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COMMON FIXED POINTS FOR SEMIGROUPS OF POINTWISE LIPSCHITZIAN MAPPINGS IN BANACH SPACES

W. M. KOZLOWSKI

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Abstract

Let C be a bounded, closed, convex subset of a uniformly convex Banach space X. We investigate the existence of common fixed points for pointwise Lipschitzian semigroups of nonlinear mappings $T_t : C \to C$, where each T_t is pointwise Lipschitzian. The latter means that there exists a family of functions $\alpha_t : C \to [0, \infty)$ such that $\|T_t(x) - T_t(y)\| \le \alpha_t(x)\|x - y\|$ for $x, y \in C$. We also demonstrate how the asymptotic aspect of the pointwise Lipschitzian semigroups can be expressed in terms of the respective Fréchet derivatives.

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1. Introduction

Let C be a bounded, closed, convex subset of a Banach space X. The purpose of this paper is to prove the existence of common fixed points for pointwise Lipschitzian semigroups of nonlinear mappings, that is, families of mappings $T_t : C \to C$ satisfying the following conditions: $T_0(x) = x$, $T_{s+t} = T_s(T_t(x))$, and each T_t is pointwise Lipschitzian. The latter means that there exists a family of functions $\alpha_t : C \to [0, \infty)$ such that $\|T_t(x) - T_t(y)\| \le \alpha_t(x)\|x - y\|$ for $x, y \in C$ (see Definitions 2.1, 2.2, and 2.3 for more details).

Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the Banach space X would define the state space and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system. The question about the existence of common fixed points, and about the structure of the set of common fixed points, can be interpreted as a question whether there exist points that are fixed during the state space transformation T_t at any given point of time t, and, if yes, what the structure of a set of such points may look like. Our results cater for both the continuous and the discrete time cases. In the setting of this paper, the state space may be an infinite-dimensional Banach space. Therefore, it is natural to apply these result not only to deterministic dynamical systems but also to stochastic dynamical systems.

To prove that the set of common fixed points is nonempty, we need to restrict our considerations to the asymptotic pointwise Lipschitzian semigroups, that is, where $\limsup_{t\to\infty} \alpha_t(x) = 1$ for every $x \in C$. It is intriguing that, *a priori*, none of the mappings T_t need to have fixed points in C (T_t need not be nonexpansive as $\alpha_t(x)$ may be strictly bigger than 1), but, *a posteriori*, all mappings T_t have fixed points and actually they share the same fixed points! This seemingly paradoxical result is due of course to the asymptotic behavior of the semigroup $\{T_t\}$.

The existence of common fixed points for families of contractions and nonexpansive mappings has been investigated since the early 1960s; see for example DeMarr [6], Browder [3], Belluce and Kirk [1, 2], Lim [14], Bruck [4]. The asymptotic approach for finding common fixed points of semigroups of Lipschitzian (but not pointwise Lipschitzian) mappings has also been investigated for some time; see for example Tan and Xu [15]. It is worthwhile mentioning recent studies on the special case, when the parameter set for the semigroup is equal to $\{0, 1, 2, 3, \ldots\}$ and $T_n = T^n$, the nth iterate of an asymptotic pointwise nonexpansive mapping, that is, a mapping $T : C \to C$ such that there exists a sequence of functions $\alpha_n : C \to [0, \infty)$ with $\|T^n(x) - T^n(y)\| \le \alpha_n(x)\|x - y\|$. Kirk and Xu [11] proved the existence of fixed points for asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings in Banach spaces, while Hussain and Khamsi extended this result to metric spaces [7], and Khamsi and Kozlowski to modular function spaces [8, 9]. Recently, Kozlowski proved the existence of common fixed points for semigroups of nonlinear contractions and nonexpansive mappings in modular function spaces [13].

Kozlowski in [12] proved convergence to fixed points of some iterative algorithms applied to asymptotic pointwise nonexpansive mappings. It is an interesting question whether similar iterative algorithms can be used for the construction of a common fixed point for an asymptotically nonexpansive semigroup of pointwise Lipschitzian mappings.

The paper is organized as follows.

(a) Section 2 provides necessary preliminary material.
(b) Section 3 starts with a fixed-point theorem for asymptotically contractive pointwise Lipschitzian semigroups (Theorem 3.1) and the analogous result for the specific case when all mappings $S_t \in \mathcal{F}$ are continuously Fréchet differentiable (Theorem 2.5). This result may be very useful for applications provided the Fréchet derivatives can be easily estimated.
(c) Theorem 3.4 proves the existence of common fixed points for asymptotically nonexpansive pointwise Lipschitzian semigroups. The Fréchet differentiable case is described in Theorem 3.5.

2. Preliminaries

Throughout this paper X will denote a Banach space, and C a nonempty, bounded, closed and convex subset of X. Throughout this paper J will be a fixed parameter semigroup of nonnegative numbers, that is, a subsemigroup of $[0, \infty)$ with normal addition. We assume that $0 \in J$ and that there exists $t > 0$ such that $t \in J$. The latter assumption implies immediately that $+\infty$ is a cluster point of J in the sense of the natural topology inherited by J from $[0, \infty)$. Typical examples are: $J = [0, \infty)$ and $J = \{0, 1, 2, 3, \ldots\}$, and also ideals of the form $J = \{n\alpha; n = 0, 1, 2, 3, \ldots\}$ for a given $\alpha > 0$.

Let us start with more formal definitions of pointwise Lipschitzian mappings and pointwise Lipschitzian semigroups of mappings, and associated notational conventions.

Definition 2.1. We say that $T : C \to C$ is a pointwise Lipschitzian mapping if there exists a function $\alpha : C \to [0, \infty)$ such that

$$\|T(x) - T(y)\| \leq \alpha(x)\|x - y\| \quad \text{for all } x, y \in C. \tag{2.1}$$

If the function $\alpha(x) < 1$ for every $x \in C$, then we say that T is a pointwise contraction. Similarly, if $\alpha(x) \leq 1$ for every $x \in C$, then T is said to be a pointwise nonexpansive mapping.

Definition 2.2. A one-parameter family $\mathcal{F} = \{T_t : t \in J\}$ of mappings from C into itself is said to be a pointwise Lipschitzian semigroup on C if $\mathcal{F}$ satisfies the following conditions.

(i) $T_0(x) = x$ for $x \in C$.
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in J$.
(iii) For each $t \in J$, T_t is a pointwise Lipschitzian mapping, that is, there exists a function $\alpha_t : C \to [0, \infty)$ such that

$$\|T_t(x) - T_t(y)\| \leq \alpha_t(x)\|x - y\| \quad \text{for all } x, y \in C.$$

(iv) For each $x \in C$, the mapping $t \to T_t(x)$ is strong continuous.

For each $t \in J$, let $F(T_t)$ denote the set of fixed points of T_t. Define the set of all common set points for mappings from $\mathcal{F}$ as the following intersection

$$F(\mathcal{F}) = \bigcap_{t \in J} F(T_t).$$

Definition 2.3. Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup.

(1) $\mathcal{F}$ is said to be asymptotically contractive if $\limsup_{t\to\infty} \alpha_t(x) < 1$ for every $x \in C$.
(2) $\mathcal{F}$ is said to be asymptotically nonexpansive if $\limsup_{t\to\infty} \alpha_t(x) \leq 1$ for every $x \in C$.

The following result was proved by Kirk in [10, Proposition 2.1].

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Lemma 2.4. *Let $T : A \to A$ be continuously differentiable at any point of A where A is a bounded open convex set. Then T is a pointwise contraction if and only if for each $x \in A$, $\|T_x'\| < 1$.*

Using Lemma 2.4 it is not difficult to prove the following proposition. Please note that by $t \to \infty$ we will always mean that $t \to \infty$ over J.

Proposition 2.5. *Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup on C. Assume that all mappings $S_t \in \mathcal{F}$ are continuously Fréchet differentiable on an open convex set A containing C. Then the following statements hold.*

(1) *$\mathcal{F}$ is asymptotically contractive on C if and only if for each $x \in C$*

$$\limsup_{t\to\infty} \|(T_t)_x'\| < 1.$$

(2) *$\mathcal{F}$ is asymptotically nonexpansive on C if and only if for each $x \in C$*

$$\limsup_{t\to\infty} \|(T_t)_x'\| \le 1.$$

3. Results

Theorem 3.1. *Assume X is uniformly convex. Let $\mathcal{F}$ be an asymptotically contractive pointwise Lipschitzian semigroup on C. Then $\mathcal{F}$ has a unique common fixed point $z \in C$ and for each $x \in C$, $\|T_t(x) - z\| \to 0$ as $t \to \infty$.*

Proof. Fix $x \in C$ and define the type function φ by

$$\varphi(y) = \limsup_{t\to\infty} \|T_t(x) - y\|.$$

Since φ is lower semicontinuous on a weakly compact set C then it attains its minimum at a point $z \in C$. Observe that for each $y \in C$ and each $s \in J$

$$\begin{aligned}\varphi(T_s(y)) &= \limsup_{t\to\infty} \|T_t(x) - T_s(y)\| = \limsup_{t\to\infty} \|T_{s+t}(x) - T_s(y)\| \\ &\le \limsup_{t\to\infty} \alpha_s(y)\|T_t(x) - y\| = \alpha_s(y)\varphi(y).\end{aligned} \tag{3.1}$$

Applying (3.1) to z and using the minimality of z,

$$\varphi(z) \le \varphi(T_s(z)) \le \alpha_s(z)\varphi(z).$$

Letting $s \to \infty$, we get

$$\varphi(z) \le \alpha(z)\varphi(z),$$

which, in view of $\alpha(z) < 1$, implies that $\varphi(z) = 0$. Then, by (3.1), for every $s \in J$,

$$\lim_{t\to\infty} \|T_t(x) - T_s(z)\| = 0. \tag{3.2}$$

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Taking $s = 0$ we get

$$\lim_{t\to\infty} \|T_t(x) - z\| = 0. \tag{3.3}$$

From (3.2) and (3.3) it follows immediately that $T_s(z) = z$, that is, $z \in F(\mathcal{F})$ as claimed. To prove the uniqueness let us take another $w \in F(\mathcal{F})$. Hence for each $t \in J$,

$$\|z - w\| = \|T_t(z) - T_t(w)\| \le \alpha_t(z)\|z - w\|.$$

Letting $t \to \infty$,

$$\|z - w\| \le \alpha(z)\|z - w\|,$$

which gives us $w = z$ since $\alpha(z) < 1$. This completes the proof of the theorem. □

Combining Theorem 3.1 with Proposition 2.5 we immediately obtain the following fixed-point result expressed in a form convenient for applications.

Theorem 3.2. *Assume X is uniformly convex. Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup on C. Assume that all mappings $T_t \in \mathcal{F}$ are continuously Fréchet differentiable on an open convex set A containing C and for each $x \in C$*

$$\limsup_{t\to\infty} \|(T_t)'_x\| < 1.$$

Then $\mathcal{F}$ has a unique common fixed point $z \in C$ and for each $x \in C$, $\|T_t(x) - z\| \to 0$ as $t \to \infty$.

The following lemma is a generalization of the result for nonexpansive mapping obtained by Bruck [5]. Please note that the function γ_2 is constructed for all pointwise Lipschitzian mappings defined in C. In the context of this paper, Lemma 3.3 will be used for proving the convexity of the set of common fixed points for mappings from $\mathcal{F}$. There are, however, many other possible applications of this result.

Lemma 3.3. *Let X be a uniformly convex Banach space, and let $C \subset X$ be nonempty, bounded, closed and convex. There exists a strictly increasing, convex continuous function $\gamma_2 : [0, \infty) \to [0, \infty)$ with $\gamma_2(0) = 0$, depending only on* diam(C), *such that for every pointwise Lipschitzian mapping $T : C \to C$, every $c \in [0, 1]$ and every $x, y \in C$*

$$\gamma_2\Big(\frac{\|T(cx + (1-c)y) - cT(x) - (1-c)T(y)\|}{\alpha(cx + (1-c)y)}\Big) \le \|x - y\| - \frac{\|T(x) - T(y)\|}{\alpha(cx + (1-c)y)}. \tag{3.4}$$

Proof. Let us fix a pointwise Lipschitzian mapping T, and hence fix its Lipschitzian parameter function α. Fix also $c \in [0, 1]$ and $x, y \in C$. Let $\delta : [0, 2] \to [0, 1]$ be the modulus of convexity for X. From the definition of the modulus of convexity it follows easily that

$$2 \min(c, 1-c)\delta(\|u - v\|) \le 1 - \|cu + (1-c)v\| \tag{3.5}$$

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for $\|u\|, \|v\| \leq 1$. Define a function

$$d(t) = \frac{1}{2} \int_0^t \delta(s)\, ds, \qquad t \in [0, 2],$$

and extend it to the interval $(2, \infty)$ by $d(t) = d(2) + \frac{1}{2}\delta(2)(t-2)$. It is easy to prove that d is a strictly increasing, continuous, convex function with $d(0) = 0$ such that

$$d(t) \leq \delta(t), \qquad t \in [0, 2], \tag{3.6}$$

and the function

$$s \mapsto \frac{d(s)}{s} \tag{3.7}$$

is increasing in $[0, 2]$. Using (3.6) and the fact that $c(1-c) \leq \min(c, 1-c)$ we deduce from (3.5) that

$$2c(1-c)d(\|u-v\|) \leq 1 - \|cu + (1-c)v\|. \tag{3.8}$$

Define $w = cx + (1-c)y$ and

$$u = \frac{T(y) - T(w)}{c\|x-y\|\alpha(w)}, \qquad v = \frac{T(w) - T(x)}{(1-c)\|x-y\|\alpha(w)}.$$

Using (2.1) it is easy to prove that $\|u\|, \|v\| \leq 1$. It follows from the definitions of u and v that

$$u - v = \frac{cT(x) + (1-c)T(y) - T(w)}{c(1-c)\alpha(w)\|x-y\|}, \tag{3.9}$$

$$cu + (1-c)v = \frac{T(x) - T(y)}{\alpha(w)\|x-y\|}. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), and multiplying both sides by $\|x-y\|$ we arrive at

$$2c(1-c)\|x-y\|d\left(\frac{\|cT(x) + (1-c)T(y) - T(w)\|}{c(1-c)\alpha(w)\|x-y\|}\right) \leq \|x-y\| - \frac{\|T(x) - T(y)\|}{\alpha(w)}. \tag{3.11}$$

Since

$$c(1-c)\|x-y\| \leq \frac{\operatorname{diam}(C)}{4},$$

we deduce from (3.11), using (3.7), that

$$\frac{1}{2}\operatorname{diam}(C)d\left(\frac{4\|T(w) - cT(x) - (1-c)T(y)\|}{\operatorname{diam}(C)\alpha(w)}\right) \leq \|x-y\| - \frac{\|T(x) - T(y)\|}{\alpha(w)}.$$

Defining

$$\gamma_2(t) = \frac{1}{2}\operatorname{diam}(C)d\left(\frac{4t}{\operatorname{diam}(C)}\right)$$

we get (3.4) which completes the proof. □

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THEOREM 3.4. *Assume X is uniformly convex. Let $\mathcal{F}$ be an asymptotically nonexpansive pointwise Lipschitzian semigroup on C. Then $\mathcal{F}$ has a common fixed point and the set $F(\mathcal{F})$ of common fixed points is closed convex.*

PROOF. Fix an arbitrary $x \in C$. Let us define the type function φ by the formula

$$\varphi(y) = \limsup_{t\to\infty} \|T_t(x) - y\|^2$$

and note that φ is lower semicontinuous in C. Hence, it attains its minimum in C, that is, there exists a $z \in C$ such that $\varphi(z) = \min\{\varphi(y) : y \in C\}$. We will prove now that $T_t(z)$ satisfies the following Cauchy type condition: to every $\epsilon > 0$ there exists a $t_\epsilon \in J$ such that

$$\|T_s(z) - T_u(z)\| \leq \epsilon$$

for all $s, u \in J$ such that $s, u \geq t_\epsilon$. To this end, let us fix any $t, s, u \in J$ and any $\epsilon > 0$. By [11, Proposition 3.4] (see also [16, Theorem 2]) for each $d > 0$ there exists a continuous function $\lambda : [0, \infty) \to [0, \infty)$ such that $\lambda(t) = 0$ if and only if $t = 0$, and

$$\|cw + (1 - c)v\|^2 \leq c\|w\|^2 + (1 - c)\|v\|^2 - c(1 - c)\lambda(\|w - v\|), \tag{3.12}$$

for any $c \in [0, 1]$ and all $w, v \in X$ such that $\|w\| \leq d$ and $\|v\| \leq d$. Applying (3.12) to $w = T_{s+u+t}(x) - T_s(z)$, $v = T_{s+u+t}(x) - T_u(z)$, $d = \operatorname{diam}(C) + \|x\|$ and $c = \frac{1}{2}$ we obtain the following inequality:

$$\begin{aligned}&\|T_{s+u+t}(x) - \tfrac{1}{2}(T_s(z) + T_u(z))\|^2\\ &\quad\leq \tfrac{1}{2}\|T_{s+u+t}(x) - T_s(z)\|^2 + \tfrac{1}{2}\|T_{s+u+t}(x) - T_u(z)\|^2 - \tfrac{1}{4}\lambda(\|T_s(z) - T_u(z)\|).\end{aligned}$$

Letting $t \to \infty$ and using the definition of φ and the pointwise Lipschitzian properties of $\mathcal{F}$, we obtain

$$\varphi\Big(\frac{T_s(z) + T_u(z)}{2}\Big) \leq \frac{1}{2}(\alpha_s(z)^2 + \alpha_u(z)^2)\varphi(z) - \frac{1}{4}\lambda(\|T_s(z) - T_u(z)\|).$$

Using the convexity of C and the minimality property of φ at z we deduce that

$$\varphi(z) \leq \varphi\Big(\frac{T_s(z) + T_u(z)}{2}\Big)$$

for all $s, u \in J$. Hence

$$\varphi(z) \leq \tfrac{1}{2}(\alpha_s(z)^2 + \alpha_u(z)^2)\varphi(z) - \tfrac{1}{4}\lambda(\|T_s(z) - T_u(z)\|),$$

and then

$$\lambda(\|T_s(z) - T_u(z)\|) \leq 2(\alpha_s(z)^2 + \alpha_u(z)^2)\varphi(z) - 4\varphi(z).$$

Keeping in mind that $\alpha_s(z) \to 1$ as $s \to \infty$, that the same holds for u, and that $\lambda(t) = 0$ if and only if $t = 0$ and λ is continuous, we can select $t_\epsilon \in J$ such that

$$\|T_s(z) - T_u(z)\| \leq \epsilon$$

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for all $s, u \in J$ with $s, u \geq t_\epsilon$, as claimed. Then, using the completeness of X and closedness of C, there exists $f \in C$ such that

$$\lim_{s\to\infty} \|f - T_s(z)\| = 0. \tag{3.13}$$

We claim now that $f \in F(\mathcal{F})$. Indeed, let us fix any $t, s \in J$ and observe that

$$\|T_t(f) - T_{t+s}(z)\| \leq \alpha_t(f)\|f - T_s(z)\|.$$

Letting $s \to \infty$ and using (3.13), we get $\|T_t(f) - f\| \leq 0$ for every $t \in J$. Hence $f \in F(T_t)$ for every $t \in J$. Therefore, $f \in F(\mathcal{F})$, which means that f is a common fixed point.

To prove that $\mathcal{F}$ is closed it suffices to demonstrate that every $F(T_t)$ is closed. To this end, let $\{v_n\} \subset F(T_t)$ and $v_n \to v$. Observe that

$$\begin{aligned} \|T_t(v) - v\| &\leq \|T_t(v) - v_n\| + \|v_n - v\| = \|T_t(v) - T_t(v_n)\| + \|v_n - v\| \\ &\leq \alpha_t(v)\|v - v_n\| + \|v_n - v\| \to 0 \end{aligned}$$

as $n \to \infty$. Hence $v \in F(T_t)$, which proves that $F(T_t)$ is closed.

It remains to prove that $\mathcal{F}$ is convex. Let $f_1, f_2 \in \mathcal{F}$ and let $c \in [0, 1]$. We need to prove that $f = cf_1 + (1 - c)f_2 \in \mathcal{F}$. Fix any $t \in J$. Using Lemma 3.3 we have

$$\begin{aligned} \|T_t(f) - f\| &= \|T_t(f) - cf_1 - (1 - c)f_2\| = \|T_t(f) - cT_t(f_1) - (1 - c)T_t(f_2)\| \\ &\leq \alpha_t(f)\gamma_2\Big(\|f_1 - f_2\| - \frac{\|T_t(f_1) - T_t(f_2)\|}{\alpha_t(f)}\Big) \\ &= \alpha_t(f)\gamma_2\Big(\|f_1 - f_2\| - \frac{\|f_1 - f_2\|}{\alpha_t(f)}\Big). \end{aligned}$$

Since $\alpha_t(f) \to 1$ as $t \to \infty$, it follows that

$$\|T_t(f) - f\| \to 0 \tag{3.14}$$

as $t \to \infty$, which implies that $f \in F(\mathcal{F})$. Indeed, take any $s, t \in J$ and note that by (3.14)

$$\begin{aligned} \|T_s(f) - f\| &\leq \|T_s(f) - T_{t+s}(f)\| + \|T_{t+s}(f) - f\| \\ &\leq \alpha_s(f)\|f - T_t(f)\| + \|T_{t+s}(f) - f\| \to 0 \end{aligned}$$

as $t \to \infty$. Hence $f \in F(T_s)$ for any $s \in J$, that is, $f \in F(\mathcal{F})$. This completes the proof of the theorem. □

Combining Theorem 3.4 with Proposition 2.5 we immediately obtain the following fixed-point result.

THEOREM 3.5. *Assume X is uniformly convex. Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup on C. Assume that all mappings $T_t \in \mathcal{F}$ are continuously Fréchet differentiable on an open convex set A containing C and for each $x \in C$*

$$\limsup_{t\to\infty} \|(T_t)'_x\| \leq 1.$$

Then $\mathcal{F}$ has a common fixed point and the set $F(\mathcal{F})$ of common fixed points is closed convex.

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W. M. KOZLOWSKI, School of Mathematics and Statistics,
University of New South Wales, Sydney, NSW 2052, Australia
e-mail: w.m.kozlowski@unsw.edu.au

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