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ON 4-DIMENSIONAL GENERALIZED COMPLEX SPACE FORMS

UN KYU KIM

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Abstract

We characterize four-dimensional generalized complex forms and construct an Einstein and weakly $*$-Einstein Hermitian manifold with pointwise constant holomorphic sectional curvature which is not globally constant.

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1. Introduction

Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold with Riemannian connection ∇ and let the curvature tensor of M is given by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

for $X, Y, Z, W \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on M.

The holomorphic sectional curvature is defined by $H(X) = -R(X, JX, X, JX)$ for $X \in T_pM$ ($p \in M$) with $g(X, X) = 1$. If $H(X)$ is constant $\mu(p)$ for all $X \in T_pM$ at each point p of M, then M is said to be of pointwise constant holomorphic sectional curvature. Further, if μ is constant on all of M, then M is said to be of constant holomorphic sectional curvature.

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An almost Hermitian manifold (M, J, g) is said to be a *generalized complex space form* if the Riemannian curvature tensor R satisfies the condition $R = f\pi_1 + h\pi_2$ for some functions f and h, where π_1 and π_2 are given by

$$\pi_1(X, Y, Z) = g(X, Z)Y - g(Y, Z)X,$$
$$\pi_2(X, Y, Z) = 2g(JX, Y)JZ + g(JX, Z)JY - g(JY, Z)JX,$$

for $X, Y, Z \in \chi(M)$.

In [8, p. 389], Tricerri and Vanhecke stated the following problem: Do there exist 4-dimensional manifolds (M, J, g) with $R = f\pi_1 + h\pi_2$, where h is a nonconstant C^∞ function? They remarked that if h is a nonzero constant, then M is a complex space form. Also they proved that $f + h$ is a constant and M must be Hermitian on $U = \{m \in M | h(m) \neq 0\}$. Olszak showed that the above question has a positive answer [5]. One of his results is the following.

THEOREM 1.1 ([5]). *Let $(M, J, \tilde{g})$ be a Bochner flat Kaehlerian manifold of dimension 4. Assume, additionally, that the scalar curvature $\tilde{\tau}$ of $\tilde{g}$ is nonzero everywhere on M and nonconstant. Let $g = e^\sigma \tilde{g}$, where $\sigma = -\log(C(\tilde{\tau})^2)$, C is a positive constant. Then the Hermitian manifold (M, J, g) is a generalized complex space form for which the function $h \neq 0$ everywhere on M and $h = C/24(\tilde{\tau})^3 \neq$ constant.*

Curvature identities are a key to understanding the geometry of various classes of almost Hermitian manifolds. In this paper we shall be concerned with the following curvature identity:

$$(*) \qquad R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW),$$

which implies

$$R(X, Y, Z, W) = R(JX, JY, JZ, JW).$$

Gray and Vanhecke [2] posed the following question: Let $\mathscr{L}$ be a given class of almost Hermitian manifolds. Suppose that $M \in \mathscr{L}$ with $\dim M \geq 4$ and assume that M is of pointwise constant holomorphic sectional curvature $\mu = \mu(p)$ $(p \in M)$. Must μ be a constant function? In [2] Gray and Vanhecke gave a negative answer to the question for the class of Hermitian manifolds. They have constructed an example of a 4-dimensional Hermitian manifold with pointwise constant holomorphic sectional curvature which is not globally constant. In [4], the present author, Kim, and Jun showed that this example is a weakly $*$-Einstein manifold, but it is not Einstein.

In the present paper we characterize 4-dimensional generalized complex space forms as the almost Hermitian manifolds with pointwise constant holomorphic sectional curvature whose curvature tensor satisfies the identity $(*)$. Also we construct a 4-dimensional Einstein and weakly $*$-Einstein Hermitian manifold with pointwise constant holomorphic sectional curvature which is not globally constant.

2. Preliminaries

Let (M, J, g) be a 4-dimensional almost Hermitian manifold. Then we have

$$\begin{aligned} &J^2X = -X, && g(JX, JY) = g(X, Y), \\ &(\nabla_X J)JY = -J(\nabla_X J)Y, && g((\nabla_X J)Y, Z) = -g(Y, (\nabla_X J)Z), \\ &g((\nabla_X J)Y, Y) = 0, && g((\nabla_X J)Y, JY) = 0 \end{aligned} \tag{2.1}$$

for $X, Y, Z \in \chi(M)$. The $*$-Ricci tensor ρ^* and the $*$-scalar curvature τ^* of M are defined respectively by

$$\begin{aligned} &\rho^*(X, Y) = g(Q^*X, Y) = \operatorname{trace}(Z \mapsto R(X, JZ)JY) \\ &\rho^*(X, Y) = \operatorname{trace} Q^* \end{aligned} \tag{2.2}$$

for all $X, Y, Z \in T_pM$, $p \in M$. For a Kaehler manifold ($\nabla J = 0$), ρ^* coincides with the Ricci tensor ρ but this does not necessarily hold on a general almost Hermitian manifold. Furthermore, M is said to be a weakly $*$-Einstein manifold if $\rho^* = (\tau^*/4)g$ holds. In particular, M is called a $*$-Einstein manifold if, in addition, τ^* is constant. We define three linear operators L_i, $i = 1, 2, 3$ as the following [8]:

$$\begin{aligned} (L_1R)(X, Y, Z, W) &= \frac{1}{2}\{R(JX, JY, Z, W) + R(Y, JZ, JX, W) \\ &\quad + R(JZ, X, JY, W)\}, \\ (L_2R)(X, Y, Z, W) &= \frac{1}{2}\{R(X, Y, Z, W) + R(JX, JY, Z, W) \\ &\quad + R(JX, Y, JZ, W) + R(JX, Y, Z, JW)\}, \\ (L_3R)(X, Y, Z, W) &= R(JX, JY, JZ, JW). \end{aligned}$$

Triceri and Vanhecke proved the following.

THEOREM 2.1 ([8]). *Let M be an almost Hermitian manifold with real dimension four and curvature R. Then we have the following identities*:

$$\begin{aligned} &(I - L_1)(I + L_2)(I + L_3)R = -\frac{1}{4}(\tau - \tau^*)(3\pi_1 - \pi_2), \\ &\rho(R + L_3R) - \rho^*(R + L_3R) = \frac{1}{2}(\tau - \tau^*)g \end{aligned} \tag{2.3}$$

where I is the identity transformation.

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On the other hand Gray and Vanhecke obtained the following.

LEMMA 2.2 ([2]). *Let M be any almost Hermitian manifold which satisfies curvature identity* $(*)$, *and assume that* M *has pointwise constant holomorphic sectional curvature* μ. *Then*

$$(2.4)\qquad \begin{aligned} R(W,X,Y,Z) = \frac{\mu}{4}\{&g(W,Z)g(X,Y) - g(W,Y)g(X,Z) \\ &+ g(JW,Z)g(JX,Y) - g(JW,Y)g(JX,Z) \\ &- 2g(JW,X)(JY,Z)\} + \frac{1}{4}\{2\lambda(W,X,Y,Z) \\ &- \lambda(W,Z,X,Y) - \lambda(W,Y,Z,X)\}, \end{aligned}$$

where $\lambda(W,X,Y,Z) = R(W,X,Y,Z) - R(W,X,JY,JZ)$.

The Kaehler form Ω of the almost Hermitian manifold is defined by $\Omega(X,Y) = g(X,JY)$, $X, Y \in \chi(M)$. The Nijenhuis tensor N of the almost complex structure J is a tensor field of type $(1,2)$ defined by

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$$

for $X, Y \in \chi(M)$. The Lee form of (M,J,g) is the 1-form defined by

$$(2.5)\qquad d\Omega = \omega \wedge \Omega, \quad \omega = \delta\Omega \cdot J.$$

We denote by B the Lee vector field which is defined by $g(B,X) = \omega(X)$ for $X \in \chi(M)$. In an almost Hermitian manifold, it is known that the following equality holds:

$$(2.6)\qquad \begin{aligned} g((\nabla_X J)Y,Z) = \frac{1}{2}\{&d\Omega(X,JY,JZ) - d\Omega(X,Y,Z) \\ &+ g(N(Y,Z),JX)\} \end{aligned}$$

for $X, Y, Z \in \chi(M)$ [7, 10]. Thus by (2.5) and (2.6), we get

$$(2.7)\qquad \begin{aligned} 2g((\nabla_X J)Y,Z) = {}&\omega(JY)g(X,Z) + \omega(Y)\Omega(X,Z) \\ &- \omega(JZ)g(X,Y) - \omega(Z)\Omega(X,Y) \\ &- \Omega(X,N(Y,Z)). \end{aligned}$$

From (2.7) and (2.1) we obtain

$$(2.8)\qquad \begin{aligned} 2g((\nabla_X J)Y,Z) = {}&g(B,JY)g(X,Z) - g(B,Y)g(JX,Z) \\ &+ g(X,Y)g(JB,Z) - g(X,JY)g(B,Z) \\ &- g((\nabla_{JY}J)JX,Z) - g(JX,(\nabla_{JZ}J)Y) \\ &+ g(X,(\nabla_Z J)Y) + g((\nabla_Y J)X,Z). \end{aligned}$$

If the Lee form ω of (M, J, g) is closed (that is, $d\omega = 0$), then a 4-dimensional Hermitian manifold (M, J, g) is said to be a locally conformal Kaehler manifold.

3. A characterization of generalized complex space forms

Let $M = (M, J, g)$ be a 4-dimensional almost Hermitian manifold and let the curvature tensor of M satisfy the condition

$$(3.1)\qquad R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$$

for $X, Y, Z, W \in \chi(M)$. Then we have, with the help of Theorem 2.1, $L_2R = R$ and $L_3R = R$,

$$(3.2)\qquad \begin{aligned} &4R(X, Y, Z, W) - 2\{R(JX, JY, Z, W) + R(Y, JZ, JX, W) + R(JZ, X, JY, W)\} \\ &= \frac{1}{4}(\tau^* - \tau)\{3g(X, Z)g(Y, W) - 3g(Y, Z)g(X, W) \\ &\quad - 2g(JX, Y)g(JZ, W) - g(JX, Z)g(JY, W) + g(JY, Z)g(JX, W)\}. \end{aligned}$$

Using Bianchi's identity, (3.1), $L_2R = R$ and $L_3R = R$, we obtain

$$(3.3)\qquad \begin{aligned} &R(JX, JY, Z, W) + R(Y, JZ, JX, W) + R(JZ, X, JY, W) \\ &= -R(X, Y, Z, W) + 2R(X, Y, JZ, JW) - R(X, W, JY, JZ) - R(X, Z, JW, JY). \end{aligned}$$

Moreover, we assume that M is of pointwise constant holomorphic sectional curvature μ. Then we have, by the Lemma 2.2,

$$(3.4)\qquad \begin{aligned} R(X, Y, Z, W) = \mu\{&g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ &+ g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ &- 2g(JX, Y)g(JZ, W)\} - \{2R(X, Y, JZ, JW) \\ &- R(X, Z, JW, JY) - R(X, W, JY, JZ)\}. \end{aligned}$$

Comparing (3.2), (3.3) and (3.4), we obtain

$$(3.5)\qquad \begin{aligned} &R(X, Y, Z, W) \\ &= \left\{\frac{3}{32}(\tau^* - \tau) - \frac{\mu}{4}\right\}\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\} \end{aligned}$$

$$+\left\{\frac{1}{32}(\tau^* - \tau) + \frac{\mu}{4}\right\}\{g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) - 2g(JX, Y)g(JZ, W)\}.$$

By the assumption (3.1) (and hence $L_3 R = R$), M is a RK-manifold with pointwise constant holomorphic sectional curvature μ. Hence it is known [6] that

$$\rho(X, Y) + 3\rho^*(X, Y) = 6\mu g(X, Y), \quad \tau + 3\tau^* = 24\mu. \tag{3.6}$$

From (3.5) and (3.6), we get

$$\begin{aligned} R(X, Y, Z, W) &= \left(\frac{\mu}{2} - \frac{\tau}{8}\right)\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} \\ &\quad + \left(\frac{\tau}{24} - \frac{\mu}{2}\right)\{2g(JX, Y)g(JZ, W) + g(JZ, W)g(JY, W) \\ &\quad - g(JX, W)g(JY, Z)\}, \end{aligned} \tag{3.7}$$

that is,

$$R = f\pi_1 + h\pi_2,$$

where the functions f and g are given by

$$f = \frac{\mu}{2} - \frac{\tau}{8}, \quad h = \frac{\tau}{24} - \frac{\mu}{2}. \tag{3.8}$$

Thus M is a generalized complex space form.

REMARK. In [3], the present author and Jun obtained (3.7) in another way.

Conversely, suppose that (M, J, g) is a 4-dimensional generalized complex space form whose curvature tensor R is given by

$$R = f\pi_1 + h\pi_2,$$

where f and h are certain smooth functions on M. We can easily check that R satisfies the condition (3.1) and the holomorphic sectional curvature is given by

$$H(X) = -R(X, JX, X, JX) = -(f + 3h),$$

which shows that $H(X)$ is constant for each unit tangent vector $X \in T_pM$ ($p \in M$). Hence $H(X)$ depends only on $p \in M$. Therefore, M is an almost Hermitian manifold with pointwise constant holomorphic sectional curvature $-(f + 3h)$. Thus we have, from (3.8),

$$R = \left(\frac{\mu}{2} - \frac{\tau}{8}\right)\pi_1 + \left(\frac{\tau}{24} - \frac{\mu}{2}\right)\pi_2, \tag{3.9}$$

where we have put $-(f + 3h) = \mu$. Thus we have the following characterization.

THEOREM 3.1. *A 4-dimensional almost Hermitian manifold (M, J, g) is a generalized complex space form if and only if M is of pointwise constant holomorphic sectional curvature and the curvature tensor R of M satisfies*

$$R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$$

for $X, Y, Z, W \in \chi(M)$.

Now let M be a 4- dimensional generalized complex space form, or equivalently M is a 4-dimensional almost Hermitian manifold with pointwise constant holomorphic sectional curvature μ whose curvature tensor satisfies (3.1). Then the curvature tensor R is given by

$$R = f\pi_1 + h\pi_2,$$

where f and h are given by (3.8). And M is both Einstein and weakly $*$-Einstein [3, 8]. If the function h is nonzero constant, then M is a complex space form [8]. If the function $h \neq 0$ at each point of M and $h \neq$ constant, then M is globally conformal to a Bochner flat Kaehler surface [5]. Now let $O = \{p \in M \mid h(p) \neq 0\}$ and $\Gamma = \{q \in M \mid h(q) = 0\}$. Suppose that $O \neq \emptyset$ and $\Gamma \neq \emptyset$. If we put $Y = X$ in (2.8), then we have, with the help of (2.1),

$$(\nabla_X J)X + (\nabla_{JX} J)JX = g(B, JX)X - g(B, X)JX + g(X, X)JB. \tag{3.10}$$

We choose two unit vectors W and X which define orthogonal holomorphic planes $\{W, JW\}$ and $\{X, JX\}$. In [8, Equation (12.5)], it is shown that

$$2W(h) + 3hg((\nabla_X J)X + (\nabla_{JX} J)JX, JW) = 0. \tag{3.11}$$

Substituting (3.10) into (3.11), we obtain

$$2W(h) + 3h\omega(W) = 0,$$

which implies

$$3h\omega + 2dh = 0, \quad dh \wedge \omega + hd\omega = 0. \tag{3.12}$$

Hence we have $\omega = -\frac{1}{3}d\log(h^2)$ and $d\omega = 0$ on O. Since (M, J, g) is a generalized complex space form, the Bochner curvature tensor vanishes on M [8]. And the Bochner curvature tensor is an invariant of a conformal transformation. Therefore the open set O is locally conformal to a Bochner flat Kaehler manifold.

Let q be any point of Γ and let $d\omega \neq 0$ at q. Then $d\omega \neq 0$ on an open neighborhood $\mathcal{U}$ of q in M. If there exists a point p in $\mathcal{U}$ such that $h \neq 0$ on an open neighborhood

$\mathscr{V}$ of p, then $d\omega = 0$ on $\mathscr{V}$ by the previous argument. But this is impossible. Therefore $h = 0$ holds on all of $\mathscr{U}$ and $R = f\pi_1$ on $\mathscr{U}$. Hence $\mathscr{U}$ is locally conformal to the 4-dimensional Euclidean space.

Summing up the above results and Olszak's [5, Theorem 2] we have the following

THEOREM 3.2. *Let (M, J, g) be a 4-dimensional almost Hermitian manifold with pointwise constant holomorphic sectional curvature μ and let the curvature tensor of M satisfies*

$$R(X, Y, Z, W) = R(JX, JY, Z, W) + R(TX, Y, JZ, W) + R(JX, Y, Z, JW)$$

for $X, Y, Z, W \in \chi(M)$.

(1) *If $h = \tau/24 - \mu/2 = 0$ holds everywhere on M, then M is of constant sectional curvature μ.*

(2) *If $h = \tau/24 - \mu/2$ is a nonzero constant, then M is a complex space form, that is, a Kaehlerian manifold with constant holomorphic sectional curvature.*

(3) *If $h = \tau/24 - \mu/2 \neq 0$ at each point of M and h is not constant, then $(M, J, \tilde{g})$ is a Bochner flat Kaehler manifold, where we have put $\tilde{g} = e^{-\sigma}g$, $\sigma = -\frac{1}{3}\log(C_1 h^2)$, C_1 is a positive constant.*

(4) *If $\{p \in M \mid h(p) \neq 0\} \neq \emptyset$, $\{p \in M \mid h(p) = 0\} \neq \emptyset$ and $\{p \in M \mid h = 0$ and $d\omega = 0$ at $p\} = \emptyset$, then M is locally conformal to Bochner flat Kahler manifold or Euclidean space.*

4. An example

In this section, using Derdzinski's results and Olszak's theorem we shall give an example of an Einstein and a weakly $*$-Einstein Hermitian manifold with pointwise constant holomorphic sectional curvature.

Let $(M, J, \tilde{g})$ be a Bochner flat Kaehlerian manifold of dimension four. Assume, additionally, that the scalar curvature $\tilde{\tau}$ of $\tilde{g}$ is nonzero everywhere on M and non-constant. Such an example was constructed by Derdzinski in [1, 5]. Let $g = e^{\sigma}\tilde{g}$, where $\sigma = -\log(C\tilde{\tau}^2)$, C is a positive constant. Then (M, J, g) is a Hermitian manifold and M is a generalized complex space form for which $h = (C/24^3)\tilde{\tau}^3 \neq 0$. h is not constant since $\tilde{\tau}$ is not constant. Since h is not constant, we have from (3.9) $\tau/24 - \mu/2 \neq$ const. Since M is a generalized complex space form, it is Einstein and weakly $*$-Einstein. Hence we have τ is a constant. Therefore the holomorphic sectional curvature μ is not constant. Thus (M, J, g) is an Einstein and weakly $*$-Einstein Hermitian manifold with pointwise constant holomorphic sectional curvature which is not globally constant.

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Department of Mathematics Education
Sung Kyun Kywan University
Seoul 110-745
Korea
e-mail: kimuk@yurim.skku.ac.kr