A BASIC ANALOGUE OF MACROBERT'S E-FUNCTION

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(Received 10 December, 1959)

1. Introduction and notation. MacRobert [2] in 1937 defined the E-function as

$$E(\alpha, \beta :: z) = \sum_{\alpha, \beta} \Gamma(\alpha)\Gamma(\beta-\alpha)z^{\alpha}F(\alpha ; \alpha-\beta+1 ; z), \tag{1}$$

where the symbol $\sum_{\alpha, \beta}$ denotes that to the expression following it, a similar expression with α and β interchanged is to be added. For (1) he also gave the integral representation

$$E(\alpha, \beta :: z) = \Gamma(\alpha) \int_0^{\infty} e^{-\lambda}\lambda^{\beta-1}(1+\lambda/z)^{-\alpha}\, d\lambda, \tag{2}$$

where Re $\beta > 0$, $|\arg z| < \pi$.

Since 1937 the E-function has been generalized and studied in detail by MacRobert and others. In this paper, I give a basic analogue of (1) and study some of its fundamental properties.

The following notation is used throughout the paper. Let

$$|q|<1, \ \log q = -w = -(w_1+iw_2),$$

where w, w_1, w_2 are constants, w_1 and w_2 being real. Also, let

$$(q^a)_n \equiv (a)_n = (1-q^a)(1-q^{a+1}) \dots (1-q^{a+n-1}),$$

$$(q^a)_0 = 1, \quad (q^a)_{-n} = (-)^n q^{\frac{1}{2}n(n+1)} q^{-na}/(q^{1-a})_n \,;$$

then we define the generalized basic hypergeometric function as

$$_{r+1}\Phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \,; z \\ b_1, b_2, \dots, b_r \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_{r+1})_n z^n}{(q)_n(b_1)_n \dots (b_r)_n} \quad (|z|<1,$$

and the " confluent " hypergeometric function as

$$_1\Phi_1(a; b \,; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n(b)_n} z^n q^{\frac{1}{2}n(n-1)}.$$

Also

$$E_q(x) = \prod_{n=0}^{\infty}(1-xq^n) = \sum_{n=0}^{\infty} \frac{(-)^n q^{\frac{1}{2}n(n-1)}}{(q)_n} x^n,$$

$$(x+y)_\alpha = x^\alpha(1+yx^{-1})_\alpha = \prod_{n=0}^{\infty} \frac{(1+yx^{-1}q^n)}{(1+yx^{-1}q^{\alpha+n})},$$

$$\frac{1}{(1+x)_\alpha} = {}_1\Phi_0(\alpha \,; -x), \quad \text{for} \quad |x|<1,$$

and

$$G(\alpha) = \left\{\prod_{n=0}^{\infty}(1-q^{\alpha+n})\right\}^{-1}.$$

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Further, following Hahn [1], the basic integral of a function under suitable conditions is defined as

$$\overset{x}{\underset{0}{\mathrm{S}}} f(y)\, d(qy) = x(1-q) \sum_{i=0}^{\infty} q^i f(q^i x),$$

$$\overset{\infty}{\underset{x}{\mathrm{S}}} f(y)\, d(qy) = x(1-q) \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x)$$

and thus

$$\overset{\infty}{\underset{0}{\mathrm{S}}} f(y)\, d(qy) = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j).$$

As above we denote the basic integrals by the symbol $\overset{b}{\underset{a}{\mathrm{S}}}\, d(qu)$.

2. Definitions. We define the basic analogue of the E-function by the integral

$$E_q(\alpha, \beta :: z) \equiv \frac{G(\alpha)}{1-q} \overset{1}{\underset{0}{\mathrm{S}}} E_q(q\lambda) \lambda^{\beta-1} {}_1\Phi_0(\alpha\,; -\lambda/z)\, d(q\lambda), \tag{3}$$

where Re $\beta > 0$, and let arg $\lambda = 0$, for simplicity.

We now proceed to evaluate the integral on the right of (3). We know that†

$$\frac{G(\alpha)}{G(1)}\, {}_1\Phi_0(\alpha; -\lambda/z) = \frac{1}{2\pi i} \int_C \frac{G(\alpha-s)\pi(z/\lambda)^s}{G(1-s)\sin \pi s}\, ds, \tag{4}$$

where the contour C is a line parallel to Re $(ws) = 0$ with loops, if necessary, to include the poles of $G(\alpha - s)$. The integral converges if Re$[s \log(z/\lambda) - \log \sin \pi s] < 0$ for large values of $|s|$ on the contour, i.e. if $|\{\arg z - w_1{}^{-1} w_2 \log |z|\}| < \pi$, since $0 < \lambda < 1$. Hence (3) gives

$$E_q(\alpha, \beta :: z) = \frac{1}{2\pi i} \frac{G(1)}{1-q} \overset{1}{\underset{0}{\mathrm{S}}} E_q(q\lambda) \lambda^{\beta-1} d(q\lambda) \int_C \frac{G(\alpha-s)}{G(1-s)} \frac{\pi(z/\lambda)^s}{\sin \pi s}\, ds.$$

Changing the order of integration, which is justified for Re $(\beta - s) > 0$ and the above argument of z, we get

$$\begin{aligned} E_q(\alpha, \beta :: z) &= \frac{1}{2\pi i} \frac{G(1)}{1-q} \int_C \frac{G(\alpha-s)}{G(1-s)} \frac{\pi z^s}{\sin \pi s}\, ds\, \overset{1}{\underset{0}{\mathrm{S}}} E_q(q\lambda) \lambda^{\beta-s-1}\, d(q\lambda) \\ &= \frac{1}{2\pi i} \int_C \frac{G(\alpha-s) G(\beta-s)}{G(1-s)} \frac{\pi z^s}{\sin \pi s}\, ds, \end{aligned} \tag{5}$$

valid by analytic continuation when Re $\beta > 0$ and $|\{\arg z - w_2 w_1{}^{-1} \log |z|\}| < \pi$.

The contour integral (5) gives another integral representation for the E_q-function. Evaluating (5) by considering the residues at the poles of $G(\alpha - s)$ and $G(\beta - s)$ [3], we get

$$E_q(\alpha, \beta :: z) = \sum_{\alpha, \beta} \frac{G(\alpha) G(\beta-\alpha)}{G(1)} \prod_{n=0}^{\infty} \frac{(1+z^{-1}q^{\alpha+n})(1+zq^{1-\alpha+n})}{(1+z^{-1}q^n)(1+zq^{1+n})}\, {}_1\Phi_1(\alpha\,; \alpha-\beta+1\,; zq^{2-\beta}). \tag{6}$$

(6) gives the series definition for the E_q-function and shows that it is symmetrical in α and β.

† Cf. [3] Evaluating the integral (4) by considering the residues at the poles of $G(\alpha - s)$, we get an expression which is identically equal to the left-hand side.

3. Recurrence relations. We now prove the following recurrence relations:

$$(1-q^{\alpha})E_q(\alpha, \beta :: z)-E_q(\alpha+1, \beta :: z) = \frac{q^{\alpha}}{z}E(\alpha+1, \beta+1 :: z), \tag{7}$$

$$(q^{\beta}-q^{\alpha})E_q(\alpha, \beta :: z)+q^{\alpha}E_q(\alpha, \beta+1 :: z) = q^{\beta}E_q(\alpha+1, \beta :: z), \tag{8}$$

$$(1-q^{\beta})E_q(\alpha, \beta :: z) = z^{-1}q^{\beta}(1-q^{\alpha-\beta-1})E_q(\alpha, \beta+1 :: z)+(1-q^{\alpha-1})E_q(\alpha-1, \beta+1 :: z). \tag{9}$$

To prove (7) we observe that the left-hand side is equal to

$$(1-q^{\alpha})\frac{G(\alpha)}{1-q}\overset{1}{\underset{0}{S}}E_q(q\lambda)\lambda^{\beta-1}{}_1\Phi_0(\alpha\,;-\lambda/z)\,d(q\lambda)-\frac{G(\alpha+1)}{1-q}\overset{1}{\underset{0}{S}}E_q(q\lambda)\lambda^{\beta-1}{}_1\Phi_0(\alpha+1\,;-\lambda/z)\,d(q\lambda)$$

$$=\frac{G(\alpha+1)}{1-q}\overset{1}{\underset{0}{S}}E_q(q\lambda)\lambda^{\beta-1}[{}_1\Phi_0(\alpha\,;-\lambda/z)-{}_1\Phi_0(\alpha+1\,;-\lambda/z)]\,d(q\lambda)$$

$$=\frac{q^{\alpha}G(\alpha+1)}{z(1-q)}\overset{1}{\underset{0}{S}}E_q(q\lambda)\lambda^{\beta}{}_1\Phi_0(\alpha+1\,;-\lambda/z)\,d(q\lambda)$$

$$=z^{-1}q^{\alpha}E_q(\alpha+1, \beta+1 :: z).$$

To prove (8), we take (7) and a similar relation with α replaced by β. Eliminating $E_q(\alpha+1, \beta+1 :: z)$ between these two relations we get (8).

To prove (9), we multiply (7) by $q^{\beta}-q^{\alpha}$ and (8) by $1-q^{\alpha}$ and subtract. Changing α to $\alpha-1$ in the result so obtained we obtain (9).

4. A generalization of (9). We next prove the following formula:

$$\sum_{r=0}^{n}\frac{(q^{-n})_r(q^{\alpha-\beta-n})_r(q^{\alpha-n})_n}{(q^{\alpha-n})_r(q)_r}(-z)^{-r}q^{r(\beta+n)}E_q(\alpha-n+r, \beta+n :: z) = (q^{\beta})_nE_q(\alpha, \beta :: z). \tag{10}$$

To prove (10) we consider its left-hand side and use the contour integral (5) for the E_q-function in it. This gives

$$\frac{1}{2\pi i}\sum_{r=0}^{n}\frac{(q^{-n})_r(q^{\alpha-\beta-n})_r(q^{\alpha-n})_n}{(q^{\alpha-n})_r(q)_r}(-z)^{-r}q^{r(\beta+n)}\int_C\frac{G(\alpha-n+r-s)G(\beta+n-s)}{G(1-s)}\frac{\pi z^s}{\sin\pi s}\,ds.$$

Putting $s=t+r$, and changing the order of integration, which is obviously justified, we get on simplification

$$\frac{1}{2\pi i}\int_C\frac{G(\alpha-n-t)G(\beta+n-t)}{G(1-t)}\frac{\pi z^t}{\sin\pi t}\;(q^{\alpha-n})_n\,{}_3\Phi_2\left[\begin{matrix}q^{-n}, q^{\alpha-\beta-n}, q^t\,;q\\ q^{\alpha-n}, q^{1+t-\beta-n}\end{matrix}\right]dt.$$

Summing the ${}_3\Phi_2$ by the basic analogue of Saalschütz's theorem, we get

$$\frac{(q^{\beta})_n}{2\pi i}\int_C\frac{G(\alpha-t)G(\beta-t)}{G(1-t)}\frac{\pi z^t}{\sin\pi t}\,dt = (q^{\beta})_nE_q(\alpha, \beta :: z),$$

which proves (10). For $n=1$, (10) reduces to (9).

5. An integral representation for $E_q(\alpha, \beta :: z)$. We show that†

† We write $f([x+h])$ to denote the series $\sum a_r(x+h)_r$, where $f(x)=\sum a_rx^r$.

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$$E_q(\alpha, \beta :: z) = \frac{1}{1-q}\int_0^1 E_q(\alpha, \beta+1 :: z/[1-q^{\beta}t])[1-qt]_{\beta-1}d(tq), \tag{11}$$

where Re $\alpha > 0$, Re $\beta > 0$ and $|q| < 1$.

Proof. The right-hand integral is given by

$$\frac{1}{1-q}\int_0^1 E_q(\alpha, \beta+1 :: z/[1-q^{\beta}t])[1-qt]_{\beta-1}d(tq)$$

$$= \frac{1}{(1-q)^2}G(\beta+1)\int_0^1[1-qt]_{\beta-1}d(tq)\int_0^1 E_q(q\lambda)\lambda^{\alpha-1}{}_1\Phi_0(\beta+1, -\lambda z^{-1}[1-q^{\beta}t])\,d(\lambda q).$$

On changing the order of integration, which is valid for Re $\alpha > 0$, Re $\beta > 0$ and $|\lambda(1-q^{\beta}t)| < |z|$, this becomes

$$\frac{1}{(1-q)^2}G(\beta+1)\int_0^1 E_q(q\lambda)\lambda^{\alpha-1}d(\lambda q)\int_0^1[1-qt]_{\beta-1}\,{}_1\Phi_0(\beta+1, \lambda z^{-1}[1-q^{\beta}t])\,d(tq).$$

Expanding the ${}_1\Phi_0$ and integrating term by term the t-integral, for Re $\beta > 0$, with the help of the result

$$\frac{1}{(1-q)}\int_0^1 x^{\alpha-1}(1-qx)_{\lambda-1}d(qx) = \prod_{n=0}^{\infty}\frac{(1-q^{\alpha+\lambda+n})(1-q^{1+n})}{(1-q^{\alpha+n})(1-q^{\lambda+n})} \quad (\text{Re}\,\alpha>0,\ \text{Re}\,\lambda>0),$$

we get

$$\frac{1}{1-q}G(\beta)\int_0^1 E_q(\lambda q)\lambda^{\alpha-1}\,{}_1\Phi_0(\beta\,;\,-\lambda/z)\,d(q\lambda) = E_q(\alpha, \beta :: z).$$

6. An asymptotic expansion for $E_q(\alpha, \beta :: z)$ for $|z| \to \infty$. If we evaluate the integral (5) by considering the residues at the poles of $\Gamma(s)$, we deduce the behaviour of $E_q(\alpha, \beta :: z)$ for large values of $|z|$. In particular, we find that

$$E_q(\alpha, \beta :: z) \sim \frac{G(\alpha)G(\beta)}{G(1)}\,{}_2\Phi_0(\alpha, \beta\,;\,-1/z).$$

7. It may be of interest to generalize the $E_q(\alpha, \beta :: z)$ function and also to define the basic analogues of the Whittaker functions $W_{k,m}$ and $M_{k,m}$ with its help, as in the case of MacRobert's function, and then to study further properties of such functions. I hope to deal with these functions in a subsequent paper.

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