrow V(Q)$. The elements of $V(Q)$ are called vertices or points, those of $A(Q)$ arrows. If $\alpha \in A(Q)$, then $s_Q(\alpha)$ is called its start vertex, $e_Q(\alpha)$ its end vertex, and we write $\alpha \colon s_Q(\alpha) \longrightarrow e_Q(\alpha)$. A quiver Q is said to have no multiple arrows provided for any pair x, y of vertices there is at most one arrow $\alpha \colon x \longrightarrow y$. Let Q be a quiver and $x \in V(Q)$, then x^+ is the set of end points with start point x, and x^- is the set of start points of arrows with end point x. We say that Q is locally finite provided both x^+ and x^- are finite sets, for any $x \in V(Q)$. A valued quiver is of the form $\mathfrak{S} = \big(V(\mathfrak{S}), A(\mathfrak{S}), s_{\mathfrak{S}}, e_{\mathfrak{S}}, d_{\mathfrak{S}}, d'_{\mathfrak{S}}\big)$, where $\big(V(\mathfrak{S}), A(\mathfrak{S}), s_{\mathfrak{S}}, e_{\mathfrak{S}}\big)$ is a quiver without multiple arrows, and $d_{\mathfrak{S}}, d'_{\mathfrak{S}} \colon A(\mathfrak{S}) \longrightarrow \mathbb{N}_1$ are maps. In case $d_{\mathfrak{S}}(\alpha) = 1 = d'_{\mathfrak{S}}(\alpha)$ for all $\alpha \in A(\mathfrak{S})$, then we say that $\mathfrak{S}$ has trivial valuation, and we may consider any quiver as a valued quiver with trivial valuation. For $\alpha \colon x \longrightarrow y$ in $A(\mathfrak{S})$, we write $d_{xy} = d_{\mathfrak{S}}(\alpha)$, $d'_{xy} = d'_{\mathfrak{S}}(\alpha)$.

A *stable translation quiver* $\Gamma = \big(V(\Gamma), A(\Gamma), s_\Gamma, e_\Gamma, \tau_\Gamma\big)$ is given by a quiver $\big(V(\Gamma), A(\Gamma), s_\Gamma, e_\Gamma\big)$ which is locally finite and has no multiple arrows, and a bijection $\tau_\Gamma \colon V(\Gamma) \longrightarrow V(\Gamma)$, satisfying $z^- = (\tau_\Gamma z)^+$ for any $z \in V(\Gamma)$. (Note that we allow loops!) Given an arrow $\alpha \colon y \longrightarrow z$, we denote by $\sigma\alpha$ the unique arrow: $\tau z \longrightarrow y$. A valued stable translation quiver is of the form $\mathfrak{C} = \big(V(\mathfrak{C}), A(\mathfrak{C}), s_{\mathfrak{C}}, e_{\mathfrak{C}}, \tau_{\mathfrak{C}}, d_{\mathfrak{C}}, d'_{\mathfrak{C}}\big)$, where, on the one hand, $\big(V(\mathfrak{C}), A(\mathfrak{C}), s_{\mathfrak{C}}, e_{\mathfrak{C}}, d_{\mathfrak{C}}, d'_{\mathfrak{C}}\big)$ is a valued quiver, whereas $\big(V(\mathfrak{C}), A(\mathfrak{C}), s_{\mathfrak{C}}, e_{\mathfrak{C}}, \tau_{\mathfrak{C}}\big)$ is a stable translation quiver, and $d_{\mathfrak{C}}(\alpha) = d'_{\mathfrak{C}}(\sigma\alpha)$, $d'_{\mathfrak{C}}(\alpha) = d_{\mathfrak{C}}(\sigma\alpha)$ for any $\alpha \in A(\mathfrak{C})$. A valued stable translation quiver $\mathfrak{C}$ is said to be connected provided $V(\mathfrak{C}) \neq \emptyset$, and $\mathfrak{C}$ cannot be written as a disjoint union of non-empty valued stable translation quivers. Let $\mathfrak{C}$ be a connected valued stable translation quiver. A vertex $x \in V(\mathfrak{C})$ is said to be periodic provided there is $t > 0$ with $x = \tau^t x$. Note that the existence of any periodic vertex implies that all vertices are periodic and, in this case, $\mathfrak{C}$ is said to be periodic, otherwise non-periodic [HPR]. We say that $\mathfrak{C}$ is *smooth* provided its valuation is trivial and x^+ consists of precisely two vertices, for any $x \in V(\mathfrak{C})$. A *subadditive*

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function ℓ on $\mathfrak{C}$ with values in $\mathbb{N}_0$ is a map $\ell\colon V(\mathfrak{C}) \longrightarrow \mathbb{N}_0$ satisfying

$$\ell(z) + \ell(\tau z) \geq \sum_{y \in z^-} d'_{yz}\ell(y), \text{ for all } z \in V(\mathfrak{C});$$

and ℓ is said to be additive provided we always have equality.

Given a valued quiver $\mathfrak{S}$, we construct a valued stable translation quiver $\mathbb{Z}\mathfrak{S}$ (following Riedtmann [Ri]) as follows: let $V(\mathbb{Z}\mathfrak{S}) = \mathbb{Z} \times V(\mathfrak{S})$; given an arrow $\alpha\colon x \longrightarrow y$ in $\mathfrak{S}$, there are arrows $(i,\alpha)\colon (i,x) \longrightarrow (i,y)$ and $\sigma(i,\alpha)\colon (i-1,y) \longrightarrow (i,x)$, for all $i \in \mathbb{Z}$; the translation τ is defined by $\tau(i,x) = (i-1,x)$, the valuations are given by $d_{\mathbb{Z}\mathfrak{S}}(i,\alpha) = d_{\mathfrak{S}}(\alpha), d'_{\mathbb{Z}\mathfrak{S}}(i,\alpha) = d'(\alpha)$.

THEOREM. *Let $\mathfrak{C}$ be a non-periodic connected valued stable translation quiver with a non-zero subadditive function ℓ with values in $\mathbb{N}_0$. Then, either $\mathfrak{C}$ is smooth and ℓ is both additive and bounded, or else $\mathfrak{C}$ is of the form $\mathbb{Z}\mathfrak{S}$ for some valued quiver $\mathfrak{S}$.*

The precise structure of those stable translation quivers $\mathfrak{C}$ which are not of the form $\mathbb{Z}\mathfrak{S}$ for some valued quiver $\mathfrak{S}$, will be given below, see Section 4; they are of the form Π_{st} with $-s < t < 0$.

Let Q be a quiver. A *path* in Q of length $t \geq 1$ is of the form $(x \mid \alpha_1, \ldots, \alpha_t \mid y)$ with $s_Q(\alpha_1) = x, s_Q(\alpha_i) = e_Q(\alpha_{i-1})$ for $2 \leq i \leq t$, and $e_Q(\alpha_t) = y$; in case $x = y$, this is called a cyclic path. If Γ is a translation quiver, a path $(x \mid \alpha_1, \ldots, \alpha_t \mid y)$ in Γ is said to be sectional, provided $\sigma\alpha_i \neq \alpha_{i-1}$, for $2 \leq i \leq t$. An important property of valued stable translation quiver of the form $\mathbb{Z}\mathfrak{S}$ is the following:

LEMMA. *Let $\mathfrak{S}$ be a valued quiver. Then any cyclic path in $\mathbb{Z}\mathfrak{S}$ is sectional. If $\mathfrak{S}$ has no cyclic path, then $\mathbb{Z}\mathfrak{S}$ has no cyclic path.*

PROOF. Given $x, y \in V(\mathfrak{S})$ and $i, j \in \mathbb{Z}$ with $i > j$, there is no path from (i,x) to (j,y) in $\mathbb{Z}\mathfrak{S}$. Thus any cyclic path in $\mathbb{Z}\mathfrak{S}$ involves only arrows of the form (i,α) with $\alpha \in A(\mathfrak{S})$ and some fixed $i \in \mathbb{Z}$, and thus it is a sectional path and is obtained from a cyclic path in $\mathfrak{S}$.

COROLLARY. *Let $\mathfrak{C}$ be a non-periodic component of the stable Auslander-Reiten quiver of an artin algebra A. Then $\mathfrak{C}$ is of the form $\mathbb{Z}\mathfrak{S}$ for some valued quiver $\mathfrak{S}$ without cyclic path. In particular, $\mathfrak{C}$ has no cyclic path.*

PROOF OF THE COROLLARY. We note that $\mathfrak{C}$ is a non-periodic connected valued stable translation quiver (see [HPR]), and the length function ℓ is a subadditive function on $\mathfrak{C}$ with values in $\mathbb{N}_1$. Either $\mathfrak{C}$ is a complete component of Auslander-Reiten quiver of A, then ℓ is additive and unbounded (by a theorem of Auslander [A], since $\mathfrak{C}$ cannot be finite), or else ℓ is not additive. It follows that $\mathfrak{C}$ is of the form $\mathbb{Z}\mathfrak{S}$ for some valued quiver $\mathfrak{S}$. Any cyclic path in $\mathfrak{S}$ would yield a sectional cyclic path in $\mathfrak{C}$, but this is impossible according to a theorem of Bautista-Smalø[BS]. The last assertion is a direct consequence of the Lemma.

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We recall that the structure of the periodic components of the stable Auslander-Reiten quiver of an artin algebra has been determined by Happel-Preiser-Ringel [HPR]. In particular, we see that a regular component of the Auslander-Reiten quiver of an artin algebra is either a stable tube or else of the form $\mathbb{Z}\mathfrak{S}$, with $\mathfrak{S}$ a valued quiver.

For the proof of the main theorem, we may assume that we deal with a stable translation quiver with trivial valuation. For, asssume the assertion of the main theorm has been shown for all $\mathfrak{C}$ with trivial valuation, and consider now a general $\mathfrak{C}$. Let $\Gamma = \big(V(\mathfrak{C}), A(\mathfrak{C}), s_{\mathfrak{C}}, e_{\mathfrak{C}}, \tau_{\mathfrak{C}}\big)$ be the corresponding stable translation quiver (with trivial valuation). Since ℓ is subadditive on $\mathfrak{C}$, it is subadditive on Γ. We can assume that the valuation of $\mathfrak{C}$ is non-trivial, thus ℓ cannot be additive on Γ (assume $d'_{yz} > 1$, for some arrow $y \longrightarrow z$; according to Lemma 5.4 below, we find t such that $\ell(\tau^t y) \neq 0$, thus $\sum_{y\in z^-} d'_{yz}\ell(\tau^t y) > \sum_{y\in z^-} \ell(\tau^t y)$). This shows that $\Gamma = \mathbb{Z}Q$ for some quiver Q, according to the main theorem. Of course, we can transfer the valuation of $\mathfrak{C}$ to Q in order to obtain a valued quiver $\mathfrak{S} = \big(V(Q), A(Q), s_Q, e_Q, d_{\mathfrak{S}}, d'_{\mathfrak{S}}\big)$, namely let $d_{\mathfrak{S}}(\alpha) = d_{\mathfrak{C}}(0,\alpha), d'_{\mathfrak{S}}(\alpha) = d'_{\mathfrak{C}}(0,\alpha)$ for $\alpha \in A(Q)$ (and $(0,\alpha) \in A(\mathbb{Z}Q) = A(\Gamma) = A(\mathfrak{C})\big)$, then $\mathfrak{C} = \mathbb{Z}\mathfrak{S}$.

2. **Preliminaries: quivers and graphs.** Let Q be a quiver. An arrow α with $s_Q(\alpha) = e_Q(\alpha)$ is called a *loop,* and we denote by $L(Q)$ the set of all loops of Q. The paths of length ≥ 1 have been defined in Section 1; we should add that we also have to consider paths of length 0, they are of the form $(x|x)$ with $x \in V(Q)$. Note that any arrow $\alpha: x \longrightarrow y$ may be considered as a path $\alpha = (x \mid \alpha \mid y)$ of length 1. Given two paths $w = (x \mid \alpha_1, \dots, \alpha_t \mid y)$ and $w' = (x' \mid \beta_1, \dots, \beta_s \mid y')$ in Q, the product ww' is defined provided $y = x'$, and then $ww' = (x \mid \alpha_1, \dots, \alpha_t, \beta_1, \dots, \beta_s \mid y')$. If $w = (x \mid \alpha_1, \dots, \alpha_t \mid y)$ is a path, we write $s_Q(w) = x$, and $e_Q(w) = y$, and say that w is a path from x to y in case $x = y$, and the length of w is at least 1, then w is called a cyclic path (starting at x). A cyclic path $(x \mid \alpha_1, \dots, \alpha_t \mid y)$ is said to be *elementary* provided $s_Q(\alpha_i) \neq s_Q(\alpha_j)$ for $i \neq j$. If $\mathcal{U}$ is a subset of $A(Q)$, we denote by $\langle \mathcal{U} \rangle$ the smallest subquiver of Q containing $\mathcal{U}$, thus $V(\langle \mathcal{U} \rangle)$ is the set of all start vertices and all end vertices of arrows in $\mathcal{U}$, and $A(\langle \mathcal{U} \rangle) = \mathcal{U}$.

A *graph* $Y = \big(V(Y), A(Y), s_Y, e_Y, \iota_Y\big)$ is given by a quiver $\big(V(Y), A(Y), s_Y, e_Y\big)$ and a fixpoint free involution ι_Y of $A(Y)$. Thus $\iota_Y: A(Y) \longrightarrow A(Y)$ is a map with $\iota_Y(\alpha) \neq \alpha$ and $\iota_Y^2(\alpha) = \alpha$ for all $\alpha \in A(Y)$. A path $(x \mid \alpha_1, \dots, \alpha_t \mid y)$ in Y is said to be *reduced* provided $\alpha_{i+1} \neq \iota\alpha_i$ for all $1 \leq i \leq t-1$. A graph Y is called a *tree* provided for every pair $x, y \in V(Y)$ there is a unique reduced path from x to y, and, in this case, the unique reduced path from x to y is called the *geodesic* from x to y [D]. We will consider cyclic paths which are both reduced and elementary. Note that for a reduced, elementary, cyclic path $(x \mid \alpha_1, \dots, \alpha_t \mid x)$ in Y, we also have $\alpha_1 \neq \iota\alpha_t$ (this is trivially true for $t = 1$; if $t \geq 2$ and $\alpha_1 = \iota\alpha_t$ then $s_Y(\alpha_t) = e_Y(\alpha_1) = s_Y(\alpha_2)$, thus $t = 2$, but we assume $\alpha_2 \neq \iota\alpha_1$).

(Remark: the definition of a graph may look rather clumsy, the usual definition just identifies the arrows α and $\iota\alpha$ and calls this identified pair $\{\alpha, \iota\alpha\}$ an edge. There are several reasons for using the definition as given above: an edge joining two vertices x, y with $x \neq y$ can easily be oriented by specifying the order of x and y. However, we also will want to change the orientation of loops, replacing a loop α by $\iota\alpha$. Note that the orbit graph of a non-periodic stable translation quiver (as defined in Section 5 below) is always a graph in the sense defined above. The definition of a graph as specified above is due to Reidemeister [Re] who called this a "Streckenkomplex".)

Given a graph Y, an *orientation* Ω of Y is given by a subset Ω of $A(Y)$ such that Ω intersects any ι-orbit of $A(Y)$ in precisely one element. We denote by (Y, Ω) the subquiver $\langle \Omega \rangle$.

On the other hand, let Q be a quiver. The *underlying graph* $\bar{Q}$ of Q is the graph $\bar{Q} = \left(V(Q), A(\bar{Q}), s_{\bar{Q}}, e_{\bar{Q}}, \iota_{\bar{Q}}\right)$, where $A(\bar{Q})$ consists of the disjoint union of two copies of $A(Q)$, one denoted by $A(Q)$, the other by $\{\alpha^* \mid \alpha \in A(Q)\}$, with $s_{\bar{Q}}(\alpha^*) = e_Q(\alpha), e_{\bar{Q}}(\alpha^*) = s_Q(\alpha)$, and with $\iota_{\bar{Q}}(\alpha^*) = \alpha, \iota_{\bar{Q}}(\alpha) = \alpha^*$. By definition, a *walk* in Q is a path in $\bar{Q}$. (Note that we also will consider walks in $\bar{Q}$, thus paths in $\bar{\bar{Q}}$). Walks in Q will be denoted in the form $w = (x \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n} \mid y)$, where $\alpha_1, \ldots, \alpha_n$ belong to $A(Q), \varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$,

$$\alpha_i^{\varepsilon_i} = \begin{cases} \alpha_i & \text{for } \varepsilon_i = 1 \\ \alpha_i^* & \text{for } \varepsilon_i = -1 \end{cases}$$

(and $s_Q(\alpha_1^{\varepsilon_1}) = x, s_Q(\alpha_i^{\varepsilon_i}) = e_Q(\alpha_{i-1}^{\varepsilon_{i-1}})$ for $2 \leq i \leq n$, $e_Q(\alpha_n^{\varepsilon_n}) = y$), and we write $x = s_Q(w), y = e_Q(w)$. Such a walk $(x \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n} \mid y)$ is reduced provided $\alpha_i = \alpha_{i+1}$ implies $\varepsilon_i = \varepsilon_{i+1}$, and it is elementary provided $s_Q(\alpha_i^{\varepsilon_i}) \neq s_Q(\alpha_j^{\varepsilon_j})$ for $i \neq j$.

Let Y be a graph. We are going to define its *first homology group* $H_1(Y)$ (see [S]). Let $C_0(Y)$ be the free abelian group with basis $V(Y)$. Let $C_1(Y)$ be the factor group of the free abelian group with basis $A(Y)$ by the subgroup generated by all elements of the form $\alpha + \iota\alpha$, with $\alpha \in A(Y)$. Thus, in $C_1(Y)$, the element $\iota\alpha$, for $\alpha \in A(Y)$, is identified with $-\alpha$. If Ω is an orientation of Y, then $C_1(Y)$ may be identified with the free abelian group with basis $A(\langle \Omega \rangle) = \Omega$. We define $\delta: C_1(Y) \longrightarrow C_0(Y)$ by $\delta(\alpha) = e_Y(\alpha) - s_Y(\alpha)$, for $\alpha \in A(Y)$, and $H_1(Y)$ is, by definition, the kernel of δ. The elements of $H_1(Y)$ are called *cycles,* thus a cycle is an element $c \in C_1(Y)$ such that $\delta(c) = 0$. If $(x \mid \alpha_1, \ldots, \alpha_t \mid x)$ is a cyclic path in Y, then $\sum_{i=1}^t \alpha_i$ belongs to $H_1(Y)$. An element of $H_1(Y)$ of the form $\sum_{i=1}^t \alpha_i$, where $(x \mid \alpha_1, \ldots, \alpha_t \mid x)$ is an elementary reduced cyclic path in Y, is called an *elementary cycle.*

Given a quiver Q, let $H_1(Q) = H_1(\bar{Q})$, thus $H_1(Q)$ is the kernel of the map $\delta: C_1(Q) \longrightarrow C_0(Q)$, where $C_1(Q)$ is the free abelian group with basis $A(Q)$, and $C_0(Q)$ is the free abelian group with basis $V(Q)$, with $\delta(\alpha) = e_Q(\alpha) - s_Q(\alpha)$ for $\alpha \in A(Q)$. The elements in $C_1(Q)$ will be written in the form $c = \sum_{\alpha \in A(Q)} c(\alpha)\alpha$ with $c(\alpha) \in \mathbb{Z}$ and almost all $c(\alpha) = 0$; the set $\operatorname{supp}(c) = \{\alpha \mid c(\alpha) \neq 0\}$ will be called its *support* and we write $|c| := \sum_{\alpha \in A(Q)} |c(\alpha)|$. Since we will have to deal with elementary cycles in $H_1(Q)$ rather frequently, let us repeat: let $(x \mid \alpha_1^{c(\alpha_1)}, \ldots, \alpha_t^{c(\alpha_t)} \mid x)$ be an elementary reduced cyclic walk in Q (where $c(\alpha_i) \in \{\pm 1\}$), then $\sum_{i=1}^t c(\alpha_i)\alpha_i$ is an elementary cycle in $H_1(Q)$, and any elementary cycle is obtained in this way.

Let Y be a graph. On the set of paths in Y, we define the homotopy relation as the smallest equivalence relation $\sim$ with the following property: if w, w' are paths, with $e_Y(w) = s_Y(w') = s_Y(\alpha)$ for some arrow α, then

$$w \cdot \alpha \cdot \iota\alpha \cdot w' \sim w \cdot w'.$$

The equivalence class of the path w will be denoted by $\bar{w}$, and called its homotopy class. For $y \in V(Y)$, let $\pi_1(Y, y)$ be the set of homotopy classes of paths starting and ending at y, it is a group with respect to the composition $\overline{w} \cdot \overline{w'} = \overline{ww'}$, the fundamental group of Y at y. It is well-known that the fundamental group of a graph Y is a free group. Note that there is a canonical group homomorphism $\pi_1(Y, y) \longrightarrow H_1(Y)$, sending the homotopy class of the path $(x \mid \alpha_1, \ldots, \alpha_n \mid x)$ to $\sum_{i=1}^{n} \alpha_i$. The kernel of the homomorphism is the commutator subgroup, and this homomorphism is surjective provided Y is connected [S].

3. **Tempered maps.** Let Y be a graph. A linear map $\vartheta : H_1(Y) \longrightarrow \mathbb{Z}$ is called *tempered* provided it satisfies the following conditions: given any elementary cycle c in $H_1(Y)$, then

$$\vartheta(c) \equiv |c| \pmod 2 \text{ and } |\vartheta(c)| \leq |c|.$$

Let Ω be any orientation of Y. We define $\vartheta_\Omega : H_1(Y) \longrightarrow \mathbb{Z}$ by $\vartheta_\Omega\big(\sum_{\alpha\in\Omega} c(\alpha)\alpha\big) = \sum_{\alpha\in\Omega} c(\alpha)$. Then clearly ϑ_Ω is a tempered map on $H_1(Y)$. We will need the converse statement:

PROPOSITION. *Let Y be a graph, and $\vartheta : H_1(Y) \longrightarrow \mathbb{Z}$ a tempered map. Then there exists an orientation Ω on Y such that $\vartheta = \vartheta_\Omega$.*

The proof of the proposition will be given in this section. We will start with a quiver Q such that $Y = \bar{Q}$, and we will construct a function $\eta : A(Q) \longrightarrow \{\pm 1\}$ such that

$$\sum_{\gamma \in A(Q)} \eta(\gamma) c(\gamma) = \vartheta(c)$$

for any $c \in H_1(Q)$. Thus

$$\Omega = \{\alpha \mid \alpha \in A(Q), \eta(\alpha) = 1\} \cup \{\alpha^* \mid \alpha \in A(Q), \eta(\alpha) = -1\}$$

will be the required orientation of Y.

3.1. Shrinking of arrows. Let Q be a quiver, $\mathcal{U} \subseteq A(Q)$ some set of arrows. We define $Q/\mathcal{U}$, the quiver obtained from Q by shrinking the arrow in $\mathcal{U}$, as follows: given $x, y \in V(Q)$, write $x \sim y$ provided there exists a walk $(x \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n} \mid y)$ from x to y with all arrows $\alpha_i \in \mathcal{U}$. The equivalence class of x in $V(Q)$ with respect to $\sim$ is denoted by $[x]_\mathcal{U}$, and $V(Q/\mathcal{U})$ is the set of these equivalence classes. Let $A(Q/\mathcal{U}) = A(Q) \setminus \mathcal{U}$ and given $\alpha \in A(Q/\mathcal{U})$, let $s_{Q/\mathcal{U}} = [s_Q(\alpha)]_\mathcal{U}$, $e_{Q/\mathcal{U}} = [e_Q(\alpha)]_\mathcal{U}$. We stress that, by definition, the arrow set $A(Q/\mathcal{U})$ of $Q/\mathcal{U}$ is a subset of the arrow set $A(Q)$.

We define $\phi_i\colon C_i(Q) \longrightarrow C_i(Q/\mathcal{U})$ for $i = 0$ and 1 as follows: given $x \in V(Q)$, let $\phi_0(x) = [x]_{\mathcal{U}}$; given $\alpha \in A(Q)$, let $\phi_1(\alpha) = \alpha$ provided $\alpha \notin \mathcal{U}$ and $\phi_1(\alpha) = 0$ otherwise. Since the diagram

$$\begin{array}{ccc} C_1(Q) & \xrightarrow{\phi_1} & C_1(Q/\mathcal{U}) \\ {\scriptstyle\delta_Q}\downarrow & & \downarrow{\scriptstyle\delta_{Q/\mathcal{U}}} \\ C_0(Q) & \xrightarrow[\phi_0]{} & C_0(Q/\mathcal{U}) \end{array}$$

obviously commutes, we obtain an induced map

$$H_1(Q) \xrightarrow{\phi} H_1(Q/\mathcal{U})$$

between the kernels of the two δ-maps, we may call it the canonical map.

Note that the inclusion $A(Q/\mathcal{U}) \subseteq A(Q)$ yields an embedding $C_1(Q/\mathcal{U}) \subseteq C_1(Q)$, but $\delta_{Q/\mathcal{U}}$ is usually *not* the restriction of δ_Q to $C_1(Q/\mathcal{U})$.

We say that $T \subseteq A(Q)$ is *cycle-free* provided there does not exist a non-zero cycle c with $\operatorname{supp}(c) \subseteq T$, (or, equivalently, provided $H_1(\langle T\rangle) = 0$). Assume that T is cycle-free, then one easily shows that $H_1(Q/\mathcal{U}) \simeq H_1(Q)$. We want to construct an explicit map

$$\xi_T\colon H_1(Q/T) \longrightarrow H_1(Q)$$

(which will be shown to be an isomorphism). For any connected component $\langle T_t\rangle$ of $\langle T\rangle$ (where $T_t \subseteq T$), choose some vertex a_t of $\langle T_t\rangle$. If x is a vertex of $\langle T_t\rangle$, choose some walk $(a_t \mid \alpha_1^{\varepsilon_1}, \dots, \alpha_n^{\varepsilon_n} \mid x)$ from a_t to x inside T_t, and let $c(x) = \sum_{i=1}^{n} \varepsilon_i\alpha_i$, whereas for $x \notin V(\langle T\rangle)$, let $c(x) = 0$. We define $\xi_0^a\colon C_0(Q/T) \longrightarrow C_0(Q)$ by

$$\xi_0^a([x]_T) = \begin{cases} a_i & \text{if } x \in V(\langle T_i\rangle) \\ x & \text{if } x \in V(Q)\setminus V(\langle T\rangle), \end{cases}$$

for $x \in V(Q)$. And $\xi_1^a\colon C_1(Q/T) \longrightarrow C_1(Q)$ by

$$\xi_1^a(\alpha) = \alpha + c\big(s(\alpha)\big) - c\big(e(\alpha)\big),$$

for $\alpha \in A(Q/T)$. We claim that the diagram

$$\begin{array}{ccc} C_1(Q/T) & \xrightarrow{\xi_1^a} & C_1(Q) \\ {\scriptstyle\delta_{Q/T}}\downarrow & & \downarrow{\scriptstyle\delta_Q} \\ C_0(Q/T) & \xrightarrow[\xi_0^a]{} & C_0(Q) \end{array}$$

commutes: given $\alpha \in A(Q/T)$, we have

$$(\delta_Q\xi_1^a)(\alpha) = \begin{cases} e(\alpha) - s(\alpha) & \text{if } s(\alpha) \notin V(\langle T\rangle), e(\alpha) \notin V(\langle T\rangle), \\ e(\alpha) - a_i & \text{if } s(\alpha) \in V(\langle T_i\rangle), e(\alpha) \notin V(\langle T\rangle), \\ a_j - s(\alpha) & \text{if } s(\alpha) \notin V(\langle T\rangle), e(\alpha) \in V(\langle T_j\rangle), \\ a_j - a_i & \text{if } s(\alpha) \in V(\langle T_i\rangle), e(\alpha) \in V(\langle T_j\rangle). \end{cases}$$

Consequently, ξ_1^a maps $H_1(Q/T)$ into $H_1(Q)$, thus we obtain an induced map

$$\xi_T: H_1(Q/T) \longrightarrow H_1(Q),$$

which does not depend on the chosen vertices a_i (note that the map ξ_1^a already does not depend on the chosen walks, but, of course, it depends on the vertices a_i).

We stress that given $x \in C_1(Q/T)$, the element $\xi_1^a(c) - c$ has support in T, and thus the composition

$$C_1(Q/T) \xrightarrow{\xi_1^a} C_1(Q) \xrightarrow{\phi_1} C_1(Q/T)$$

is the identity; in particular, the composition

$$H_1(Q/T) \xrightarrow{\xi_T} H_1(Q) \xrightarrow{\phi} H_1(Q/T)$$

is the identity. Also the composition

$$H_1(Q) \xrightarrow{\phi} H_1(Q/T) \xrightarrow{\xi_T} H_1(Q)$$

is the identity, since for $c \in H_1(Q)$, the cycle $\xi_T\phi(c) - c$ has support in T, and thus vanishes.

Given a quiver Q, let $L(Q) \subseteq A(Q)$ be the set of loops. The quiver Q will be said to be *reduced* provided the support of any elementary cycle is a loop. If Q is a reduced quiver, then, clearly, $H_1(Q)$ is just the free abelian group generated by $L(Q)$. Thus, if Q is an arbitrary quiver, and T is a cycle-free subset of $A(Q)$ such that Q/T is reduced, then one obtains a free generating system for $H_1(Q)$ by

$$\{\xi_T(\alpha) \mid \alpha \in L(Q/T)\}.$$

In fact, since for $\alpha \neq \beta \in L(Q/T)$, we have $\big(\xi_T(\alpha)\big)(\beta) = 0$, whereas $\big(\xi_T(\alpha)\big)(\alpha) = 1$, we see that any $c \in H_1(Q)$ can be written in the form $c = \sum_{\alpha \in L(Q/T)} c(\alpha)\xi_T(\alpha)$.

3.2. Dealing with one elementary cycle. Let Q be a quiver and $\vartheta: H_1(Q) \longrightarrow \mathbb{Z}$ a tempered map. Note that for any loop α of Q, we have $|\vartheta(\alpha)| = 1$, since $|\vartheta(\alpha)| \leq 1$ and $\vartheta(\alpha) \equiv 1 \pmod 2$, thus $|\alpha| - |\vartheta(\alpha)| = 0$.

We consider an elementary cycle c which is not a loop. By the definition of a tempered map, we know that $u := \frac{1}{2}\big(|c| - \vartheta(c)|\big)$ is a non-negative integer. Replacing, if necessary, c by $-c$, we assume $\vartheta(c) \geq 0$.

LEMMA A. *Let c be an elementary cycle, not a loop, and assume $\vartheta(c) \geq 0$. Let $S =$ supp(c), and $n = |c|$, and $u = \frac{1}{2}\big(n - \vartheta(c)\big)$. Then there are $\binom{n}{u}$ functions $\eta: S \longrightarrow \{\pm 1\}$ such that*

$$(*) \qquad \sum_{\alpha \in S} \eta(\alpha)c(\alpha) = \vartheta(c).$$

PROOF. Given $\eta: S \longrightarrow \{\pm 1\}$ satisfying $(*)$, we have $\eta(\alpha)c(\alpha) \in \{\pm 1\}$ for all $\alpha \in S$. There are n summands on the left, all are in $\{\pm 1\}$, thus u of the summand

$\eta(\alpha)c(\alpha)$ have to be -1, the remaining ones have to be 1, since $n - u = u + \vartheta(c)$. Thus $\mathcal{U} = \{\alpha \in S \mid \eta(\alpha)c(\alpha) = -1\}$ is a subset of S with u elements. Conversely, let $\mathcal{U}$ be a subset of S with u elements, then we define

$$\eta(\alpha) = \begin{cases} -c(\alpha) & \text{for } \alpha \in \mathcal{U} \\ c(\alpha) & \text{for } \alpha \in S \setminus \mathcal{U}. \end{cases}$$

and we see that

$$\sum_{\alpha\in S} \eta(\alpha)c(\alpha) = \sum_{\alpha\in\mathcal{U}} (-1) + \sum_{\alpha\in S\setminus\mathcal{U}} 1 = -u + n - u = \vartheta(c),$$

thus $(*)$ is satisfied.

Let $u_\vartheta = \min\{\frac{1}{2}\big(|c| - |\vartheta(c)|\big) \bigm| c \text{ an elementary cycle of } Q, \text{ not a loop}\}$.

LEMMA B. *Let c be an elementary cycle of Q with $u = u_\vartheta$. Let $\eta: S \longrightarrow \{\pm 1\}$ be a function such that $\sum_{\alpha\in S}\eta(\alpha)c(\alpha) = \vartheta(c)$. Let $\alpha_0 \in S$ be a fixed arrow and $T := S\setminus\{\alpha_0\}$. Define $\eta_T: H_1(Q) \longrightarrow \mathbb{Z}$ by*

$$\eta_T(d) = \sum_{\beta\in T} \eta(\beta)d(\beta)$$

for $d \in H_1(Q)$. Then

$$\vartheta' = (\vartheta - \eta_T)\xi_T : H_1(Q/T) \longrightarrow \mathbb{Z}$$

is a tempered map on $H_1(Q/T)$.

PROOF. Let d be an elementary cycle of Q/T, we want to show that

$$\vartheta'(d) \equiv |d| \pmod 2 \text{ and } |\vartheta'(d)| \le |d|.$$

Note that instead of d, we also may consider $-d$. We will consider the support of d as a subset of $A(Q/T) \subseteq A(Q)$, and we write ξ instead of ξ_T.

First, we assume $d = \alpha_0$. Since $\xi(d)$ is a cycle in $H_1(\langle S\rangle)$ with $\xi(d)(\alpha_0) = 1$, we have $\xi(d) = c(\alpha_0)c$. Now

$$\begin{aligned}\vartheta'(d) &= (\vartheta - \eta_T)\big(c(\alpha_0)c\big) = c(\alpha_0)\Big(\sum_{a\in S}\eta(\alpha)c(\alpha) - \sum_{\alpha\in T}\eta(\alpha)c(\alpha)\Big)\\ &= c(\alpha_0)\eta(\alpha_0)c(\alpha_0) = \eta(\alpha_0),\end{aligned}$$

therefore, $|\vartheta'(d)| = 1$.

Thus, we can assume that $\operatorname{supp}(d) \neq \{\alpha_0\}$, and therefore $\alpha_0 \notin \operatorname{supp}(d)$, since d is an elementary cycle of Q/T. Let $\big([y]_T \mid \beta_1^{d(\beta_1)}, \dots, \beta_m^{d(\beta_m)} \mid [y]_T\big)$ be a cyclic walk in Q/T with $\operatorname{supp}(d) = \{\beta_1, \dots, \beta_m\}$. Let $y_i = s_Q\big(\beta_i^{d(\beta_i)}\big)$. There is at most one index i such that $y_i \in V(\langle T\rangle)$, and we can assume $y_i \notin V(\langle T\rangle)$ for all $2 \le i \le m$. Let $y = y_1$, and $z = e_Q\big(\beta_m^{d(\beta_m)}\big)$. In case $z = y$, we see that d is an elementary cycle of Q itself and $\xi(d) = d$. Thus, we only have to consider the case $y \neq z$. In this case, both y, z belong to $\langle T\rangle$. Let $(x_1 \mid \alpha_1^{c(\alpha_1)}, \dots, \alpha_n^{c(\alpha_n)} \mid x_1)$ be a cyclic walk in Q with $\operatorname{supp}(c) = \{\alpha_1, \dots, \alpha_n\}$

and let $x_i = s_Q(\alpha_i^{c(\alpha_i)})$. Let $z = x_r$, $y = x_s$ with $1 \leq r, s \leq n$. We can assume $r < s$, otherwise we consider $-d$ instead of d. It follows that

$$\xi(d) = d + \sum_{i=r}^{s-1} c(\alpha_i)\alpha_i,$$

in particular, $|\xi(d)| = m + n'$, where $n' = s - r$. Let u' be the number of arrows $\alpha_i \in \mathcal{U}$ with $r \leq i \leq s-1$, where $\mathcal{U} = \{\alpha \in S \mid \eta(\alpha)c(\alpha) = -1\}$. Note that

$$\begin{aligned}\vartheta'(d) &= (\vartheta - \eta_T)\big(\xi(d)\big) = \vartheta\big(\xi(d)\big) - \sum_{i=r}^{s-1} \eta(\alpha_i)c(\alpha_i) \\ &= \vartheta\big(\xi(d)\big) + u' - (n' - u') \\ &= \vartheta\big(\xi(d)\big) + 2u' - n',\end{aligned}$$

since u' of the summands $\eta(\alpha_i)c(\alpha_i)$ are -1, the remaining ones are 1. Now, $\xi(d)$ is an elementary cycle of Q, and therefore $\vartheta\big(\xi(d)\big) \equiv |\xi(d)| = m + n' \pmod 2$, thus

$$|\vartheta'(d)| = \vartheta\big(\xi(d)\big) + 2u' - n' \equiv m + n' + 2u' - n' \equiv m = |d| \pmod 2.$$

Since $\xi(d)$ is an elementary cycle of Q and not a loop, we have

$$|\xi(d)| - \big|\vartheta\big(\xi(d)\big)\big| \geq 2u_\vartheta,$$

thus

$$\vartheta\big(\xi(d)\big) \leq \big|\vartheta\big(\xi(d)\big)\big| \leq |\xi(d)| - 2u_\vartheta = m + n' - 2u_\vartheta,$$

therefore

$$\vartheta'(d) = \vartheta\big(\xi(d)\big) + 2u' - n' \leq m + n' - 2u_\vartheta + 2u' - n' = m - 2u_\vartheta + 2u' \leq m = |d|$$

where we have used $u' \leq u = u_\vartheta$.

In order to show that we also have $-|d| \leq \vartheta'(d)$, let us consider $\xi(d) - c$. We have

$$\xi(d) - c = d - \sum_{i=1}^{r-1} c(\alpha_i)\alpha_i - \sum_{i=s}^{n} c(\alpha_i)\alpha_i,$$

in particular, also $\xi(d) - c$ is an elementary cycle of Q and not a loop. We have $|\xi(d) - c| = m + n''$, where $n'' = n - n'$. Since $\xi(d) - c$ is an elementary cycle of Q and not a loop,

$$|\xi(d) - c| - \big|\vartheta\big(\xi(d) - c\big)\big| \geq 2u_\vartheta,$$

thus

$$\vartheta\big(\xi(d) - c\big) \geq -\big|\vartheta\big(\xi(d) - c\big)\big| \geq -|\xi(d) - c| + 2u_\vartheta = -m - n'' + 2u_\vartheta.$$

We also will use that

$$\vartheta(c) = |c| - 2u_\vartheta = n - 2u_\vartheta.$$

Altogether, we have

$$\begin{aligned}\vartheta'(d) &= \vartheta\big(\xi(d)\big) + 2u' - n' \\ &= \vartheta\big(\xi(d) - c\big) + \vartheta(c) + 2u' - n' \text{ (since } \vartheta \text{ is additive)} \\ &\geq -m - n'' + 2u_\vartheta + n - 2u_\vartheta + 2u' - n' \\ &= -m + 2u' \geq -m = -|d|.\end{aligned}$$

This shows that $|\vartheta'(d)| \leq |d|$, and therefore finishes the proof.

3.3. Transfinite induction.

LEMMA. *Let $\vartheta: H_1(Q) \longrightarrow \mathbb{Z}$ be a tempered map for a quiver Q. Then there exists a cycle-free subset T of $A(Q)$ and a function $\eta: T \longrightarrow \{\pm 1\}$ such that*

(1) X/T is reduced, and

(2) the map $\vartheta_T: H_1(Q/T) \longrightarrow \mathbb{Z}$ defined by

$$\vartheta_T(c) = \vartheta(\bar{c}) - \sum_{\alpha \in T} \eta(\alpha)\bar{c}(\alpha)$$

for $c \in H_1(Q/T)$ and $\bar{c} = \xi_T(c)$ is tempered.

PROOF. Let $T_1 = \emptyset$, and $\vartheta_1 = \vartheta$. Assume there is some ordinal number λ such that for any ordinal number $\mu < \lambda$, we have constructed a subset $T_\mu \subseteq A(Q)$ and a map $\eta_\mu: T_\mu \longrightarrow \{\pm 1\}$ with the following properties:

(a) If $\nu < \mu$ is an ordinal, then $T_\nu \underset{\neq}{\subset} T_\mu$, and $\eta_\nu = \eta_\mu | T_\nu$,

(b) T_μ is cycle-free,

(c) the map $\vartheta_\mu: H_1(Q/T_\mu) \longrightarrow \mathbb{Z}$ defined by $\vartheta_\mu(c) = \vartheta(\bar{c}) - \sum_{\alpha \in T_\mu} \eta_\mu(\alpha)\bar{c}(\alpha)$ for $c \in H_1(Q/T_\mu)$ and $\bar{c} = \xi_{T_\mu}(c)$, is tempered.

First, assume λ is a limit ordinal. Let $T_\lambda = \bigcup_{\mu<\lambda} T_\mu$, and define η_λ by $\eta_\lambda | T_\mu = \eta_\mu$. By definition (a) is satisfied for $\mu = \lambda$. Since a filtered union of cycle-free subsets of $A(Q)$ is cycle-free, also (b) is satisfied for $\mu = \lambda$. In order to show (c) for $\mu = \lambda$, let c be an elementary cycle of $H_1(Q/T_\lambda)$, and $\bar{c} = \xi_{T_\lambda}(c)$. The support of $\bar{c} - c$ is a finite subset of T_λ, thus it lies in T_μ for some $\mu < \lambda$, and $\vartheta_\lambda(c) = \vartheta_\mu(c)$. Therefore $\vartheta_\lambda(c) \equiv |c|$ (mod 2) and $|\vartheta_\lambda(c)| \leq |c|$.

Now assume $\lambda > 1$ and that λ is not a limit ordinal, thus $\lambda - 1$ exists. Let $Q_{\lambda-1} := Q/T_{\lambda-1}$. In case $Q_{\lambda-1}$ is reduced, let $T = T_{\lambda-1}$, $\eta = \eta_{\lambda-1}$; clearly, all assertions of the lemma are satisfied in this way. So assume $Q_{\lambda-1}$ is not reduced. We write $\xi_{\lambda-1}$ instead of $\xi_{T_{\lambda-1}}$, and $u_{\lambda-1}$ instead of $u_{\vartheta_{\lambda-1}}$. We choose some elementary cycle of $Q_{\lambda-1}$, not a loop, with $\frac{1}{2}\big(|c| - |\vartheta_{\lambda-1}(c)|\big) = u_{\lambda-1}$, and we can assume $\vartheta_{\lambda-1}(c) \geq 0$. Let S be the support of c, fix some $\alpha_0 \in S$, and let $T := S \setminus \{\alpha_0\}$. According to Section 3.2, there exists a function $\eta: S \longrightarrow \{\pm 1\}$ such that

$$\sum_{\alpha \in S} \eta(\alpha)c(\alpha) = \vartheta_{\lambda-1}(c),$$

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for all $c \in H_1(X_\lambda)$, and the function $\vartheta'_{\lambda-1}: H_1(Q_{\lambda_1}/T) \longrightarrow \mathbb{Z}$ defined by

$$\vartheta'_{\lambda-1}(d) = \vartheta_{\lambda-1}\big(\xi_T(d)\big) - \sum_{\alpha\in T} \eta(\alpha)\big(\xi_T(d)\big)(\alpha),$$

for $d \in H(Q_{\lambda-1}/T)$, is a tempered map. Let $T_\lambda = T_{\lambda-1} \cup T \subseteq A(Q)$, let $\eta_\lambda : T_\lambda \longrightarrow \{\pm 1\}$ be defined by $\eta_\lambda | T_{\lambda-1} = \eta_{\lambda-1}$, and $\eta_\lambda | T = \eta | T$, thus (a) is satisfied for $\mu = \lambda$. In order to see that T_λ is cycle-free, let c be a cycle with support in T_λ. Shrinking of $T_{\lambda-1}$ produces the cycle $c|T$ in $H_1(Q_{\lambda-1})$, however T is cycle-free in $Q_{\lambda-1}$, thus $c|T = 0$, or, equivalently, $\operatorname{supp}(c) \subseteq T_{\lambda-1}$. Since $T_{\lambda-1}$ is cycle-free, it follows that $c = 0$. For the proof of (c), we first note that clearly $Q/T_\lambda = Q_{\lambda-1}/T$ and we claim that $\vartheta_\lambda = \vartheta'_{\lambda-1}$, so that ϑ_λ is tempered. Let $\xi_\lambda = \xi_{T_\lambda}$. Let $d \in H_1(Q/T_\lambda)$ and note that $\xi_\lambda(d) = \xi_{\lambda-1}\xi_T(d)$, so that $\big(\xi_\lambda(d)\big)(\alpha) = \big(\xi_T(d)\big)(\alpha)$ for $\alpha \in T$. Thus

$$\begin{aligned}\vartheta_\lambda(d) &= \vartheta\big(\xi_\lambda(d)\big) - \sum_{\alpha\in T_\lambda} \eta_\lambda(\alpha)\big(\xi_\lambda(d)\big)(\alpha)\\ &= \vartheta\big(\xi_{\lambda-1}\xi_T(d)\big) - \sum_{\alpha\in T_{\lambda-1}} \eta_{\lambda-1}(\alpha)\big(\xi_{\lambda-1}\xi_T(d)\big)(\alpha) - \sum_{\alpha\in T}\eta(\alpha)\big(\xi_T(d)\big)(\alpha)\\ &= \vartheta_{\lambda-1}\big(\xi_T(d)\big) - \sum_{\alpha\in T}\eta(\alpha)\big(\xi_T(d)\big)(\alpha)\\ &= \vartheta'_{\lambda-1}(d).\end{aligned}$$

This finishes the proof of the induction step.

Since our algorithm produces a strictly increasing chain of subsets T_λ in $A(Q)$, it must stop. Thus, for some ordinal $\lambda > 1$, not a limit ordinal, $Q/T_{\lambda-1}$ has to be reduced. This yields the proof of the lemma.

3.4. Proof of the Proposition. Let X be a quiver, $\vartheta: H_1(Q) \longrightarrow \mathbb{Z}$ a tempered map. Let T be a cycle-free subset of $A(Q)$ such that Q/T is reduced, and let $\eta: T \longrightarrow \{\pm 1\}$ be a map such that ϑ_T (as defined in Section 3.3) is tempered. It remains to extend η to all of $A(Q)$.

If α is a loop of Q/T, let $\eta(\alpha) = \vartheta_T(\alpha)$, whereas for $\beta \in Q/T$, not a loop, we may choose $\eta(\beta)$ arbitrarily, say let $\eta(\beta) = 1$. Thus $\eta: A(Q) \longrightarrow \{\pm 1\}$ is defined. Recall that $L(Q/T)$ denotes the set of loops of Q/T, and given $\alpha \in L(Q/T)$, let $\bar\alpha = \xi_T(\alpha) \in H_1(Q)$. We have

$$\begin{aligned}\sum_{\gamma\in A(Q)} \eta(\gamma)\bar\alpha(\gamma) &= \eta(\alpha) + \sum_{\gamma\in T}\eta(\gamma)\bar\alpha(\gamma) \quad (\text{since } \bar\alpha(\alpha) = 1)\\ &= \eta(\alpha) + \vartheta(\bar\alpha) - \vartheta_T(\alpha) \quad (\text{by the definition of } \vartheta_T)\\ &= \vartheta(\bar\alpha).\end{aligned}$$

Note that $\{\bar\alpha \mid \alpha \in L(Q/T)\}$ is a basis of $H_1(Q)$ as a free abelian group, in fact, given $c \in H_1(Q)$, we have $c = \sum_{\alpha\in L(Q/T)} c(\alpha)\bar\alpha$, and therefore

$$\begin{aligned}\sum_{\gamma\in A(Q)} \eta(\gamma)c(\gamma) &= \sum_\gamma\sum_\alpha \eta(\gamma)c(\alpha)\bar\alpha(\gamma)\\ &= \sum_\alpha c(\alpha)\Big(\sum_\gamma \eta(\gamma)\bar\alpha(\gamma)\Big)\\ &= \sum_\alpha c(\alpha)\vartheta(\bar\alpha) = \vartheta(c).\end{aligned}$$

This completes the proof.

4. **Smooth non-periodic stable translation quivers.** We are going to give a complete list of the non-periodic stable translation quivers which are smooth and to exhibit some of their properties.

4.1. The stable translation quiver Π_{st}. First, we note that an isomorphism $f\colon \Gamma \longrightarrow \Gamma'$ of two stable translation quivers Γ, Γ' is given by two bijections $V(f)\colon V(\Gamma) \longrightarrow V(\Gamma')$, $A(f)\colon A(\Gamma) \longrightarrow A(\Gamma')$ which are compatible with s, e, τ, (thus, $s_{\Gamma'}A(f) = V(f)s_{\Gamma}, e_{\Gamma'}A(f) = V(f)e_{\Gamma}, \tau_{\Gamma'}V(f) = V(f)\tau_{\Gamma}$). Let Γ be a stable translation quiver. A group G of automorphisms of Γ is said to be admissible provided for $x \in V(\Gamma)$, any orbit of $V(\Gamma)$ under G intersects x^+ in at most one vertex. For an admissible group G of automorphisms of Γ we may form Γ/G, this is the translation quiver defined as follows: $V(\Gamma/G)$ is the set of G-orbits on $V(\Gamma)$, $A(\Gamma/G)$ is the set of G-orbits on $A(\Gamma)$, and $s_{\Gamma/G}, e_{\Gamma/G}, \tau_{\Gamma/G}$ are induced by $s_{\Gamma}, e_{\Gamma}, \tau_{\Gamma}$, respectively. (Note that we follow Riedtmann [Ri], but with some slight changes in order to take care of the fact that our translation quivers are allowed to have loops).

Smooth stable translation quivers have been considered before. Following Butler-Ringel [BR], let us consider the stable translation quiver Π defined as follows: $V(\Pi) = \{(a,b) \in \mathbb{Z}^2 \mid a \equiv b \pmod 2\}$ with arrows $(a,b) \longrightarrow (a+1, b+1)$, and $(a,b) \longrightarrow (a+1, b-1)$ and with translation $\tau_{\Pi}(a,b) = (a-2, b)$, for all $(a,b) \in V(\Pi)$. Observe that Π is isomorphic to $\mathbb{Z}A_{\infty}^{\infty}$ as defined in [HPR]. Given $(s,t) \in \mathbb{Z}^2$, consider the automorphism g_{st} of Π defined by $g_{st}(a,b) = (a+s-t, b+s+t)$, let $\langle g_{st} \rangle$ be the group of automorphisms generated by g_{st}, and define $\Pi_{st} = \Pi/\langle g_{st} \rangle$. Since $g_{st}^{-1} = g_{-s,-t}$, we have $\Pi_{st} = \Pi_{-s,-t}$; also, the automorphism ι of Π defined by $(a,b) \longrightarrow (a,-b)$ yields an isomorphism of Π_{st} and $\Pi_{-t,-s}$; thus any stable translation quiver of the form Π_{st} is isomorphic to one satisfying $0 \leq s$ and $-s \leq t \leq s$. Of course, $\Pi_{00} = \Pi$.

PROPOSITION. *The smooth non-period stable translation quivers are of the form* Π_{st} *with* $0 < s$ *and* $-s < t \leq s$, *or of the form* Π.

PROOF. Let Γ be a smooth stable translation quiver. Clearly, $\Gamma \simeq \Pi/G$ for some admissible group of automorphisms of Π (see Riedtmann [Ri]). Note that elements of the form $g_{ss}\iota$ with $s \in \mathbb{Z}$ cannot belong to G, since $g_{ss}\iota$ interchanges the two elements of $(s,s)^+$. On the other hand, for $s \neq t$ in Z, we have $(g_{ss}\iota)^2(0,0) = (2s-2t, 0)$, thus $g_{st}\iota$ cannot belong to G in case Π/G is non-periodic. Thus, assume Π/G is non-periodic. Let $g \in G$ and write $g(0,0)$ in the form $g(0,0) = (s-t, s+t)$ for some $s,t \in \mathbb{Z}$, thus $g_{st}^{-1}g$ fixes $(0,0)$, therefore $g = g_{st}$ or $g = g_{st}\iota$, but as we have seen, the latter is impossible. Therefore all elements of G are of the form g_{st}. Also, if $(s,t), (s',t') \in \mathbb{Z}^2$, then $g_{st}^{-(s'+t')}g_{s't'}^{s+t}(0,0) = (2s't - 2st', 0)$. Both g_{st} and $g_{s't'}$ can belong to G only in case $s't = st'$, since otherwise Π/G would have a periodic vertex. It follows that G is a cyclic group, thus $\Gamma \simeq \Pi_{st}$ for some $0 \leq s$ and $-s \leq t \leq s$. However, the cases $0 < s = -t$ are impossible, since in these cases Π_{st} has periodic vertices.

Recall that $\tilde{A}_{st}$ denotes the quiver

$$\begin{array}{ccccc} & \circ \xrightarrow{\alpha_2} \circ - \cdots \longrightarrow \circ & \\ \alpha_1 \nearrow & & \searrow \alpha_s \\ \circ & & \circ \\ \beta_1 \searrow & & \nearrow \beta_t \\ & \circ \underset{\beta_2}{\longrightarrow} \circ - \cdots \longrightarrow \circ & \end{array}$$

where $0 \leq t \leq s$ and $0 < s$ (note that we allow $t = 0$; in this case we deal with an oriented cycle!). Also, A_∞^∞ is the quiver with $V(A_\infty^\infty) = \mathbb{Z}$, with arrows $z \longrightarrow z+1$, for all $z \in \mathbb{Z}$. We observe that

$$\Pi_{st} \simeq \mathbb{Z}\tilde{A}_{st} \text{ for } 0 < s \text{ and } 0 \leq t \leq s,$$

whereas, as we have noted above,

$$\Pi \simeq \mathbb{Z}A_\infty^\infty.$$

4.2. The cases $-s < t < 0$. It remains to consider the stable translation quivers of the form Π_{st} with $-s < t < 0$. They have a cyclic path which is not sectional, and thus they cannot be written in the form $\mathbb{Z}Q$, with Q a quiver.

LEMMA. *Let $-s < t < 0$. Let ℓ be a subadditive function on Π_{st} with values in $\mathbb{N}_0$. Then ℓ is additive and bounded.*

PROOF. We write $\Pi_{st} = \Pi / \langle g_{st} \rangle$ and consider ℓ as a g_{st}-invariant function $V(\Pi) \longrightarrow \mathbb{N}_0$, thus ℓ is a subadditive function on Π.

Given vertices x, y of Π, write $x \preceq y$ provided there is a path from x to y, and we denote by $[x, y]$ the set of all vertices z of Π satisfying $x \preceq z \preceq y$. The four neighbours of $x = (a, b) \in V(\Pi)$ will be denoted as follows: $x^{\cdot} = (a+1, b+1)$, $x_{\cdot} = (a+1, b-1)$, ${}^{\cdot}x = (a-1, b+1)$, and ${}_{\cdot}x = (a-1, b-1)$. If W is a finite subset of $V(\Pi)$, let $\ell W = \sum_{w \in W} \ell(w)$.

Let $x \in V(\Pi)$. We claim that

$$\ell(x) + \ell(gx) \geq \ell\big(x + (s, s)\big) + \ell\big(x + (-t, t)\big),$$

and that we have equality only in case

$$\ell(y) + \ell(\tau y) = \ell({}^{\cdot}y) + \ell({}_{\cdot}y) \text{ for all } y \in [\tau^- x, gx].$$

For the proof, we add up the inequalities

$$\ell(y) + \ell(\tau y) \geq \ell({}^{\cdot}y) + \ell({}_{\cdot}y),$$

with $y \in [\tau^- x, z]$, where $z = gx$. We obtain

$$\ell[\tau^- x, z] + \ell[x, \tau z] \geq \ell[x^{\cdot}, {}^{\cdot}z] + \ell[x_{\cdot}, {}_{\cdot}z].$$

The left hand side of this inequality is equal to

$$\ell(x) + \ell(z) + \ell[x^{\cdot}, .z] + \ell[x., {}^{\cdot}z],$$

the right hand side is equal to

$$\ell\big(x + (s,s)\big) + \ell\big(x + (-t,t)\big) + \ell[x^{\cdot}, .z] + \ell[x., {}^{\cdot}z].$$

We can subtract $\ell[x^{\cdot}, .z] + \ell[x., {}^{\cdot}z]$ from both sides in order to obtain the required inequality. Also, we see that we obtain an equality only in case we have started with equalities for all $y \in [\tau^{-}x, z]$.

Next, we claim $2\ell(x) \geq \ell\big(x + (s,s)\big) + \ell\big(x - (s,s)\big)$. For, we have

$$\begin{aligned} 2\ell(x) &= \ell(x) + \ell(gx) \geq \ell\big(x + (s,s)\big) + \ell\big(x + (-t,t)\big) \\ &= \ell\big(x + (s,s)\big) + \ell\big(x - (s,s)\big), \end{aligned}$$

where we use that $g\big(x - (s,s)\big) = x + (-t,t)$ and the fact that ℓ is g-invariant.

It follows that for a fixed x, the map $f\colon \mathbb{Z} \longrightarrow \mathbb{N}_0$, defined by $f(i) = \ell\big(x + (i \cdot s, i \cdot s)\big)$ for $i \in \mathbb{Z}$ is constant: choose i_0 with $f(i_0) \leq f(i)$ for all $i \in \mathbb{Z}$ and conclude that $f(i_0 + 1) = f(i_0) = f(i_0 - 1)$, as in the proof of Lemma 3 of [HPR].

Thus, for all $x \in V(\Pi)$, we have $\ell\big(x + (s,s)\big) = \ell(x)$. Similarly, we also have $\ell\big(x + (-t,t)\big) = \ell(x)$. In particular, the only values taken by ℓ are the numbers $\ell(y)$ with $y \in [\tau^{-}x, gx]$, and therefore ℓ is bounded. Also, the equality $\ell(x) + \ell(gx) = \ell\big(x + (s,s)\big) + \ell\big(x + (-t,t)\big)$ implies $\ell(y) + \ell(\tau y) = \ell({}^{\cdot}y) + \ell(.y)$ for $y \in [\tau^{-}x, gx]$ thus ℓ is additive.

REMARK. The results of this section establish the main theorem in case we deal with a smooth stable translation quiver. We summarize the discussion above as follows:

Conditions	Shape of Π_{st}	Is there a subadditive non-additive ℓ?	Is additive ℓ bounded?
$0 < s,\ 0 < t \leq s$	$\mathbb{Z}\tilde{A}_{st}$	yes	yes
$0 < s,\ t = 0$	$\mathbb{Z}\tilde{A}_{s0}$	no	can be unbounded
$0 < s,\ -s < t < 0$		no	yes
$0 < s,\ -s = t$	$\mathbb{Z}A_\infty^\infty/(s)$	no	yes
$s = 0,\ t = 0$	$\mathbb{Z}A_\infty^\infty$	yes	can be unbounded

(In fact, when $s > 0, t = 0$, we consider $[x, gx.]$. Since $\ell(x) + \ell(gx.) = \ell(gx) + \ell(x.)$, we have $\ell(y) + \ell(\tau\ell) = \ell({}^{\cdot}y) + \ell(.y)$, for any $y \in [\tau^{-1}x, gx.]$. Therefore ℓ is additive.)

5. The non-periodic stable connected translation quivers.

5.1. The orbit graph. Let Γ be a non-periodic connected stable translation quiver. Give $x \in V(\Gamma)$, we denote by x^τ its τ-orbit, given $\alpha \in A(\Gamma)$, we denote by α^τ its τ-orbit (by definition, $\tau\alpha = \sigma^2\alpha$). The *orbit graph* Γ^τ of Γ is defined as follows: $V(\Gamma^\tau)$ is the set of τ-orbits of vertices of Γ, and $A(\Gamma^\tau)$ is the set of τ-orbits of arrows of Γ, the maps s_{Γ^τ} and e_{Γ^τ} are induced by s_Γ and e_Γ, respectively, and $\iota_{\Gamma^\tau}(\alpha^\tau) = (\sigma\alpha)^\tau$, for $\alpha \in A(\Gamma)$. In order to see that Γ^τ is a graph, we only have to observe that $\alpha^\tau \neq (\sigma\alpha)^\tau$ for any $\alpha \in A(\Gamma)$. (For, assume $\sigma\alpha = \tau^t\alpha$ for $t \in \mathbb{Z}$, then $\tau\alpha = \sigma^2\alpha = \sigma\tau^t\alpha = \tau^t\sigma\alpha = \tau^{2t}\alpha$, therefore $\alpha = \tau^{2t-1}\alpha$. But this implies that $s_\Gamma(\alpha)$ and $e_\Gamma(\alpha)$ are periodic vertices).

REMARK. In case the group generated by the automorphism τ is admissible, we may consider instead of the orbit graph also the stable translation quiver $\Gamma_{\langle\tau\rangle}$. However, Π_{10} is an example of a non-periodic translation quiver where $\langle\tau\rangle$ is not admissible.

LEMMA. *Let $w = (y \mid \beta_1^{\varepsilon_1}, \dots, \beta_m^{\varepsilon_m} \mid y')$ be a walk in Γ^τ. Let $x \in V(\Gamma)$ with $x^\tau = y$. Then there exists a unique walk $w_x = (x \mid \alpha_2^{\varepsilon_1}, \dots, \alpha_m^{\varepsilon_m} \mid x')$ in Γ with $\alpha_i^\tau = \beta_i$, for all $1 \leq i \leq m$.*

PROOF. First, we note the following: let $\alpha\colon x \longrightarrow x'$ be an arrow of Γ. Then $\tau^t\alpha$ is the only arrow in α^τ starting at $\tau^t x$, and the only arrow in α^τ ending in $\tau^t x'$, since x and x' are non-periodic.

The proof of the lemma is by induction on m. The assertion is clear for $m = 0$. Consider now a walk $w = (y \mid \beta_1^{\varepsilon_1}, \dots, \beta_m^{\varepsilon_m} \mid y')$ in Γ^τ with $m \geq 1$ and take some $x \in V(\Gamma)$ with $x^\tau = y$. By induction there is a unique walk $(x \mid \alpha_1^{\varepsilon_1}, \dots, \alpha_{m-1}^{\varepsilon_{m-1}} \mid x'')$ in Γ with $\alpha_i^\tau = \beta_i$ for all $1 \leq i \leq m-1$. Now, $x'' = e_\Gamma(\alpha_{m-1}^{\varepsilon_{m-1}})$, therefore $(x'')^\tau = e_{\Gamma^\tau}(\beta_{m-1}^{\varepsilon_{m-1}}) = s_{\Gamma^\tau}(\beta_m^{\varepsilon_m})$. Our first observation yields a unique arrow $\alpha_m \in \Gamma$ such that $\alpha_m^\tau = \beta_m$ and $s_\Gamma(\alpha_m^{\varepsilon_m}) = x''$. Let $e_\Gamma(\alpha_m^{\varepsilon_m}) = x'$; then $w_x = (x \mid \alpha_1^{\varepsilon_1}, \dots, \alpha_m^{\varepsilon_m} \mid x')$ is the required walk.

5.2. The map ϑ_Γ. Fix some vertex x of Γ. Therefore we have fixed some $x^\tau \in \Gamma^\tau$, and we may consider the fundamental group $\pi_1(\Gamma^\tau, x^\tau)$.

LEMMA. *Given a cyclic path $w = (x^\tau \mid \beta_1, \dots, \beta_m \mid x^\tau)$ in Γ^τ. Let $\vartheta_\Gamma(w) = m + 2t$, where $e_\Gamma(w_x) = \tau^t x$. Then $\vartheta_\Gamma\colon \pi_1(\Gamma^\tau, x^\tau) \longrightarrow \mathbb{Z}$ is a group homomorphism.*

PROOF. Consider $w = (x^\tau \mid \beta_1, \dots, \beta_m \mid x^\tau)$ and $v = (x^\tau \mid \beta_{m+1}, \dots, \beta_n \mid x^\tau)$. Let $w_x = (x \mid \alpha_1, \dots, \alpha_m \mid \tau^t x)$ and $v_x = (x \mid \alpha_{m+1}, \dots, \alpha_n \mid \tau^s x)$. Then $(wv)_x = (x \mid \alpha_1, \dots, \alpha_m, \tau^t\alpha_{m+1}, \dots, \tau^t\alpha_n \mid \tau^{s+t} x)$. Therefore

$$\vartheta_\Gamma(wv) = m + n + s + t = \vartheta_\Gamma(w) + \vartheta_\Gamma(v).$$

PROPOSITION. *The map $\vartheta_\Gamma\colon \pi_1(\Gamma^\tau, x^\tau) \longrightarrow \mathbb{Z}$ induces a group homomorphism $H_1(\Gamma^\tau) \longrightarrow \mathbb{Z}$ which is independent of x (and which will be denoted by ϑ_Γ, again).*

PROOF. Since $H_1(\Gamma^\tau)$ is the commutator factor group of $\pi_1(\Gamma^\tau, x^\tau)$, we see that $\vartheta_\Gamma\colon \pi_1(\Gamma^\tau, x^\tau) \longrightarrow \mathbb{Z}$ factors through $H_1(\Gamma^\tau)$. In order to see that the induced map is independent of x, consider an arrow $\alpha : x' \longrightarrow x$. Let $w = (x^\tau \mid \beta_1, \ldots, \beta_m \mid x^\tau)$ be a cycle in Γ^τ, and let $w_x = (x \mid \alpha_1, \ldots, \alpha_m \mid \tau^t x)$. Thus $\vartheta_\Gamma(w) = m + 2t$. Similarly, let $w' = \big((x')^\tau \mid \alpha^\tau, \beta_1, \ldots, \beta_m, (\sigma\alpha)^\tau \mid (x')^\tau\big)$. Then the canonical images of $\bar{w} \in \pi_1(\Gamma^\tau, x^\tau)$ and $\bar{w'} \in \pi_1(\Gamma^\tau, x'^\tau)$ in $H_1(\Gamma^\tau)$ coincide. On the other hand we have $w'_{x'} = (x' \mid \alpha, \alpha_1, \ldots, \alpha_m, \tau^{t-1}\sigma\alpha \mid \tau^{t-1}x')$, since $\tau^{t-1}\sigma\alpha : \tau^t x \longrightarrow \tau^{t-1}x'$ is the only arrow in $(\sigma\alpha)^\tau$ starting at $\tau^t x$, therefore $\vartheta_\Gamma(w') = m + 2 + 2(t-1) = \vartheta_\Gamma(w)$.

5.3. A special subquiver of Γ. Let $w = (x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_m^{\varepsilon_m} \mid x^\tau)$ be a cyclic walk in Γ^τ. We define a corresponding cyclic path in Γ^τ by

$$\tilde{w} = (x^\tau \mid \gamma_1, \ldots, \gamma_m \mid x^\tau), \text{ where } \gamma_i = \begin{cases} \beta_i & \text{for } \varepsilon_i = 1 \\ \iota\beta_i & \text{for } \varepsilon_i = -1, \end{cases}$$

and we define $\vartheta_\Gamma(w) := \vartheta_\Gamma(\tilde{w})$. Given w and x, we have defined above the walk w_x in Γ, and we can use $\vartheta_\Gamma(w)$ in order to determine the endpoint of w_x.

LEMMA A. *Let $x \in V(\Gamma)$, and $w = (x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_m^{\varepsilon_m} \mid x^\tau)$ a cyclic walk in Γ^τ. Let $t = \frac{1}{2}\big(\vartheta_\Gamma(w) - \sum_{i=1}^m \varepsilon_i\big)$. Then $t \in \mathbb{Z}$ and $e(w_x) = \tau^t x$.*

PROOF. We use induction on the number of indices i with $\varepsilon_i = -1$. If all $\varepsilon_i = 1$, then $w = \tilde{w}$, and by definition of $\vartheta_\Gamma(w)$ we have $\vartheta_\Gamma(w) = m + 2t$ where $e(w_x) = \tau^t x$. Consider now some $w = (x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_m^{\varepsilon_m} \mid x^\tau)$ with $\varepsilon_r = -1$ for some $1 \le r \le m$. Let $v = (x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_{r-1}^{e_{r-1}}, \iota\beta_r, \beta_{r+1}^{\varepsilon_{r+1}}, \ldots, \beta_m^{\varepsilon_m} \mid x^r)$; thus $\tilde{w} = \tilde{v}$, and $\vartheta_\Gamma(w) = \vartheta_\Gamma(v)$. By induction, $e(v_x) = \tau^s x$, where $s = \frac{1}{2}\Big(\vartheta_\Gamma(v) - \big(\sum_{i=1}^{r-1} \varepsilon_i + 1 + \sum_{i=r+1}^m \varepsilon_i\big)\Big) = \frac{1}{2}\big(\vartheta_\Gamma(w) - \sum_{i=1}^m \varepsilon_i - 2\big)$. Let $v_x = (x \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_{r-1}^{\varepsilon_{r-1}}, \alpha_r, \alpha_{r+1}^{\varepsilon_{r+1}}, \ldots, \alpha_m^{\varepsilon_m} \mid \tau^s x)$. Then $w_x = \big(x \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_{r-1}^{\varepsilon_{r-1}}, (\sigma\alpha_r)^{-1}, (\tau\alpha_{r+1})^{\varepsilon_{r+1}}, \ldots, (\tau\alpha_m)^{\varepsilon_m} \mid \tau^{s+1}x\big)$, therefore $e(w_x) = \tau^{s+1}x$, and $s + 1 = \frac{1}{2}\big(\vartheta_\Gamma(w) - \sum_{i=1}^m \varepsilon_i - 2\big) + 1 = \frac{1}{2}\big(\vartheta_\Gamma(w) - \sum_{i=1}^m \varepsilon_i\big)$.

Let Ω be an orientation on Γ^τ. A cyclic walk $w = (x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_m^{\varepsilon_m} \mid x^\tau)$ in (Γ^τ, Ω) is a cyclic walk in Γ^τ with all $\beta_i \in \Omega$; if we consider the corresponding cycle $\sum_{i=1}^m \varepsilon_i\beta_i$ in $H_1(\Gamma^\tau)$, then $\vartheta_\Omega\big(\sum_{i=1}^m \varepsilon_i\beta_i\big) = \sum_{i=1}^m \varepsilon_i$. Thus we may define $\vartheta_\Omega(w) = \sum_{i=1}^m \varepsilon_i$. The previous lemma can be reformulated in this case as follows:

LEMMA B. *Let Ω be an orientation on Γ^τ. Let $x \in V(\Gamma)$, and $(x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_m^{\varepsilon_m} \mid x^\tau)$ a cyclic walk in (Γ^τ, Ω). Then $e(w_x) = \tau^t x$, where $t = \frac{1}{2}\big(\vartheta_\Gamma(w) - \vartheta_\Omega(w)\big)$.*

Note that both maps ϑ_Γ, ϑ_Ω are defined on $H_1(\Gamma^\tau)$, and the case when $\vartheta_\Gamma = \vartheta_\Omega$ will be of great importance:

PROPOSITION. *Let Ω be an orientation on Γ^τ with $\vartheta_\Gamma = \vartheta_\Omega$. Let $x \in V(\Gamma)$. If w is a cyclic walk in (Γ^τ, Ω) starting at x^τ, then w_x is a cyclic walk in Γ starting at x.*

COROLLARY. *Let Ω be an orientation on Γ^τ with $\vartheta_\Gamma = \vartheta_\Omega$. Let $x \in V(\Gamma)$. Let Q be the following subquiver of Γ: its vertices are the end vertices of walks of the form w_x, its arrows are the arrows occurring in the walks of the form w_x, with w a walk in (Γ^τ, Ω) starting at x^τ. Then $V(Q)$ contains precisely one vertex of each τ-orbit of $V(\Gamma)$, and $A(Q)$ contains precisely one arrow of each σ-orbit of $A(\Gamma)$.*

PROOF. Let w, v be walks in (Γ^τ, Ω) starting at x^τ and having the same end vertex. Let $e(v_x) = y$, $e(w_x) = \tau^t y$. We claim that $t = 0$. Now wv^{-1} is a cyclic walk in (Γ^τ, Ω)

(or of length zero), and $(wv^{-1})_x$ ends in $\tau^t x$. It follows from the previous proposition that $t = 0$.

5.4. If Γ is not smooth, then ϑ_Γ is tempered. Let Γ be a non-periodic connected stable translation quiver, and let ℓ be a non-zero subadditive function on Γ with values in $\mathbb{N}_0$.

LEMMA. *For any vertex $x \in V(\Gamma)$, there is t with $\ell(\tau^t x) \neq 0$.*

PROOF. Let $\ell(y) \neq 0$, for some $y \in V(\Gamma)$. If there is an arrow $y \longrightarrow z$. Then $\ell(\tau z) + \ell(z) \geq \ell(y) > 0$. Thus $\ell(z) \neq 0$ or $\ell(\tau z) \neq 0$. It follows that the set of vertices y^τ in Γ^τ with $\ell(\tau^t y) \neq 0$ for at least one $t \in \mathbb{Z}$ is non-empty and closed under neighbours. Thus it is all of $V(\Gamma^\tau)$.

A *stable translation subquiver* Γ' of Γ is a stable translation quiver Γ' such that $V(\Gamma') \subseteq V(\Gamma)$, $A(\Gamma') \subseteq A(\Gamma)$ and $s_{\Gamma'}, e_{\Gamma'}, \tau_{\Gamma'}$ are the restrictions of $s_\Gamma, e_\Gamma, \tau_\Gamma$, respectively.

COROLLARY. *If Γ' is a stable translation subquiver of Γ, with $\Gamma' \neq \emptyset$ and $\Gamma' \neq \Gamma$, then $\ell \,|\, \Gamma'$ is not additive.*

PROOF. Choose $y \in V(\Gamma) \setminus V(\Gamma')$, $z \in V(\Gamma')$ with an arrow $y \longrightarrow z$. We can assume in addition, that $\ell(y) \neq 0$ (otherwise shift by some power of t). But then $\ell(\tau z) + \ell(z) \geq \sum_{y_i \in z^-} \ell(y_i)$, where z^- is the set of vertices y_i in $V(\Gamma)$ with an arrow $y_i \longrightarrow z$. On Γ', we have to delete on the right side of the inequality at least $\ell(y)$. Thus we obtain a proper inequality.

PROPOSITION. *Assume that Γ is not smooth. Then $\vartheta_\Gamma \colon H_1(\Gamma^\tau) \longrightarrow \mathbb{Z}$ is tempered.*

PROOF. Let c be an elementary cycle in Γ^τ, say $\sum_{i=1}^m \beta_i$, where $w = (y \mid \beta_1, \ldots, \beta_m \mid y)$ is a reduced, elementary, cyclic path in Γ^τ. Let $y_i = s_{\Gamma^\tau}(\beta_i)$ for $1 \leq i \leq m$. Let $\Gamma(c)$ be the stable translation subquiver of Γ with $V\big(\Gamma(c)\big)$ the set of all vertices $x \in y_i$ for some $1 \leq i \leq m$, and with $A\big(\Gamma(c)\big)$ the set of all arrows α such that α or $\sigma\alpha$ belongs to β_i for some $1 \leq i \leq m$. Clearly, $\Gamma(c)$ is smooth. Since we assume that Γ is not smooth, it follows that $\Gamma(c) \neq \Gamma$, thus $\ell \,|\, \Gamma(c)$ is not additive. According to the table of Section 4, we see that $\Gamma(c)$ is isomorphic to Π_{st} with $0 < s$ and $0 < t \leq s$. Since the definition of ϑ_Γ does not depend on the chosen base point, we may assume that our base point x satisfies $x^\tau = y$. Without loss of generality, we can assume that w_x is either the image in Π_{st} of the path $(0,0) \longrightarrow (1,1) \longrightarrow \cdots \longrightarrow (s+t, s+t)$ in Π, this is a path of length $m = s+t$, and $e(w_x) = \tau^{-t}x$, therefore $\vartheta_\Gamma(w) = s + t - 2t = s - t$; or else that w_x is the image in Π_{st} of the path $(0,0) \longrightarrow (1,-1) \longrightarrow \cdots \longrightarrow (s+t, -s-t)$ in Π, this is a path of length $m = s + t$, and $e(w_x) = \tau^{-s}x$, therefore $\vartheta_\Gamma(w) = s + t - 2s = t - s$. Altogether, we see that $|\vartheta_\Gamma(w)| \equiv m \pmod 2$ and that $|\vartheta_\Gamma(w)| \leq m$.

5.5. Proof of the main theorem. For the proof of the main theorem, we can assume that Γ is not smooth. By Proposition 5.4, the map $\vartheta_\Gamma \colon H_1(\Gamma^\tau) \longrightarrow \mathbb{Z}$ is tempered. According to Section 3, there exists an orientation Ω on Γ^τ such that $\vartheta_\Gamma = \vartheta_\Omega$. Corollary 5.3 yields a subquiver Q of Γ such that $V(Q)$ intersects every τ-orbit of $V(\Gamma)$ in precisely one vertex, and such that $A(Q)$ contains precisely one arrow of each σ-orbit of $A(\Gamma)$. Since Γ is non-periodic, it follows that $\Gamma \simeq \mathbb{Z}Q$. This completes the proof.

5.6. The existence of cyclic paths.

COROLLARY. *Let $\mathcal{C}$ be a non-periodic connected valued stable translation quiver, with a non-zero subadditive function ℓ with values in $\mathbb{N}_0$. Assume that $\mathcal{C}$ has a cyclic path. Then $\mathcal{C}$ is smooth and ℓ is additive.*

REMARK. More precisely, we will show that under the given assumptions, either $\mathcal{C} = \Pi_{st}$ with $-s < t < 0$, and the cyclic path is non-sectional, or else $\mathcal{C} = \mathbb{Z}\tilde{A}_{s0}$ for some $0 < s$, and the cyclic path is sectional.

PROOF. According to the main theorem mentioned in Section 1, and the table in Section 4, either $\mathcal{C} = \Pi_{st}$ for $0 < s$, and $-s < t < 0$, or $\mathcal{C} = \mathbb{Z}\mathfrak{S}$, for some valued quiver $\mathfrak{S}$. In the first case, there exits a cyclic path which is non-sectional. When $\mathcal{C} = \mathbb{Z}\mathfrak{S}$, the cyclic path has to be sectional and $\mathfrak{S}$ includes a cyclic path by the lemma of Section 1. Let $\Gamma = \big(V(\mathcal{C}), A(\mathcal{C}), s_{\mathcal{C}}, e_{\mathcal{C}}, \tau_{\mathcal{C}}\big)$ be the corresponding stable translation quiver, with trivial valuation, and $\Gamma = \mathbb{Z}Q$, so that Q corresponds to $\mathfrak{S}$. Assume that $w_x = (x \mid \alpha_1, \ldots, \alpha_n \mid x)$ is a cyclic path in Q, and let $s(\alpha_i), s(\alpha_{i+1}), \ldots, s(\alpha_j)$ be pairwise different, but $s(\alpha_i) = e(\alpha_j)$ for some $1 \leq i \leq j \leq n$. Let $y = s(\alpha_i)$, and consider $w = (y^\tau \mid \beta_i, \ldots, \beta_j \mid y^\tau)$ in Γ^τ with $\beta_k = \alpha_k^\tau$ for $k = i, \ldots, j$. We want to show that w is reduced and elementary. First of all, $\beta_{k+1} \neq \iota\beta_k$, for $k = i, \ldots, j-1$, and $\beta_j \neq \iota\beta_i$, since $A(Q)$ meets each σ-orbit only once; and second, $s(\beta_k) \neq s(\beta_\ell)$, for $k \neq \ell$, since $V(Q)$ meets each τ-orbit only once (5.3, Corollary). Therefore $c = \sum_{k=1}^{j} \beta_k$ is an elementary cycle. Consider the stable translation subquiver $\Gamma(c)$, with $V\big(\Gamma(c)\big) = \{x \mid x^\tau = s(\beta_k)$, for some $i \leq k \leq j\}$, $A\big(\Gamma(c)\big) = \{\alpha \mid \alpha^\tau$ or $(\sigma\alpha)^\tau = \beta_k$ for some $i \leq k \leq j\}$. If $\mathcal{C} \neq \Gamma(c)$, $\ell|_{\Gamma(c)}$, is not additive. By the table of Section 4, we have $\Gamma(c) = \mathbb{Z}\tilde{A}_{st}$, for $0 < s$, $0 < t \leq s$. But there is no cyclic path in $\tilde{A}_{st}$, a contradiction. Thus $\mathcal{C} = \Gamma(c)$ is smooth. Since $\mathfrak{S}$ includes a cyclic path, the only possibility is $\mathfrak{S} = \tilde{A}_{s0}$.

APPENDIX 1. THE MAP $\vartheta_\Gamma: H_1(\Gamma^\tau) \longrightarrow \mathbb{Z}$

There is a more sophisticated way to construct ϑ_Γ. Following Bongartz-Gabriel [BG], we consider Γ as a simplicial set (the loops which we allow do not lead to difficulties). Under the assumption that Γ is connected and non-periodic, we easily see that the geometric realizations $|\Gamma|$ of Γ and of the orbit graph Γ^τ are homotopic; just copy the proof in [BG]: first, we choose one fixed arrow in any σ-orbit of $A(\Gamma)$, and let X be the subspace of $|\Gamma|$ formed by these arrows and all "degree 2-arrows", so that X is a strong deformation retract; finally, we shrink all "degree 2-arrows" and obtain the geometric realization of orbit graph Γ^τ (of course, for the geometric realization of a graph Y we have to identify α and $\sigma\alpha$, for any $\alpha \in A(Y)$.)

The first homology group $H_1(|\Gamma|)$ can be calculated as the homology group of the following complex

$$C_2(\Gamma) \xrightarrow{\delta_2} C_1(\Gamma) \xrightarrow{\delta_1} C_0(\Gamma),$$

where $C_2(\Gamma)$ is the free abelian group on the set of triangles, $C_1(\Gamma)$ the free abelian group on the set of all (degree 1 and degree 2) arrows, and $C_0(\Gamma)$ the free abelian group on the set of vertices, and δ_2, δ_1 are the corresponding boundary maps. To be more precise: any vertex z yields a "degree 2-arrow" $\tau z \dashrightarrow z$, and $\delta_1(\tau z \dashrightarrow z) = z - \tau z$, whereas $\delta_1(x \longrightarrow y) = y - x$; any arrow $y \longrightarrow z$ yields a triangle

$$\begin{array}{ccc} & y & \\ \nearrow & & \searrow \\ \tau z & \dashrightarrow & z \end{array}, \text{ and } \delta_2 \left(\begin{array}{ccc} & y & \\ \nearrow & & \searrow \\ \tau z & \dashrightarrow & z \end{array} \right) = (\tau z \longrightarrow y) + (y \longrightarrow z) - (\tau z \dashrightarrow z).$$

The degree map $C_1(\Gamma) \xrightarrow{d} \mathbb{Z}$ with $d(x \longrightarrow y) = 1$, $d(\tau z \dashrightarrow z) = 2$ vanishes on the image of δ_2, thus it yields a homomorphism $\bar{d}\colon H_1(\Gamma) \longrightarrow \mathbb{Z}$. Combining $\bar{d}$ with the canonical homotopy equivalence $|\Gamma| \longrightarrow |\Gamma^\tau|$, we obtain $\vartheta_\Gamma\colon H_1(\Gamma^\tau) \longrightarrow \mathbb{Z}$.

APPENDIX 2. G-INVARIANT COMPLETE SECTIONS

There is another way to construct the subquiver Q of Γ. A *morphism of quiver* $f\colon B \longrightarrow Q$ is the disjoint union of two maps $V(f)\colon V(B) \longrightarrow V(Q)$, and $A(f)\colon A(B) \longrightarrow A(Q)$, with $f\big(s(\alpha)\big) = s\big(f(\alpha)\big), f\big(e(\alpha)\big) = e\big(f(\alpha)\big)$. We call f a *covering* provided for any $x \in V(Q)$, the map $V(f)$ yields a bijection between x^+ and $f(x)^+$, and between x^- and $f(x)^-$. The quiver B is called an *oriented tree,* if $\bar{B}$ is a tree. A covering $f\colon B \longrightarrow Q$ is said to be *universal,* if B is an oriented tree. If $f\colon B \longrightarrow Q$ is a universal covering of quivers, then $\bar{f}\colon \bar{B} \longrightarrow \bar{Q}$ is a universal covering of graphs. A *morphism of stable translation quivers* $f\colon \Gamma_1 \longrightarrow \Gamma_2$ is a morphism of quivers, with $f(\tau x) = \tau f(x)$ for any vertex x in Γ_1. If $f\colon \Gamma_1 \longrightarrow \Gamma_2$ is both a morphism of stable translation quivers and a covering of quivers, then it is called a *covering of stable translation quivers,* and f is a *universal covering* of stable translation quivers, if, in addition, Γ_1^τ is a tree.

Let $\Gamma = \mathbb{Z}B/G$ be a non-periodic connected stable translation quiver with trivial valuation, where B is an oriented tree, and G is an admissible automorphism group. Here, $\mathbb{Z}B \xrightarrow{p} \Gamma$ is just the universal covering of Γ (see [BG]).

Let $q\colon \bar{B} \longrightarrow \Gamma^\tau$ be defined by sending a^τ to $p(a)^\tau$, thus q is a universal covering of graphs, since $\bar{B}$ is a tree and q is bijective at any star. We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}B & \xrightarrow{\pi} & \bar{B} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \Gamma & \xrightarrow[\pi]{} & \Gamma^\tau \end{array}$$

where π is the orbit map, sending the vertex x to x^τ and the arrow α to α^τ.

According to algebraic topology, there is an isomorphism $\varphi\colon G \xrightarrow{\sim} \pi_1(\Gamma, x)$ of groups [S]. The isomorphism φ is defined as follows: take a fixed base point $a \in p^{-1}(x) \subseteq \mathbb{Z}B$, for any g in G, and consider $g(a) \in p^{-1}(x)$. Let $w\big(a, g(a)\big)$ be any walk from a to $g(a)$ in $\mathbb{Z}B$. Thus $p\Big(w\big(a, g(a)\big)\Big)$ is a cyclic walk at x in Γ or of length zero, and we take $\varphi(g) = \overline{p\Big(w\big(a, g(a)\big)\Big)}$.

We define a degree map $d\colon G \longrightarrow \mathbb{Z}$, such that $d(g) = \sum_{i=1}^n \varepsilon_i d(\alpha_i)$, where $w\big(a, g(a)\big) = d\Big(w\big(a \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n} \mid g(a)\big)\Big)$ (see [BG] and Appendix 1). Since $\mathbb{Z}B$ is simply connected, d is independent of the choice of the walk. On the other hand, $d(gh) =$

$d\Big(w\big(a, gh(a)\big)\Big) = d\Big(w\big(a, g(a)\big)\cdot w\big(g(a), gh(a)\big)\Big) = d\Big(w\big(a, g(a)\big)\Big) + d\Big(w\big(g(a), gh(a)\big)\Big).$ The walk $\big(g(a) \mid g(\alpha_1)^{\varepsilon_1}, \ldots, g(\alpha_n)^{\varepsilon_n} \mid gh(a)\big)$ is obtained from $\big(a \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n} \mid h(a)\big)$ by applying g. Thus we have $d(gh) = d\Big(w\big(a, g(a)\big)\Big) + d\Big(w\big(a, h(a)\big)\Big) = d(g) + d(h)$, and therefore d is a group homomorphism.

Since Γ and Γ^τ are homotopy equivalent (see Appendix 1), we have an isomorphism $h\colon \pi_1(\Gamma, x) \simeq \pi_1(\Gamma^\tau, x^\tau)$ sending the homotopy class of $(x \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n} \mid x)$ to the homotopy class of $(x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_n^{\varepsilon_n} \mid x^\tau)$, where $\beta_i = \alpha_i^\tau$, $i = 1, \ldots, n$. We have also a map $f\colon \pi_1(\Gamma^\tau, x^\tau) \longrightarrow H_1(\Gamma^\tau)$, sending the homotopy class of $(x^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_n^{\varepsilon_n} \mid x^\tau)$ to the cycle $\sum_{i=1}^n \varepsilon_i \beta_i$, and we note that $\ker f = \pi_1(\Gamma^\tau, x^\tau)'$. Therefore the map ϑ_Γ is induced as follows:

$$\begin{array}{ccccccc} G & \xrightarrow{\varphi} & \pi_1(\Gamma, x) & \xrightarrow{h} & \pi_1(\Gamma^\tau, x^\tau) & \xrightarrow{f} & H_1(\Gamma^\tau) \\ {\scriptstyle d}\big\downarrow & & & & & \swarrow & \\ \mathbb{Z} & \longleftarrow & & \vartheta_\Gamma & & & \end{array}$$

Assume that Γ is not smooth. Then ϑ_Γ is tempered (Proposition of Section 5.4), and there exists an orientation Ω of Γ^τ, with $\vartheta_\Gamma = \vartheta_\Omega$ (Proposition of Section 3).

Let B be an oriented tree. We call a subquiver U of $\mathbb{Z}B$ a *section,* if U is connected and intersects any τ-orbit of vertices at most once. If $\bar{U} = \bar{B}$, then U is said to be *complete.* If $GU = U$, then U is called *G-invariant.*

PROPOSITION. *Let $\Gamma = \mathbb{Z}B/G$ be a non-periodic connected stable translation quiver, with a non-zero subadditive function ℓ with values in $\mathbb{N}_0$ and assume that Γ is not smooth. Let Ω be an orientation of Γ^τ, with $\vartheta_\Gamma = \vartheta_\Omega$, and let $(\bar{B}, \Omega)$ be the universal covering of (Γ^τ, Ω). Then there exists a G-invariant complete section U of $\mathbb{Z}B$, which is homomorphic to $(\bar{B}, \Omega)$.*

PROOF. Assume that we have a section U_0 of $\mathbb{Z}B$, which is isomorphic to a full oriented subtree $(\bar{B}_0, \Omega)$, of $(\bar{B}, \Omega)$, with $a \in V(U_0)$. Then $a^\tau \in V(\bar{B}_0, \Omega)$, and $\alpha \in A(U_0)$. Then $\alpha^\tau \in A(\bar{B}_0, \Omega)$. If $(\bar{B}_0, \Omega) \underset{\neq}{\subset} (\bar{B}, \Omega)$, then there exists a vertex $a \in V(U_0)$, such that $b^\tau \cap V(U_0) = \emptyset$. But there is a σ-orbit between a^τ and b^τ. Let $b = e(\alpha)$ in case we have $a^\tau \longrightarrow b^\tau$ in $(\bar{B}, \Omega)$, and α is the unique arrow between a^τ and b^τ with $s(\alpha) = a$, and let $b = s(\alpha)$ in case we have $b^\tau \longrightarrow a^\tau$ in $(\bar{B}, \Omega)$, and α is the unique arrow between a^τ and b^τ with $e(\alpha) = a$. Let U_1 be the subquiver with $V(U_1) = V(U_0) \cup \{b\}$, $A(U_1) = A(U_0) \cup \{\alpha\}$. Then U_1 is a section, and $U_1 \simeq (B_1, \Omega)$, where $V(B_1, \Omega) = V(B_0, \Omega) \cup \{b^\tau\}$, $A(B_1, \Omega) = A(B_0, \Omega) \cup \{\alpha^\tau\}$. By induction, we obtain a section $U \simeq (\bar{B}, \Omega)$. Since $\bar{U} = \bar{B}$, we see that U is a complete section of $\mathbb{Z}B$. It remains to prove that U is G-invariant.

For any $a \in V(U)$, and $g \in G$, assume that $g(a)^\tau \cap V(U) = \{b\}$. Let $w(a, b) = (a \mid \alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n} \mid b)$ be the unique geodesic from a to b inside U (see Section 2). We see that $\pi\big(w(a, b)\big) = (a^\tau \mid \beta_1^{\varepsilon_1}, \ldots, \beta_n^{\varepsilon_n} \mid b^\tau)$ is the unique geodesic from a^τ to b^τ in $(\bar{B}, \Omega)$, and $d\big(w(a, b)\big) = \sum_{i=1}^n \varepsilon_i$. On the other hand, let $w'\big(a, g(a)\big)$ be any

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walk from a to $g(a)$ in $\mathbb{Z}B$. Since $b^{\tau} = g(a)^{\tau}$, it follows that $\pi\Big(w'\big(a, g(a)\big)\Big)$ is equal to $w_1 \cdot w_1^{-1} \cdot \beta_1^{\varepsilon_1} \cdot w_2 \cdot w_2^{-1} \cdot \beta_2^{\varepsilon_2} \cdot \cdots \cdot \beta_n^{\varepsilon_n} \cdot w_{n+1} \cdot w_{n+1}^{-1}$, where w_i, $i = 1, \ldots, n+1$, are walks in $(\bar{B}, \Omega)$, and where we also consider $\beta_i^{\varepsilon_i}$, $i = 1, \ldots, n$, as walks. Therefore $\overline{q\pi(w')} = \overline{q\pi(w)}$, but $\pi p(w') = q\pi(w')$. Thus we have $\overline{\pi p(w')} = \overline{q\pi(w)}$. Now $d(g) = \vartheta_\Gamma \cdot f \cdot h \cdot \varphi(g) = \vartheta_\Gamma \cdot f \cdot h \cdot \overline{p(w')} = \vartheta_\Gamma \cdot f\big(\overline{\pi p(w')}\big) = \vartheta_\Gamma \cdot f\big(\overline{q\pi(w)}\big) = \vartheta_\Gamma\big(\sum_{i=1}^{n} q(\beta_i)^{\varepsilon_i}\big) = \sum_{i=1}^{n} \varepsilon_i$, since $\vartheta_\Gamma = \vartheta_\Omega$. Thus $d\Big(w\big(a, g(a)\big)\Big) = d\big(w(a, b)\big)$. Since $g(a)^{\tau} = b^{\tau}$, we have $g(a) = b \in V(U)$ as required.

For the construction of Q, let $Q = U/(G|U)$. Then $\Gamma = \mathbb{Z}B/G = \mathbb{Z}U/G = \mathbb{Z}\big(U/(G|U)\big) = \mathbb{Z}Q$.

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REFERENCES

[A] M. Auslander, *Applications of morphisms determined by objects*. Proc. Conf. on Representation Theory. Philadelphia (1976), Marcel Dekker (1978), 245–327.

[B] C. Berge, *Graphs and hypergraphs*. North-Holland Publishing Company, 1973.

[BG] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. **65**(1982), 331–378.

[BS] R. Bautista and S. O. Smalø, *Nonexistent cycles*, Comm. Alg. **11**(1983), 1755–1767.

[BR] M. C. R. Butler and C. M. Ringel, *AR-sequences with few middle terms and applications to string algebras*, Comm. Alg. **15**(1987), 145–179.

[D] W. Dicks, *Groups, trees and projective modules*, LNM **790**(1980).

[GZ] P. Gabriel and M. Zisman, *Calculus of fractions and homology theory*, Erg. Math. **35**, Berlin-Heildelberg-New York, Springer, 1967.

[HPR] D. Happel, U. Preiser and C. M. Ringel, *Vinberg's characterization of Dynkin diagrams using subadditive function with application to DTr-periodic modules*, LNM **832**(1979), 280–294.

[Re] K. Reidemeister, *Einführung in die kombinatorische Topologie*. Reprint, Chelsea Publishing Company, New York, N.Y., 1950.

[Ri] Chr. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comm. Math. Helv. **55** (1980), 199–224.

[R] C. M. Ringel, *Tame algebras and integral quadratic forms*. Springer LNM **1099**(1984).

[S] E. H. Spanier, *Algebraic Topology*. McGraw Hill Book Company, 1966.

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