BULL. AUSTRAL. MATH. SOC. 46B20, 47H10
VOL. 46 (1992) [367–372]

STABILITY OF WEAK NORMAL STRUCTURE IN JAMES QUASI REFLEXIVE SPACE

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We introduce a coefficient on general Banach spaces which allows us to derive the weak normal structure for those Banach spaces whose Banach-Mazur distance to James quasi reflexive space is less than $\sqrt{3/2}$.

1. INTRODUCTION

Let K be a nonempty convex subset of a Banach space X. The set K is said to have normal structure if for each bounded convex subset $C \subset X$ consisting of more than one point there is a point $x \in C$ such that $\sup\{\|x-y\| : y \in C\} < diam(C)$. We say that X has weak normal structure if each nonempty convex and weakly compact subset $K \subset X$ has normal structure.

A mapping $T\colon K \to X$ is called nonexpansive if $\|T(x) - T(y)\| \leqslant \|x - y\|$ for all $x, y \in K$. We say that X has the fixed point property (FPP) if every nonexpansive mapping $T\colon K \to K$ defined on a nonempty convex and weakly compact subset K of X has a fixed point.

A classical result of Kirk [6] states that a Banach space X has the FPP whenever X has weak normal structure, but Karlovitz [3] showed that this last property is not necessary. "Nonstandard" techniques were used to show that certain Banach spaces that lack weak normal structure have the FPP [1, 7, 9]. Since Lin [7] proved positive results concerning the FPP in Banach spaces with unconditional basis and the James quasi reflexive space J cannot be embedded in a space with unconditional basis, it is somewhat surprising that the ideas arising from Lins's paper were used by Khamsi [4] to show that J has the FPP. But Tingley [11] proved that the FPP for J can be derived from Kirk's theorem [6] by showing in a direct fashion that the space J has weak normal structure. Our main result shows that J satisfies a stronger condition than that of Tingley.

Received 6 November 1991

https://doi.org/10.1017/S0004972700012016 Published online by Cambridge University Press

2. A sufficient condition for weak normal structure

We recall the following two properties each of which implies weak normal structure.

A Banach space X has property (*) (see [11]) if for every weakly null (and not constant) sequence (x_n) we have

$$\sup_m \left(\limsup_n \|x_n - x_m\| \right) > \liminf_n \|x_n\| .$$

A Banach space X with a Schauder basis has the Gossez-Lami Dozo property (in short: GLD property — see [2]) if for each $c > 0$ there exists $r = r(c) > 0$ such that for every $x \in X$ and every positive integer n we have

$$\|P_n(x)\| = 1 \quad \text{and} \quad \|(I - P_n)(x)\| \geqslant c \quad \text{implies} \quad \|x\| \geqslant 1 + r$$

where (P_n) is the sequence of natural projections associated with the basis.

A natural generalisation of GLD can be formulated as follows. We say that a Banach space X has the generalised Gossez-Lami Dozo property (GGLD property in short) if for every weakly null sequence (x_n) such that $\lim \|x_n\| = 1$, we have that $D[(x_n)] > 1$, where

$$D[(x_n)] = \limsup_m \left(\limsup_n \|x_n - x_m\| \right).$$

Since propety (*) implies weak normal structure (see [11]), the following result shows that property GGLD also implies weak normal structure.

Theorem 1. *If a Banach space X satisfies property GGLD then X satisfies property (*) and then X has weak normal structure.*

Proof: Let (x_n) be a weakly null sequence not constantly equal to 0. We can suppose that (x_n) has a subsequence (x'_n) such that there exists $b = \lim \|x_n\|$ and $b \neq 0$, because otherwise (*) is trivially satisfied.

If we put $y_n = x'_n / b$, then we have that (y_n) converges weakly to 0 and $\lim \|y_n\| = 1$, so that the hypothesis on property GGLD implies that $D[(y_n)] > 1$. Hence $D[(x'_n)] = bD[(y_n)] > b = \lim \|x'_n\|$ and from this the conclusion follows easily. $\square$

The converse of the above theorem is not true, since the space $X = (C_0, |\cdot|)$, where $|x| = \|x\|_\infty + \sum_{i=1}^{\infty} 2^{-i} |x_i|$ satisfies property (*), but fails to have property GGLD.

In order to get stability results for GGLD in terms of the Banach-Mazur distance (Recall that for isomorphic Banach spaces X and Y, the Banach-Mazur distance from X to Y, denoted $d(X, Y)$, is defined to be the infimum of $\|U\| \, \|U^{-1}\|$ taken over all bicontinuous linear operators from X onto Y) we define the coefficient $\beta(X)$ as the infimum of $D[(x_n)]$ taken over all weakly null sequences (x_n) in X such that $\lim\|x_n\| = 1$. Obviously, X has property GGLD if $\beta(X) > 1$, and moreover, the following theorem shows that, in this case, property GGLD is in some sense 'contagious' under slight perturbations of the norm.

Theorem 2. *If $\beta(X) > 1$ and $d(Y, X) < \beta(X)$, then Y has property GGLD.*

Proof: By definition of the Banach-Mazur distance, there exists an isomorphism $U: X \to Y$ such that $\|U^{-1}\| = 1$ and $\|U\| < \beta(X)$. Let (y_n) be any weakly null sequence in Y with $\lim\|y_n\| = 1$. Put $x_n = U^{-1}(y_n)$ and let (x'_n) be a subsequence of (x_n) such that there exists $b = \lim\|x'_n\|$. For the sequence (z_n) defined by $z_n = x'_n/b$ we have that (z_n) converges weakly to 0 and $\lim\|z_n\| = 1$, so that $D[(z_n)] \geqslant \beta(X)$. On the other hand, $D[(z_n)] \leqslant (1/b)D[(y_n)]$ and $\|U\|\,b \geqslant \lim\|U(x'_n)\| = 1$. Therefore, $\beta(X) \leqslant \|U\|\,D[(y_n)]$, what shows that $\beta(X) \leqslant \|U\|\,\beta(Y)$. It follows from this that $\beta(Y) > 1$, since $\|U\| < \beta(X)$. ∎

3. James quasi reflexive space

Recall that the James space J consists of all real sequences $x = (x_n)$ for which $\lim x_n = 0$ and $\|x\| < \infty$, where

$$\|x\| = \sup\{[(x_{p_1} - x_{p_2})^2 + \cdots + (x_{p_{m-1}} - x_{p_m})^2 + (x_{p_m} - x_{p_1})^2]^{1/2}\}$$

and the supremum is taken over all choices of m and $p_1 < p_2 < \ldots p_m$. Then J is a Banach space with norm $\|\cdot\|$ and the sequence (e_n) given by $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is in the nth position, is a Schauder basis for J (see [8]).

We need the following technical lemma.

Lemma. *Let x and y be defined as*

$$x = \sum_{i=a}^{b} x_i e_i, \qquad y = \sum_{i=c}^{d} y_i e_i$$

with $1 < a \leqslant b < c - 1$ and $c \leqslant d < \infty$. Then

$$\|x + y\|^2 \geqslant \frac{3}{4}\left(\|x\|^2 + \|y\|^2\right).$$

Proof: Since $\{i: x_i \neq 0\}$ is finite, the norm of x is attained for some finite increasing sequence, say

$$(1) \qquad \|x\|^2 = \sum_{i=1}^{k-1} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_k} - x_{p_1})^2.$$

https://doi.org/10.1017/S0004972700012016 Published online by Cambridge University Press

If $p_k > b$ we would have $x_{p_k} = 0$ and then, either $p_1 = 1$ and $\|x\|^2$ would be attained for the sequence $\{p_1, \dots, p_{k-1}\}$, or $p_1 > 1$ and $\|x\|^2$ would be attained for the sequence $\{1, p_1, \dots, p_{k-1}\}$. Hence we can suppose that $p_k \leqslant b$.

By definition of the norm on J, we have that

$$\left(x_{p_1} - x_{p_k}\right)^2 \leqslant \frac{1}{2}\|x\|^2$$

and so, by (1),

$$\sum_{i=1}^{k-1}\left(x_{p_i} - x_{p_{i+1}}\right)^2 \geqslant \frac{1}{2}\|x\|^2. \tag{2}$$

The inequality $a^2 + b^2 \geqslant (a-b)^2/2$, (1) and (2), give

$$\begin{aligned} x_{p_1}^2 &+ \sum_{i=1}^{k-1}\left(x_{p_i} - x_{p_{i+1}}\right)^2 + x_{p_k}^2 \\ &\geqslant \frac{1}{2}\left[\sum_{i=1}^{k-1}\left(x_{p_i} - x_{p_{i+1}}\right)^2 + \left(x_{p_k} - x_{p_1}\right)^2\right] + \frac{1}{2}\sum_{i=1}^{k-1}\left(x_{p_i} - x_{p_{i+1}}\right)^2 \\ &\geqslant \frac{3}{4}\|x\|^2. \end{aligned} \tag{3}$$

A similar argument shows that there exists a finite increasing sequence $\{q_1, \dots, q_r\}$, with $c \leqslant q_1 \leqslant q_r \leqslant d+1$, for which $\|y\|$ is attained and

$$y_{q_1}^2 + \sum_{j=1}^{r-1}\left(y_{q_j} - y_{q_{j+1}}\right)^2 + y_{q_r}^2 \geqslant \frac{3}{4}\|y\|^2. \tag{4}$$

Using the sequence $\{p_1, \dots, p_k, b+1, q_1, \dots, q_r, d+2\}$ we obtain

$$\|x+y\|^2 \geqslant x_{p_1}^2 + \sum_{i=1}^{k-1}\left(x_{p_i} - x_{p_{i+1}}\right)^2 + x_{p_k}^2 + y_{q_1}^2 + \sum_{j=1}^{r-1}\left(y_{q_j} - y_{q_{j+1}}\right)^2 + y_{q_r}^2$$

which with (3) and (4) gives the desired inequality. ∎

Theorem 3. *If X is any Banach space isomorphic to J, with $d(X, J) < \sqrt{3/2}$ then X satisfies property GGLD.*

Proof: By Theorem 2, we only need to prove that $\beta(J) \geqslant \sqrt{3/2}$. Let (x_n) be any weakly null sequence in J with $\lim\|x_n\| = 1$. Then there exists a subsequence (x'_n) of (x_n) and a sequence (u_n) in J such that

(i) $\lim\|x'_n - u_n\| = 0$

(ii) $u_n = \sum_{i=a_n}^{b_n} u_n(i)e_i$

https://doi.org/10.1017/S0004972700012016 Published online by Cambridge University Press

with $a_n \leqslant b_n < a_{n+1} - 1$ for every positive integer n.

Applying the lemma to the sequence (u_n), we get

$$\|u_n + u_m\|^2 \geqslant \frac{3}{4}\left(\|u_n\|^2 + \|u_m\|^2\right) \qquad (n \neq m), \tag{6}$$

and then, by (i) and (6), $D[(x'_n)] = D[(u_n)] \geqslant \sqrt{3/2}$. Since $D[(x_n)] \geqslant D[(x'_n)]$ and (x_n) is arbitrary, we have $\beta(J) \geqslant \sqrt{3/2}$. ∎

REMARK 1. By Theorem 3, a Banach space X has weak normal structure whenever $d(X, J) < \sqrt{3/2}$. This result is sharp because if we renorm the space J according to

$$|x| = \max\{\|x\|, \sqrt{3} \sup_{i<j} |x_i - x_j|\}$$

we have that $\|x\| \leqslant |x| \leqslant \sqrt{3/2}\,\|x\|$ and $(J, |\cdot|)$ lacks weak normal structure. Indeed, the sequence (x_n) given by $x_n = e_{3n+1} - e_{3n+2}$ is a diametral sequence in $(J, |\cdot|)$, from which the result follows (see, for example, [10]).

REMARK 2. Khamsi [5] associated with any Banach space X with a finite dimensional Schauder decomposition the coefficient $\beta_p(X)$ defined for $p \in [1, \infty)$ as the infimum of the set of numbers λ such that

$$(\|x\|^p + \|y\|^p)^{1/p} \leqslant \lambda \|x + y\|$$

for every x and y in X with $\operatorname{supp}(x) < \operatorname{supp}(y)$.

He showed that a Banach space X has weak normal structure whenever $\beta_p(X)_< 2^{1/p}$ for some $p \in [1, \infty)$. But for the decomposition given by the usual basis in J we have that $\beta_p(J) \geqslant 2^{1/p}$ for all $p \in [1, \infty)$.

Moreover, it is not hard to see that $\beta_p(X)\beta(X) \geqslant 2^{1/p}$ for all $p \in [1, \infty)$ and for all finite dimensional decompositions of X and hence $\beta(X) > 1$ if X has a finite dimensional decomposition such that $\beta_p(X) < 2^{1/p}$ for some $p \in [1, \infty)$.

Since $\beta(J) = \sqrt{3/2}$, this leads to the following question suggested by Khamsi (personal communication): Does the space J have a basis (or a finite dimensional decomposition) such that $\beta_p(J) < 2^{1/p}$ for some $p \in [1, \infty)$?

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https://doi.org/10.1017/S0004972700012016 Published online by Cambridge University Press