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The Grothendieck Trace and the de Rham Integral

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Abstract. On a smooth n-dimensional complete variety X over $\mathbb{C}$ we show that the trace map $\tilde{\theta}_X\colon H^n(X,\Omega^n_X) \to \mathbb{C}$ arising from Lipman's version of Grothendieck duality in [L] agrees with

$$(2\pi i)^{-n}(-1)^{n(n-1)/2}\int_X : H^{2n}_{DR}(X,\mathbb{C}) \to \mathbb{C}$$

under the Dolbeault isomorphism.

This short paper is concerned with explicating the relationship between the trace that occurs in Grothendieck duality and its counterpart (the integral) in the de Rham theory. The sign $(-1)^{n(n-1)/2}$ seen in our main theorem is forced upon us by our sign conventions on spectral sequences and the fact that we use the trace in Lipman's book [L] as the concrete version of Grothendieck's trace. This sign differs from the one obtained by Conrad in [C2], primarily because of the different definitions of the trace in Grothendieck duality. We elaborate on this in Section 5. Our sign differs from the one obtained by Deligne [D] because of hidden conventions. This issue is discussed on pp. 2–3 of [C1]—where the discrepancy is traced to the fact that Conrad uses the transitivity relations [C, p. 29, (2.2.3) and (2.2.4)], and Deligne uses a different set of transitivity relations. Our results are consistent with Conrad's and hence—via the discussion in [C1] and [C2] mentioned above—with Deligne's.

The justification for one more proof is in the methods. The two traces—one arising from Grothendieck duality and the other from de Rham theory—give two versions of the fundamental class of a point p of an n-dimensional smooth variety M over the complex numbers. The Grothendieck duality version of this fundamental class is the cohomology class in $H^n_p(\Omega^n_M)$ of $dz_1 \wedge \cdots \wedge dz_n/z_1\cdots z_n$ where $z_1,\ldots,z_n$ form a local system of coordinates at p (*cf.* [G1, Section 4]). On the other hand, from the de Rham point of view this fundamental class is the class of the Dirac delta distribution at p. We relate the two via the Bochner-Martinelli kernel. The relationship follows from the fact that the distributional derivative of the Bochner-Martinelli kernel (restricted to the second factor in $\mathbb{C}^n \times \mathbb{C}^n$) is the Dirac distribution at a point. We assume that all manifolds we deal with are paracompact and Hausdorff.

1 Currents

Recall that on a complex manifold M of dimension n, the cohomology groups $H^s(M,\Omega^r_M)$ can be represented as follows:

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(a) by $\bar{\partial}$-closed forms of degree (r, s) and
(b) by $\bar{\partial}$-closed currents of degree (r, s), where a current of degree (r, s) is an element of the space of continuous linear functionals on the space of compactly supported forms of degree $(n - r, n - s)$ [dR, Chapter III], [GH, Chapter 3.1].

If $\mathcal{A}^{a,b}$ (resp. $\mathcal{D}^{a,b}$) represent the sheaf of forms (resp. currents) of degree (a, b), then the natural inclusion $\mathcal{A}^{a,b} \subset \mathcal{D}^{a,b}$ respects the $\bar{\partial}$ operation and hence we have a map of complexes (of sheaves), $\mathcal{A}^{r,\bullet} \hookrightarrow \mathcal{D}^{r,\bullet}$ between two fine resolutions of Ω^r_M, and on taking global sections the resulting map $\Gamma(M, \mathcal{A}^{r,\bullet}) \to \Gamma(M, \mathcal{D}^{r,\bullet})$ is a quasi-isomorphism giving an isomorphism

$$H^{r,s}_{\bar{\partial}}(M) \xrightarrow{\sim} H^{r,s}_{\bar{\partial},\mathrm{curr}}(M) \tag{1.1}$$

where $H^{r,s}_{\bar{\partial}}(M) = H^s\big(\Gamma(M, \mathcal{A}^{r,\bullet})\big)$ and $H^{r,s}_{\bar{\partial},\mathrm{curr}}(M) = H^s\big(\Gamma(M, \mathcal{D}^{r,\bullet})\big)$. If $\Omega^r_M \to \mathcal{J}^\bullet$ is an injective resolution, then we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{A}^{r,\bullet} & \hookrightarrow & \mathcal{D}^{r,\bullet} \\ & {\scriptstyle\alpha_A}\searrow \quad \swarrow{\scriptstyle\alpha_D} & \\ & \mathcal{J}^\bullet & \end{array}$$

of quasi-isomorphisms lifting the identity map on Ω^r_M. Up to homotopy the arrows α_A and α_D are unique. Since $\mathcal{A}^{r,\bullet}$ and $\mathcal{D}^{r,\bullet}$ are fine, α_A and α_D induce isomorphisms

$$H^{r,s}_{\bar{\partial}}(M) \xrightarrow{\sim} H^s(M, \Omega^r_M) \tag{1.2}$$

and

$$H^{r,s}_{\bar{\partial},\mathrm{curr}}(M) \xrightarrow{\sim} H^s(M, \Omega^r_M) \tag{1.3}$$

and we have a commutative diagram

$$\begin{array}{ccc} H^{r,s}_{\bar{\partial}}(M) & \xrightarrow[(1.1)]{\sim} & H^{r,s}_{\bar{\partial},\mathrm{curr}}(M) \\ {\scriptstyle(1.2)}\downarrow & \swarrow{\scriptstyle(1.3)} & \\ H^s(M, \Omega^r_M). & & \end{array}$$

We say that an (r, s)-form (resp. (r, s)-current) β *represents* a cohomology class $\mu \in H^s(M, \Omega^r_M)$ if β is $\bar{\partial}$-closed and the cohomology class of β maps to μ under (1.2) (resp. (1.3)). This gives a precise meaning to the opening sentence of this section. There is a need to get our conventions correct, for we want to get our signs "on the nose" for our theorem. Appealing to the Hodge-de Rham spectral sequence creates a sign ambiguity depending on the conventions one uses for double complexes.

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In this short note, we will use the fact that if $p \in M$ is a point, then the Dirac delta distribution at p, $\delta_{\{p\}}$, which acts on compactly supported smooth functions (compactly supported forms of degree $(0, 0)$), is an (n, n) current. Recall that if T is an (n, n) current *with compact support* (*e.g.*, $T = \delta_{\{p\}}$) then $\int_M T = T[1]$ is defined [dR, Chapter III, Section 8, pp. 42–43]. In fact $T[\varphi]$ is defined for every C^∞ function φ on M—*including those whose support is not compact* [dR, Chapter III, Section 8, p. 41]. In particular, it follows that $\int_M \delta_{\{p\}} = \delta_{\{p\}}[1] = 1$.

Let $\Phi = \Phi_M$ denote the family of compact subsets of M, and let $\Gamma_\Phi(M, \_\_)$ be the functor of compact supports on M. Let $H^i_\Phi(M, \_\_)$ denote the corresponding i-th derived functor, *i.e.*, $H^i_\Phi(M, \_\_)$ denotes *cohomology with compact support*. It is well known that any sheaf of modules over the sheaf of algebras of C^∞ functions on M is Φ-soft [I, Chapter III, 2.8, 2.9, 3.2]. Therefore the preceding discussion can be reproduced with subscript Φ and one concludes that elements in $H^n_\Phi(M, \Omega^n_M)$ can be represented by (n, n) currents with compact support[1]—two such currents representing the same class if they differ by $\bar\partial\eta$, where η is an $(n, n-1)$ current with compact support. Clearly $\int_M : H^n_\Phi(M, \Omega^n_M) \to \mathbb{C}$ is well defined (since $\bar\partial\eta = d\eta$ for an $(n, n-1)$ current η, and since Stokes theorem extends to currents [dR, p. 54]).

2 Comparison of Cohomologies

In this section we give our conventions for comparing Čech, derived functor, and Dolbeault cohomologies (the latter two have already been compared). We remind the reader about the need for care about signs, and hence the need to lay down our conventions clearly and without ambiguity.

Let $\mathfrak{U} = \{U_i\}_i$ be an open cover of our complex manifold M, i varying over a well ordered set and $\mathcal{F}$ a sheaf of abelian groups on M. Let $\mathcal{C}^\bullet = \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ be the ordered *sheaf* Čech complex associated to $\mathfrak{U}$ (*cf.* [Ha, pp. 218–220] for the definition of the Čech complex and its sheafified version). Then we have a canonical resolution $\mathcal{F} \to \mathcal{C}^\bullet$. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Let $\alpha_C : \mathcal{C}^\bullet \to \mathcal{I}^\bullet$ be the homotopy unique map of complexes lifting the identity map $\mathcal{F}$. Applying $H^s \circ \Gamma(M, \alpha_C)$ we get the well-known comparison map

$$\check{H}^s(\mathfrak{U}, \mathcal{F}) \to H^s(M, \mathcal{F}). \tag{2.1}$$

If $\mathcal{F} = \Omega^r_M$, then (2.1) can be described in a different way, as follows: Let $\mathcal{C}^{p,q} = \mathcal{C}^p(\mathfrak{U}, \mathcal{A}^{r,q})$ and $\delta : \mathcal{C}^{p,q} \to \mathcal{C}^{p+1,q}$ the Čech coboundary. Let $\mathcal{T}^\bullet$ be the complex given by $\mathcal{T}^m = \bigoplus_{p+q=m} \mathcal{C}^{p,q}$, the coboundary $d^m : \mathcal{T}^m \to \mathcal{T}^{m+1}$ being given on $\mathcal{C}^{p,q}$ by $\delta + (-1)^p \bar\partial$. Now $\mathcal{C}^\bullet = \mathcal{C}^\bullet(\mathfrak{U}, \Omega^r_M)$ and $\mathcal{A}^{r,\bullet}$ are clearly subcomplexes of $\mathcal{T}^\bullet$ under the obvious inclusions $\mathcal{C}^p \subset \mathcal{C}^{p,0}$ and $\mathcal{A}^{r,q} \hookrightarrow \mathcal{C}^{0,q}$. Now, the inclusions (of complexes)

$$\alpha'_C : \mathcal{C}^\bullet \hookrightarrow \mathcal{T}^\bullet, \quad \alpha'_A : \mathcal{A}^{r,\bullet} \hookrightarrow \mathcal{T}^\bullet \tag{2.2}$$

are quasi-isomorphisms. Moreover, the composition $\Omega^r_M \to \mathcal{C}^\bullet \subset \mathcal{T}^\bullet$ is a quasi-isomorphism (and it equals the composition $\Omega^r_M \to \mathcal{A}^{r,\bullet} \hookrightarrow \mathcal{T}^\bullet$). Let $H =$

[1] The meaning of *represented* being analogous to our earlier meaning, with the functor of *sections with compact support* $\Gamma_\Phi(M, \_\_)$ replacing $\Gamma(M, \_\_)$.

$H^s\big(\Gamma(M,\mathcal{T}^\bullet)\big)$. Applying $H^s\circ\Gamma(M,\,\_\_)$ to the inclusions in (2.2), we get maps

$$\varphi\colon \check{H}^s(\mathfrak{U},\Omega^r_M)\longrightarrow H,\qquad \psi\colon H^{r,s}_{\bar\partial}(M)\longrightarrow H.$$

Since $\mathcal{A}^{r,\bullet}$ and $\mathcal{T}^\bullet$ are complexes of fine sheaves, therefore ψ is an isomorphism. This gives us a map

$$\psi^{-1}\circ\varphi\colon \check{H}^s(\mathfrak{U},\Omega^r_M)\longrightarrow H^{r,s}_{\bar\partial}(M). \tag{2.3}$$

The map (2.1) is well known to be determined by the commutativity of:

$$\begin{array}{ccc} \check{H}^s(\mathfrak{U},\Omega^r_M) & \xrightarrow{(2.3)} & H^{r,s}_{\bar\partial}(M) \\ & \searrow^{(2.1)} & \Big\downarrow{\scriptstyle (1.2)} \\ & & H^s(M,\Omega^r_M). \end{array} \tag{2.4}$$

Indeed consider the diagram

$$\begin{array}{ccc} \mathcal{A}^{r,\bullet} & \xrightarrow{\alpha_A} & \mathcal{J}^\bullet \\ {\scriptstyle \alpha'_A}\Big\downarrow & \nearrow^{\alpha_T} & \Big\uparrow{\scriptstyle \alpha_C} \\ \mathcal{T}^\bullet & \xleftarrow[\alpha'_C]{} & \mathcal{C}^\bullet \end{array}$$

where α_T is the homotopy unique map lifting the identity on Ω^r_M. We point out that all arrows are maps between resolutions of Ω^r_M, and they all lift the identity on Ω^r_M. Therefore, since $\mathcal{J}^\bullet$ is injective, the above diagram commutes up to homotopy. Applying $H^s\circ\Gamma(M,\,\_\_)$ we get the commutativity of (2.4).

3 Key Proposition

Suppose p is a point of M, and $U\subset M$ is a coordinate neighborhood of p; $z=(z_1,\dots,z_n)\colon U\to\mathbb{C}^n$ the coordinates, $z(p)=0$. Assume further that U is Stein. Let $U^*=U\setminus\{p\}$. If $U_i=\{q\in U\mid z_i(q)\neq 0\}$, then $\mathfrak{U}=\{U_i\}$ is an ordered Stein cover of U^*. The element

$$\frac{dz_1\wedge\cdots\wedge dz_n}{z_1\cdots z_n}\in C^{n-1}(\mathfrak{U},\Omega^n_M)$$

defines an element

$$\tau_p\in H^{n-1}(U^*,\Omega^n_M).$$

If $\mathcal{J}^\bullet$ is an injective, or for that matter, flasque, resolution of Ω^n_M, we have a short exact sequence of complexes

$$0\to\Gamma_p(U,\mathcal{J}^\bullet)\to\Gamma(U,\mathcal{J}^\bullet)\to\Gamma(U^*,\mathcal{J}^\bullet)\to 0$$

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whence a connecting homomorphism[2]

$$H^{n-1}(U^*, \Omega^n_M) \to H^n_p(U, \Omega^n_M) = H^n_p(M, \Omega^n_M).$$

Composing this with the natural map $H^n_p(M, \Omega^n_M) \to H^n_\Phi(M, \Omega^n_M)$ we get a map

$$c_p \colon H^{n-1}(U^*, \Omega^n_M) \to H^n_\Phi(M, \Omega^n_M).$$

Proposition 3.1 *The image of τ_p in $H^n_\Phi(M, \Omega^n_M)$ under c_p is represented by the (n, n) current $(2\pi i)^n \epsilon(n)\delta_{\{p\}}$, where $\epsilon(n) = (-1)^{n(n-1)/2}$ and $\delta_{\{p\}}$ is the Dirac distribution at p.*

Proof As above, let $\mathcal{J}^\bullet$ be an injective resolution of Ω^n_M. We have a commutative diagram with exact rows, induced by the homotopy unique quasi-isomorphism $\mathcal{D}^{n,\bullet} \to \mathcal{J}^\bullet$ lifting the identity map on Ω^n_M,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_p(U, \mathcal{D}^{n,\bullet}) & \longrightarrow & \Gamma(U, \mathcal{D}^{n,\bullet}) & \longrightarrow & \Gamma(U^*, \mathcal{D}^{n,\bullet}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_p(U, \mathcal{J}^\bullet) & \longrightarrow & \Gamma(U, \mathcal{J}^\bullet) & \longrightarrow & \Gamma(U^*, \mathcal{J}^\bullet) & \longrightarrow & 0.
\end{array}$$

If β_0 is an $(n, n-1)$ current on U^* representing τ_p, and β an $(n, n-1)$ current on U extending β_0, then $\bar\partial\beta$ is an (n, n) current *supported at p*, and can hence be thought of as a compactly supported current on M. From the above commutative diagram and our conventions about connecting homomorphisms, the image of $\bar\partial\beta$ in $H^n_\Phi(M, \Omega^n_M)$ is precisely $c_p(\tau_p)$. Our task then is to pick β_0 and its extension β in such a way that $\bar\partial\beta = (2\pi i)^n \epsilon(n)\delta_{\{p\}}$. To do this we may as well identify U with its image in $\mathbb{C}^n$ (so that $p = 0 \in \mathbb{C}^n$). Let $\mathcal{C}^{p,q}$, $\mathcal{T}^\bullet$, *etc.*, be as in the discussion in the previous section, with $r = n$, $s = n - 1$ and $M = U^*$. Set $T^\bullet = \Gamma(U^*, \mathcal{T}^\bullet)$, and $C^{p,q} = \Gamma(U^*, \mathcal{C}^{p,q})$. Let

$$\beta_0' = c_n \frac{\sum_i (-1)^{i-1} \bar z_i \, d\bar z_1 \wedge \cdots \wedge \widehat{d\bar z_i} \wedge \cdots \wedge d\bar z_n \wedge dz_1 \wedge \cdots \wedge dz_n}{|z|^{2n}}$$

where $c_n = \epsilon(n)(n-1)!/(2\pi i)^n$ so that

$$\int_{S^{2n-1}} \beta_0' = 1,$$

where the orientation on S^{2n-1} is the one which makes the closed unit ball in $\mathbb{C}^n$ an oriented manifold with boundary, the interior having the standard orientation of $\mathbb{C}^n$.

[2] In our convention, if $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence of of complexes of abelian groups, then the connecting map $H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ sends $[\gamma]$ to $[d_B\beta]$, where
(a) $\beta \in B^i$ is a pre-image of $\gamma \in C^i$, and
(b) $d_B\beta$ is identified with an $i+1$-cocycle in $A^\bullet$, since the image of $d_B\beta$ in C^{i+1} is 0.
This convention (without signs) is compatible with the computations in the proof of [L, Lemma (8.6), pp. 79–81], a crucial ingredient in the proof of the Residue Theorem [*loc. cit.*, Theorem (0.6)(d), p. 25].

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For $n = 1$, $\beta_0' = 1/(2\pi i) \times \mathrm{d}z/z$. If $n > 1$, according to [GH, p. 654], [T, p. 907, 2.6] (see also [*ibid*, pp. 910–911, Section 4]), [TT, pp. 287, 297] and [H, p. 87], we can find elements $\xi_p \in C^{p,n-p-2}$, $p = 0, \dots, n-2$, $\omega_p \in C^{p,n-p-1}$, $p = 0, \dots, n-1$ such that

- $\omega_{n-1} = \alpha_C'\big(\frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{z_1 \cdots z_n}\big)$,
- $\omega_0 = \alpha_A'\big((2\pi i)^n \beta_0'\big)$,
- $\bar{\partial}\xi_p = \omega_p$, $p = 0, \dots, n-2$, and
- $\delta\xi_p = \omega_{p+1}$, $p = 0, \dots, n-2$.

(See p. 907, Prop. 2.6 and Section 4(i), pp. 910–911 of [T].) Now

$$
\begin{aligned}
\mathrm{d}_{T^\bullet}\Big(\sum_{p=0}^{n-2}(-1)^{p(p-1)/2}\xi_p\Big) &= \sum_{p=0}^{n-2}(-1)^{p(p-1)/2}[(-1)^p\omega_p + \omega_{p+1}] \\
&= \omega_0 + \sum_{p=0}^{n-3}[(-1)^{p(p-1)/2} + (-1)^{(p+2)(p+1)/2}]\omega_{p+1} \\
&\qquad + (-1)^{(n-2)(n-3)/2}\omega_{n-1} \\
&= \omega_0 + (-1)^{(n-2)(n-3)/2}\omega_{n-1} \\
&\qquad (\text{since } (-1)^{p(p-1)/2} + (-1)^{(p+2)(p+1)/2} = 0).
\end{aligned}
$$

It follows that ω_{n-1} and $-(-1)^{(n-2)(n-3)/2}\omega_0$ represent the same cohomology class in $T^\bullet$. Now

$$
\omega_0 = \alpha_A'\big((2\pi i)^n\beta_0'\big) \quad \text{and} \quad -(-1)^{(n-2)(n-3)/2} = (-1)^{n(n-1)/2} = \epsilon(n).
$$

Let $\beta_0 = \epsilon(n)(2\pi i)^n\beta_0'$ (the case $n = 1$ included). The above computations, together with the diagram (2.4) show that for $n > 1$, β_0 represents τ_p. This statement is obviously true for $n = 1$.

One checks—by following the growth of β_0' at the origin—that $\int_U \beta_0' \wedge \varphi$ is finite for every compactly supported $(0, 1)$ form φ on U, and that $\varphi \mapsto \int_U \beta_0' \wedge \varphi$ defines an $(n, n-1)$ current β' on U extending $\beta_0' \in \Gamma(U^*, \mathcal{D}^{n,n-1})$. It is well known that $\bar{\partial}\beta' = \delta_{\{0\}}$ (see *e.g.*, [TT, (2.13)]). Setting $\beta = \epsilon(n)(2\pi i)^n\beta'$, we see that we are done. ∎

We wish to point out that in [GH], the definition of the Dolbeault isomorphism is different from ours. In *loc. cit.*, the Dolbeault isomorphism is obtained by breaking up the Dolbeault resolution into short exact sequences and then repeatedly using connecting homomorphisms (isomorphisms in this case) to obtain $H^{n,n-1}(M) \simeq H^{n-1}(M, \Omega_M^n)$. Our procedure instead is to use the map $\mathcal{D}^{n,\bullet} \to \mathcal{I}^\bullet$. The conclusion reached in [GH] then is that $(2\pi i)^n\beta_0'$ represents τ_p. The two conventions differ by a factor of $\epsilon(n)$ as the above computations show. Similar comments apply to [H] and [T].

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4 Grothendieck Trace vs. the Integral

For an algebraic scheme X over $\mathbb{C}$, let X^{an} denote the corresponding analytic space. From now on let $M = X^{\mathrm{an}}$ where X is a smooth complete variety of dimension n over $\mathbb{C}$. Then M is compact and hence $H^n_\Phi(M, \Omega^n_M) = H^n(M, \Omega^n_M)$. Recall that we have a natural map $M \to X$ of locally ringed spaces, and the natural map $g\colon H^n(X, \Omega^n_X) \to H^n(M, \Omega^n_M)$ is an isomorphism (this follows from Grothendieck's generalization of GAGA in [G2, XII]). There is a natural isomorphism

$$H^n(M, \Omega^n_M) \xrightarrow{\sim} H^{2n}_{DR}(M, \mathbb{C}) \tag{4.1}$$

which is compatible with integration of top degree forms. In greater detail, if $\nu \in H^n(M, \Omega^n_M)$, and β is an (n, n)-form representing it, then, the image of ν under (4.1) is the de Rham class represented by β (viewed as a $2n$-form).

We point out that the above isomorphism is the standard isomorphism arising from "the" Hodge to de Rham spectral sequence, and not the isomorphism arising from the EGA conventions for spectral sequences associated with double complexes (see [C2] for the issues involved with using different conventions, which is why we put quote marks around the word "the" in the previous sentence).

For the trace map in Grothendieck duality we use Lipman's version

$$\tilde{\theta}_X\colon H^n(X, \Omega^n_X) \to \mathbb{C}$$

defined in [L, p. 25, Theorem (0.6)(d)].

Our main theorem is:

Theorem 4.1 *The following diagram commutes:*

$$\begin{array}{ccc} H^n(X, \Omega^n_X) & \xrightarrow{\ \tilde{\theta}_X\ } & \mathbb{C} \\ {\scriptstyle g}\big\downarrow{\scriptstyle \simeq} & & \big\uparrow{\scriptstyle (2\pi i)^{-n}\epsilon(n)\int_M} \\ H^n(M, \Omega^n_M) & \xrightarrow[(4.1)]{} & H^{2n}_{DR}(M, \mathbb{C}). \end{array} \tag{4.2}$$

Proof Since all vector spaces in the diagram are one dimensional over $\mathbb{C}$ and all maps are non-zero, therefore it is enough to show that for any one judiciously picked non-zero element $\mu \in H^n(X, \Omega^n_X)$, $(2\pi i)^{-n}\epsilon(n)\int_M g(\mu) = \tilde{\theta}_X(\mu)$.

Pick an element $p \in X_{\max} = |M|$ (here $X_{\max}$ means the set of closed points of X). Let $z_1, \ldots, z_n \in \mathcal{O}_{X,p}$ be a regular system of parameters. Let $V = \operatorname{Spec} A$ be an affine open neighborhood of p in X on which

(a) $z_1, \ldots, z_n$ are defined,
(b) the maximal ideal of A corresponding to p is $(z_1, \ldots, z_n)A$, and
(c) z defines an étale map $z\colon V \to \mathbb{A}^n$.

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The set $V_{\max}$ may be thought of as an open set in M, and we can find a Stein open subset $U \subset V_{\max}$ containing p such that $z\colon U \to \mathbb{C}^n$ is an open immersion. Let $V^* = V \setminus \{p\}$. Just as we defined τ_p, we can define $\tau_p^{\mathrm{alg}} \in H^{n-1}(V^*, \Omega_X^n)$ as the image of the Čech co-cycle $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n / z_1 \cdots z_n \in C^{n-1}(\mathcal{V}, \Omega_X^n)$, where $\mathcal{V} = \{V_i\}$ and $V_i = \operatorname{Spec} A_{z_i}$.

We have connecting maps $H^{n-1}(V^*, \Omega_X^n) \to H_p^n(V, \Omega_X^n) = H_p^n(X, \Omega_X^n)$, and a map $c_p^{\mathrm{alg}}\colon H^{n-1}(V^*, \Omega_X^n) \to H^n(X, \Omega_X^n)$. We define the *fundamental class* of $\{p\}$ to be the class $[p] = c_p^{\mathrm{alg}}(\tau_p^{\mathrm{alg}}) \in H^n(X, \Omega_X^n)$. One checks easily that

(a) the natural map $H^{n-1}(V^*, \Omega_X^n) \to H^{n-1}(U^*, \Omega_M^n)$ sends τ_p^{alg} to τ_p and
(b) the diagram

$$\begin{array}{ccc} H^{n-1}(V^*, \Omega_X^n) & \xrightarrow{c_p^{\mathrm{alg}}} & H^n(X, \Omega_X^n) \\ \downarrow & & \downarrow \\ H^{n-1}(U^*, \Omega_M^n) & \xrightarrow{c_p} & H^n(M, \Omega_M^n) \end{array}$$

commutes.

It follows from Proposition 3.1 that $g[p] = (2\pi i)^n \epsilon(n)[\delta_{\{p\}}]$. Clearly $[p]$ is non-zero (since g is an isomorphism), and this is our judiciously chosen μ. It is immediate that $(2\pi i)^{-n}\epsilon(n) \int_M g[p] = 1$. On the other hand, the image of τ_p^{alg} in $H_p^n(X, \Omega_X^n)$ is given by the generalized fraction $\left[\begin{smallmatrix} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \\ z_1, \dots, z_n \end{smallmatrix} \right]$ (see [L, p. 59]). It follows from the residue theorem in [*loc. cit.*, Theorem (0.5)(d), p. 25] and from the residue formulas in p. 64 of *loc. cit.* that

$$\begin{aligned} \tilde{\theta}_X[p] &= \operatorname{res}_p \begin{bmatrix} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \\ z_1, \dots, z_n \end{bmatrix} \\ &= 1 \\ &= (2\pi i)^{-n} \epsilon(n) \int_M g[p]. \end{aligned} \qquad \blacksquare$$

5 Comparison with Conrad's Results

Let X be an n-dimensional variety over $\mathbb{C}$, *i.e.*, a non-empty reduced irreducible separated $\mathbb{C}$-scheme of finite type. If X is complete and smooth over $\mathbb{C}$, let

$$\gamma_X\colon H^n(X, \Omega_X^n) \to \mathbb{C}$$

be the trace map of Conrad [C, p. 150, (3.4.11)]. We claim that

$$\gamma_X = (-1)^{n(n+1)/2} \tilde{\theta}_X. \tag{5.1}$$

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For brevity set $\tilde{\epsilon}(n) = (-1)^{n(n+1)/2}$. In [C2, p. 10, Theorem 2.2], Conrad proves (with $M = X^{\mathrm{an}}$ as before) that

$$\text{(5.2)} \qquad \begin{array}{ccc} H^n(X, \Omega^n_X) & \xrightarrow{\ \gamma_X\ } & \mathbb{C} \\ {\scriptstyle g}\Big\downarrow \simeq & & \Big\uparrow {\scriptstyle (2\pi i)^{-n}(-1)^n \int_M} \\ H^n(M, \Omega^n_M) & \xrightarrow[(4.1)]{} & H^{2n}_{DR}(M, \mathbb{C}) \end{array}$$

commutes. Since $\epsilon(n) = (-1)^n \tilde{\epsilon}(n)$, (5.1) follows from (4.2) and (5.2). It is however psychologically more reassuring if we can directly prove (5.1) without using either Theorem 4.1 or [C2, Theorem 2.2]. Such a direct proof would imply that these two theorems are compatible with each other. We give a brief outline (with relevant references) of how this can be done, without getting into too many details.

There is an important case where (5.1) can be easily verified. Let

$$\mathbb{P} = \operatorname{Proj} \mathbb{C}[T_0, \ldots, T_n] = \mathbb{P}^n_{\mathbb{C}}.$$

Set $t_i = T_i/T_0$, $i = 1, \ldots, n$, $U_i = \{T_i \neq 0\}$, $i = 0, \ldots, n$, and $\mathfrak{U} = \{U_i\}_{i=0}^n$. Then $\mathfrak{U}$ is an affine open cover of $\mathbb{P}$ and $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n / t_1 \cdots t_n$ is a Čech n-cocycle in $C^n(\mathfrak{U}, \Omega^n_{\mathbb{P}})$ and hence represents an element $c \in H^n(\mathbb{P}, \Omega^n_{\mathbb{P}})$ via the map (2.1). It is well known that c does not depend on the choice of the "coordinate system" $T_0, \ldots, T_n$.

By [L, p. 74, Remarks] and [C, p. 32, (2.3.1)] (see also [*ibid*, (2.3.2) and (2.3.3)][3]) we have:

$$\tilde{\theta}_{\mathbb{P}}(c) = 1 \quad \text{and} \quad \gamma_{\mathbb{P}}(c) = (-1)^{n(n+1)/2}$$

giving (5.1) for $X = \mathbb{P}$.

To say more we must expand our discussion to include singular varieties. We begin by putting a "canonical structure" on the dualizing structures in Conrad's book [C] (restricting ourselves to $\mathbb{C}$-varieties). For X an n-dimensional variety over $\mathbb{C}$, let $\omega_X = H^{-n}(\pi^!_X \mathbb{C})$ where $\pi_X \colon X \to \operatorname{Spec} \mathbb{C}$ is the structural map and $\pi^!_X$ is as in [C, pp. 133–136, (3.3.1), (3.3.6)—(3.3.13)]. If X is *smooth* ω_X can be identified with Ω^n_X. This is seen by setting $f = \pi_X$ and $K^\bullet = \mathbb{C}$ in [C, p. 136, (3.3.16)] and appealing to the definitions in p. 31, (2.2.7) and p. 134, (3.3.6) of *ibid*. One checks that the data $\omega = \{\omega_X\}$ over all such X is an $\mathcal{O}$-module in the sense of Lipman [L, pp. 28–32, Chapter I, Section 1]. The $\mathcal{O}$-module ω has a *canonical structure* in the sense of Lipman [*ibid*, pp. 32–33, (2.1)]. This is seen from [C, p. 31, (2.2.9)] and [*ibid*, p. 79]—especially the discussion on finite étale maps in which equations (2.7.9) and (2.7.10) of *ibid* are embedded. Therefore ω_X can be identified with $\tilde{\omega}_X$—the sheaf of regular differential forms on X (*cf.* [L, p. 34, Lemma (2.2)] as well as the Remark following it). The sheaf $\tilde{\omega}_X$ is a coherent $\mathcal{O}_X$ submodule of the sheaf of meromorphic differentials on X, and on the smooth locus X° of X, it equals $\Omega^n_{X^\circ}$.

[3] Note however that the assertion made in [C, p. 33, (2.3.4)] is off by a sign of $(-1)^n$—see [C2].

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Suppose X is *complete*. Let

$$\gamma_X\colon H^n(X,\tilde{\omega}_X) \longrightarrow \mathbb{C}$$

be the map induced by the trace map $\mathrm{Tr}_{\pi_X}\colon \mathbb{R}\pi_{X_*}\pi_X^!\mathbb{C} \to \mathbb{C}$ of [C, pp. 146–150, 3.4]. A few remarks are in order. The complex $\pi_X^!\mathbb{C}$ can be identified with a residual complex $\pi_X^\Delta\mathbb{C}$ which is concentrated in degrees $-n,\dots,0$ (the notion of π_X^Δ is already embedded in the definitions of $\pi_X^!$ referred to above), and hence we have a natural map $\tilde{\omega}_X[n] \to \pi_X^!\mathbb{C}$. The other important remark is this: in order to define γ_X in a manner compatible with its definition when X is smooth, we *must* follow Conrad's conventions for (3.4.13) of [C]. It is easy to check that $(\tilde{\omega}_X,\gamma_X)$ is an n-dualizing pair, *i.e.*, it represents the functor $\mathrm{Hom}_{\mathbb{C}}\big(H^n(X,\_\_),\mathbb{C}\big)$ of quasi-coherent sheaves.

Let $h\colon V \to W$ be a proper surjective map between n-dimensional $\mathbb{C}$-varieties. Such a map must necessarily be generically finite, and since we are working with fields of characteristic zero, this map must actually be generically étale. According to [L, p. 38, Lemma (3.2)], there is a unique map

$$t_h\colon h_*\tilde{\omega}_V \to \tilde{\omega}_W$$

(denoted $t_h^\#$ in *loc. cit.*) which localizes to trace $\otimes 1$ at the generic point of W [L, pp. 32–33, (2.1.1)]. If W is complete then by [C, p. 149, (3.4.3) (TRA1) and (TRA2)] and [*ibid*, p. 79, (2.7.10)], it is not difficult to see that

(5.3)
$$\begin{array}{ccc} H^n(W,h_*\tilde{\omega}_V) & \xrightarrow{\ L\ } & H^n(V,\tilde{\omega}_V) \\ {\scriptstyle t_h}\downarrow & & \downarrow{\scriptstyle \gamma_V} \\ H^n(W,\tilde{\omega}_W) & \xrightarrow[\gamma_W]{} & \mathbb{C} \end{array}$$

commutes, where L is the edge homomorphism from the Leray spectral sequence. We point out that in this instance the two double complex conventions (giving rise to the Leray spectral sequence) mentioned in [C1, p. 1] give rise to the same L. By [L, p. 15, (0.2.1)], (5.3) commutes after substituting (γ_V,γ_W) with $(\tilde{\theta}_V,\tilde{\theta}_W)$. We could, at this stage appeal to [L, pp. 15–16, (0.2B)], to settle (5.1). The arguments used so far from [L] are elementary, but *loc. cit.* requires the full strength of the main theorems of *ibid*. We therefore point out another way by modifying some of the arguments in *ibid*.

If X is an n-dimensional complete $\mathbb{C}$-variety, by Chow's Lemma and Noether normalization, we have a diagram

$$\begin{array}{ccccc} & & V & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle \pi} & \\ X & & & & \mathbb{P} \end{array}$$

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such that V is a projective variety, f is birational, and π is finite surjective. Let $(t_V, t_X, t_{\mathbb{P}}) \in \{(\tilde\theta_V, \tilde\theta_X, \tilde\theta_{\mathbb{P}}), (\gamma_V, \gamma_X, \gamma_{\mathbb{P}})\}$. Then (5.3) and [L, p. 15, (0.2.1)] give us a commutative diagram

$$\begin{array}{ccccc}
H^n(X, f_*\tilde\omega_V) & \xrightarrow{\ L\ } & H^n(V, \tilde\omega_V) & \xleftarrow[(\simeq)]{\ L\ } & H^n(\mathbb{P}, \pi_*\tilde\omega_V) \\
{\scriptstyle t_f}\downarrow & & {\scriptstyle t_V}\downarrow & & \downarrow{\scriptstyle t_\pi} \\
H^n(X, \tilde\omega_X) & \xrightarrow[t_X]{} & \mathbb{C} & \xleftarrow[t_{\mathbb{P}}]{} & H^n(\mathbb{P}, \Omega^n_{\mathbb{P}}).
\end{array}$$

Since $\gamma_{\mathbb{P}} = \tilde\epsilon(n)\tilde\theta_{\mathbb{P}}$, therefore, using the right half of the above diagram, especially the fact that the westward pointing L is surjective, we get

$$(*) \qquad \gamma_V = \tilde\epsilon(n)\tilde\theta_V.$$

Next, we have a *non-empty* open subscheme U of X such that under f, $f^{-1}(U)$ is isomorphic to U. By [L, p. 38, Lemma (3.2)] the map $t_f\colon f_*\tilde\omega_V \to \tilde\omega_X$ is injective and its cokernel $\mathcal{C}$ is supported in $X \setminus U$. Clearly (since U is non-empty), $H^n(X, \mathcal{C}) = 0$. It follows that $H^n(X, t_f)$ is *surjective*. The left half of the above commutative diagram now gives $\gamma_X = \tilde\epsilon(n)\tilde\theta_X$ (we are using $(*)$). Hence (5.1) is true (and not just for smooth X).

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References

[C] B. Conrad, *Grothendieck duality and base change.* Lecture Notes in Math. **1750**, Springer-Verlag, New York, 2000.

[C1] B. Conrad, *Clarifications and corrections to "Grothendieck duality and base change".* Preprint, www-math.mit.edu/˜dejong.

[C2] ——, *More Examples.* Preprint, www-math.mit.edu/˜dejong.

[D] P. Deligne, *Intégration sur un cycle évanescent.* Invent. Math. **76**(1983), 129–143.

[dR] G. de Rham, *Variétés différentiables.* Hermann, Paris, 1955.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. John Wiley & Sons, New York, 1978.

[G1] A. Grothendieck, *Théorèmes de dualité pour les faisceaux algébriques cohérents.* Séminaire Bourbaki **149**, May 1957.

[G2] ——, *Revêtements étales et groupe fondamental*. Lecture Notes in Math. **224**, Springer-Verlag, Heidelberg, 1971.

[Ha] R. Hartshorne, *Algebraic Geometry.* Graduate Texts in Math. **52**, Springer-Verlag, 1977.

[H] F. R. Harvey, *Integral formulae connected by Dolbeault's isomorphism.* Rice Univ. Studies **56**(1970), 77–97.

[I] B. Iversen, *Cohomology of sheaves.* Universitext, Springer-Verlag, 1986.

[L] J. Lipman, *Dualizing sheaves, differentials and residues on algebraic varieties.* Astérisque **117**, 1984.

[T] Y. L. L. Tong, *Integral representation formulae and the Grothendieck residue symbol.* Amer. J. Math. **95**(1973), 904–917.

[TT] D. Toledo and Y. L. L. Tong, *A parametrix for $\bar{\partial}$ and the Riemann-Roch in Čech theory.* Topology **15**(1976), 273–301.

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