Note on a Theorem of Lommel.

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1.

Among the many formulae which show special relations existing between the circular functions and the Bessel-Function $J_n(x)$, when n is half an odd integer, there is one due to Lommel

$$\frac{\sin 2x}{\pi} = \{J_{\frac{1}{2}}(x)\}^2 - 3\{J_{\frac{3}{2}}(x)\}^2 + 5\{J_{\frac{5}{2}}(x)\}^2 - \ldots\ldots$$

In connection with the paper on Basic sines and cosines in this volume of the *Proceedings*, it may be interesting to consider briefly an analogue of Lommel's theorem, which we write

$$\sum_{r=0}^{r=\infty}(-1)^r \frac{[4r+2]}{[2]} p^{r(r-1)} J_{\left[\frac{2r+1}{2}\right]}(x)\, \mathfrak{J}_{\left[\frac{2r+1}{2}\right]}(x)$$

$$= \frac{1}{[2]^2\{\Gamma_{p^2}[\frac{3}{2}]\}^2}\left[(1+p)x - \frac{2(1+p)(1+p^3)x^3}{[3]!} + \frac{2(1+p^2)(1+p)(1+p^3)(1+p^5)x^5}{[5]!} - \ldots\right], \quad (2)$$

the general term of the series within the large brackets being

$$(-1)^r \frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2}).(1+p^1)(1+p^3)\ldots(1+p^{2r+1})}{[2r+1]!} x^{2r+1}.$$

When the base p equals 1, this series reduces to $\dfrac{\sin 2x}{\pi}$.

2.

Defining $J_{[n]}(x)$ as $\displaystyle\sum_{r=0}^{r=\infty}(-1)^r \frac{x^{n+2r}}{\{2n+2r\}!\,\{2r\}!}$,

$\{2m\}!$ is in general $[2]^m \Gamma_{p^2}([m+1])$.

This reduces, when m is a positive integer, to $[2][4][6]\ldots[2m]$.

We take $\displaystyle \mathfrak{J}_{[n]}(x) = \sum_{r=0}^{r=\infty}(-1)^r \frac{x^{n+2r}}{\{2n+2r\}!\,\{2r\}!} p^{2r(n+r)}.$

This function is connected with $J_{[n]}$ by an inversion of the base p.

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In a paper shortly to be printed (*Proc. R. S.*) it is shown that

$$J_{[m]}(x)\,\mathfrak{J}_{[n]}(x)=J_{[n]}(x)\,\mathfrak{J}_{[m]}(x)=\sum_{r=0}^{r=\infty}(-1)^r\frac{\{2m+2n+4r\}_r}{\{2m+2r\}\,!\,\{2n+2r\}\,!\,\{2r\}\,!}\,x^{m+n+2r}, \quad (3)$$

where $\{2m+2n+4r\}_r=[2m+2n+4r][2m+2n+4r-2]\ldots[2m+2n+2r+2]$.

3.

Consider now the series

$$[2]J_{[\frac{1}{2}]}\mathfrak{J}_{[\frac{1}{2}]}-[6]J_{[\frac{3}{2}]}\mathfrak{J}_{[\frac{3}{2}]}+\ldots+(-1)^s p^{s(s-1)}J_{\left[\frac{2s+1}{2}\right]}\mathfrak{J}_{\left[\frac{2s+1}{2}\right]}-\ldots\ldots \quad (4)$$

This series may be written by means of (3) in the form

$$[2]\Sigma(-1)^r\frac{\{4r+2\}_r}{\{2r+1\}\,!\,\{2r+1\}\,!\,\{2r\}\,!}\,x^{2r+1}$$

$$-[6]\Sigma(-1)^r\frac{\{4r+6\}_r}{\{2r+3\}\,!\,\{2r+3\}\,!\,\{2r\}\,!}\,x^{2r+3}$$

$$\ldots$$

$$\ldots$$

$$(-1)^{r+s}[4s+2]\,\Sigma\,\frac{\{4r+4s+2\}_r}{\{2r+2s+1\}\,!\,\{2r+2s+1\}\,!\,\{2r\}\,!}\,x^{2r+2s+1}$$

$$\ldots$$

$$\ldots$$

Collecting the terms in a series of ascending powers of x, the coefficient of x arises only from the first of these series, and is

$$[2]\frac{\{2\}_0}{\{1\}\,!\,\{1\}\,!\,\{0\}\,!},$$

which reduces to

$$\frac{[2]}{[2]^{\frac{1}{2}}[2]^{\frac{1}{2}}\{\Gamma_{p^2}(\frac{3}{2})\}^2}$$

$$=\frac{1}{\{\Gamma_{p^2}([\frac{3}{2}])\}^2},$$

as is seen from the definition of the function $\{2n\}\,!$.

The coefficient of x^3 is

$$-[2]\frac{\{6\}_1}{\{3\}\,!\,\{3\}\,!\,\{2\}\,!}-[6]\frac{\{6\}_0}{\{3\}\,!\,\{3\}\,!\,\{0\}\,!}$$

and this reduces to

$$\frac{2(1+p^3)}{[3]\,!}\,\frac{1}{\{\Gamma_{p^2}[\frac{3}{2}]\}^2}.$$

4.

The reduction, however, of the series which forms the coefficient of x^{2r+1}, offers some difficulty and is effected by a theorem due to Heine (*Kugelfunctionen*, Appendix to Chap. II., Vol. I.).

The coefficient of x^{2r+1} is, after some obvious reductions,

$$(-1)^r\frac{\{4r+2\}_r}{\{2r+1\}!\{2r+1\}!\{2r\}!}\left[[2]+[6]\frac{[2r]}{[2r+4]}+p^2[10]\frac{[2r][2r-2]}{[2r+4][2r+6]}+\dots\right.$$
$$\left.+p^{s(s-1)}[4s+2]\frac{[2r][2r-2]..[2r-2s+2]}{[2r+4]......[2r+2s+2]}+\dots\right]. \quad - \quad - \quad (5)$$

Now
$$[2]=\frac{p^2-1}{p-1}=\frac{p^2}{p-1}-\frac{1}{p-1},$$
$$[6]=\frac{p^6}{p-1}-\frac{1}{p-1},$$
......
......
$$[4s+2]=\frac{p^{4s+2}}{p-1}-\frac{1}{p-1},$$
......
......

therefore we write the series which is within the large brackets of expression (5) as the difference of two series, viz.,

$$\frac{p^2}{p-1}S^1-\frac{1}{p-1}S_2=\frac{p_2}{p-1}\left\{1+p^4\frac{[2r]}{[2r+4]}+..+p^{s(s+3)}\frac{[2r].....[2r-2s+2]}{[2r+4]..[2r+2s+2]}+..\right\}$$
$$-\frac{1}{p-1}\left\{1+\frac{[2r]}{[2r+4]}+..+p^{s(s-1)}\frac{[2r].....[2r-2s+2]}{[2r+4]..[2r+2s+2]}+..\right\}. \quad (6)$$

5.

Heine has shown that if

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2+\dots=\phi[a,\ b,\ c,\ q,\ x],$$

then
$$\phi[a,\ b,\ c,\ q,\ x]=\prod_{n=0}^{n=\infty}\frac{(1-bxq^n)\left(1-\frac{c}{b}q^n\right)}{(1-xq^n)(1-cq^n)}\phi\left[b,\ \frac{abx}{c},\ bx,\ q,\ \frac{c}{b}\right].$$

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If in this transformation we put
$$a=p^2,\quad q=p^2,\quad b=p^{-2r},\quad c=p^{2r+4},\quad x=-p^{2r+4},$$
$\phi[a, b, c, q, x]$ becomes, after simple and obvious reductions, identical with S_1.

The infinite product on the right side of Heine's transformation, reduces (r integral) to the finite product

$$\frac{(1+p^4)(1+p^6)\ldots\ldots(1+p^{2r+2})}{(1-p^{2r+4})(1-p^{2r+6})\ldots(1-p^{4r+2})}$$

which we will for convenience denote by P_1.

$$\phi\left[b, \frac{abx}{c}, bx, q, \frac{c}{b}\right] \text{ becomes}$$

$$\left\{1-\frac{(p^{2r}-1)(p^{2r-2}+1)}{(p^2-1)(p^4+1)}p^6+\frac{(p^{2r}-1)(p^{2r-2}-1)(p^{2r-2}+1)(p^{2r-4}+1)}{(p^2-1)(p^4-1)(p^4+1)(p^6+1)}p^{16}-\ldots\right\}$$

$$=1-a_1+a_2-\ldots,$$

so that $$S_1=P_1\{1-a_1+a_2-a_3+\ldots\}.$$

Similarily, if we put
$$a=p^2,\quad q=p^2,\quad b=p^{-2r},\quad c=p^{2r+4},\quad x=-p^{2r},$$
in Heine's transformation, we obtain

$$\phi[a, b, c, q, x]=S_2,$$

$$\phi\left[b, \frac{abx}{c}, bx, q, \frac{c}{b}\right]$$
$$=\left\{1-\frac{(p^{2r}-1)(p^{2r+2}+1)}{(p^2-1)(1+1)}p^2+\frac{(p^{2r}-1)(p^{2r-2}-1)(p^{2r+2}+1)(p^{2r}+1)}{(p^2-1)(p^4-1)(1+1)(1+p^2)}p^8-\ldots\right\}$$
$$=1-b_1+b_2-\ldots.$$

The infinite product becomes

$$\frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2})}{(1-p^{2r+4})(1-p^{2r+6})\ldots(1-p^{4r+2})}=P_2.$$

Finally we have

$$\frac{p^2}{p-1}S_1-\frac{1}{p-1}S_2=\frac{p^2}{p-1}P_1\{1-a_1+a_2-..\}-\frac{1}{p-1}P_2\{1-b_1+b_2-..\}. \qquad (7)$$

P_1 and P_2 have most of their factors in common, so taking out the common part we may write (7)

$$\frac{(1+p^4)(1+p^6)\ldots(1+p^{2r-2})}{(1-p^{2r+4})(1-p^{2r+6})\ldots(1-p^{4r+2})}\cdot\frac{1}{(p-1)}$$

$$\left\{p^2(1+p^{2r})(1+p^{2r+2})\{1-a_1+a_2-..\}-2(1+p^2)\{1-b_1+b_2-..\}\right\}. \qquad (8)$$

We sum the series within the large brackets as follows:

$$\begin{array}{l} -2(1+p^2) \\ +2(1+p^2)b_1+p^2(1+p^{2r})(1+p^{2r+2}) \\ -2(1+p^2)b_2-p^2(1+p^{2r})(1+p^{2r+2})a_1 \\ \quad\ldots \qquad\qquad \ldots \\ \quad\ldots \qquad\qquad \ldots \end{array}$$

in general taking the term involving b_n with the term involving b_{n-1}.

Without difficulty, even in the general term, we can reduce this series to

$$-2(1+p^2)+2p^2\frac{[4r+4]}{[2]}-2p^8\frac{[4r+4][4r]}{[2][8]}+2p^{18}\frac{[4r+4][4r][4r-4]}{[2][8][12]}-\ldots$$

$$=-2(1+p^2)\left[1-p^2\frac{[4r+4]}{[4]}+p^8\frac{[4r+4][4r]}{[4][8]}-\ldots\right].$$

The product expression for the series within the large brackets is

$$(1-p^2)(1-p^6)(1-p^{10})\ldots(1-p^{4r+2}).$$

Expression (8) is thus reduced to

$$-\frac{(1+p^4)\ldots(1+p^{2r-2})}{(1-p^{2r+4})\ldots(1-p^{4r+2})}\cdot\frac{(p+1)}{(p^2-1)}2(1+p^2).(1-p^2)(1-p^6)..(1-p^{4r+2}).$$

The coefficient of x^{2r+1} is then by (5) and (9), after cancelling common factors,

$$(-1)^r\frac{2(1+p^2)(1+p^4)\ldots(1+p^{2r-2}).(1+p^3)\ldots(1+p^{2r+1})}{[2r+1]!}\frac{1}{\{\Gamma_{p^2}([\frac{3}{2}])\}^2},$$

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which establishes

$$[2]J_{[\frac{1}{2}]}\mathfrak{J}_{[\frac{1}{2}]} - [6]J_{[\frac{3}{2}]}\mathfrak{J}_{[\frac{3}{2}]} + \ldots = \frac{1}{\{\Gamma_{p^2}(\frac{3}{2})\}^2}\left[x - \frac{2(1+p^3)}{[3]}x^3 + \ldots\right].$$

Dividing throughout by [2] we have the theorem (2) as stated.

We may show also that

$$J_{[\frac{1}{2}]}\mathfrak{J}_{[\frac{1}{2}]} + J_{[-\frac{1}{2}]}\mathfrak{J}_{[-\frac{1}{2}]} = \frac{[2]}{x\{\Gamma_{p^2}(\frac{1}{2})\}^2},$$

which reduces when $p = 1$ to

$$J_{\frac{1}{2}}^2 + J_{-\frac{1}{2}}^2 = \frac{2}{\pi x}, \quad \text{(Lommel)}$$

and, in general, if $n = \frac{1}{2}(2\kappa + 1)$,

$$J_{[n]}\mathfrak{J}_{[n]} + J_{[-n]}\mathfrak{J}_{[-n]} = \frac{1}{\{\Gamma_{p^2}(\frac{1}{2})\}^2}\left[a_1 x^{-1} + a_2 x^{-2} + \ldots + a_{\kappa-1}x^{-2\kappa+1}\right],$$

where a_1, a_2, ... are simple expressions of factors of the type $[m]$.

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