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A REMARK ON PREINVEX FUNCTIONS

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In this paper, we show that the ratio of preinvex functions is invex. Hence, we give a positive answer to the open question which was proposed in a paper of Yang, Yang and Teo in (2003).

1. INTRODUCTION

Let R^n denotes n-dimension Euclidean space. In [**2**], Hanson considered the real differentiable function $f(x)$ on R^n whose gradient $\nabla f(x)$ satisfies the condition: for any $x, y \in R^n$, there exists a vector $\eta(x, y) \in R^n$ such that

$$f(x) \geqslant f(y) + \nabla f(y)\eta(x, y).$$

Craven [**3**] called this an invex function. Later, Weir and Mond [**4**] and Weir and Jeyakumar [**5**] introduced preinvex functions defined as follows.

Let $K \subset R^n$ and $f : K \to R$. Then f is preinvex if for any $x, y \in K$, there exists a vector $\eta(x, y) \in R^n$, for all $\alpha \in [0, 1]$, $y + \alpha\eta(x, y) \in K$

$$f\big(y + \alpha\eta(x, y)\big) \leqslant \alpha f(x) + (1 - \alpha)f(y).$$

It is easy to show that preinvexity is a generalisation of invexity for nondifferentiable function.

In [**6**], Yang and Chen presented a wider class of generalised convex functions, called semipreinvex functions as follows.

A set K in R^n is said to satisfy the "semi-connected" property, if for any $x, y \in K$ and $\alpha \in [0, 1]$, there exists a vector $\eta(x, y, \alpha) \in R^n$, such that $y + \alpha\eta(x, y, \alpha) \in K$. Let K be a set in R^n having the "semi-connected" property with $\eta(x, y, \alpha) : K \times K \times [0, 1] \longrightarrow R^n$ and $f(x)$ be a real function on K. Then f is called semi-preinvex with respect to $\eta(x, y, \alpha)$ if for $x, y \in K$ and $\alpha \in [0, 1]$,

$$f\big(y + \alpha\eta(x, y, \alpha)\big) \leqslant \alpha f(x) + (1 - \alpha)f(y)$$

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holds and $\lim_{\alpha\downarrow 0} \alpha\eta(x,y,a) = 0$.

The following result is due to Khan and Hanson [**7**] and Craven and Mond [**8**].

Theorem 1.1. *Let $X_0 \subset R^n$ and let f and g be real-valued functions defined on X_0. If $f(x) \geqslant 0$, $g(x) > 0$, $f(x)$ and $-g(x)$ are invex with respect to a same $\eta(x,y)$ on X_0, then $f(x)/g(x)$ is an invex function with respect to $\overline{\eta}(x,y) = \big(g(y)/g(x)\big)\eta(x,y)$.*

Yang, Yang and Teo [**1**] generalise Theorem 1.1 as follows.

Theorem 1.2. (See [**1**, Theorem 2.9].) *Let $X_0 \subset R^n$ and let f and g be real-valued differential functions defined on X_0. If $f(x) \geqslant 0$, $g(x) > 0$, $f(x)$ and $-g(x)$ are semipreinvex with respect to a same $\eta(x,y,\alpha)$ on X_0, and $\lim_{\alpha\to 0} \eta(x,y,\alpha) = \eta(x,y)$, then $f(x)/g(x)$ is an invex function with respect to $\overline{\eta}(x,y) = \big(g(y)/g(x)\big)\eta(x,y)$.*

Then, Yang, Yang and Teo [**1**] proposed an open question as follows:

Is there a similar result as that of Theorem 1.2 for preinvex functions?

In this paper, we show that the ratio of preinvex functions is invex. Hence, we give a positive answer to the open question in [**1**].

2. Main Results

First of all, we prove the following result which is a generalisation of Theorem 1.1 and a similar result with [**1**, Theorem 2.8].

Theorem 2.1. *Let $X_0 \subset R^n$ and let f and g be real-valued functions defined on X_0. If $f(x) \geqslant 0$, $g(x) > 0$, $f(x)$ and $-g(x)$ are preinvex with respect to a same $\eta(x,y)$ on X_0, then $f(x)/g(x)$ is a semipreinvex function with respect to $\eta^*(x,y,\alpha) = \big[(g(y))/(\alpha g(y)) + (1-\alpha)g(x)\big]\eta(x,y)$.*

Proof: Since $f(x)$ and $-g(x)$ are preinvex with respect to a same $\eta(x,y)$ and $f(x) \geqslant 0$, $g(x) > 0$, we have, for all $x, y \in X_0$ and $\alpha \in [0,1]$, $y + \alpha\eta^*(x,y,\alpha) \in X_0$, and

$$
\begin{aligned}
&\left(\frac{f}{g}\right)(y+\alpha\eta^*(x,y,\alpha)) \\
&\quad = \frac{f(y+\alpha\eta^*(x,y,\alpha))}{g(y+\alpha\eta^*(x,y,\alpha))} \\
&\quad = \frac{f(y+[(\alpha g(y))/(\alpha g(y)+(1-\alpha)g(x))]\eta(x,y))}{g(y+[(\alpha g(y))/(\alpha g(y)+(1-\alpha)g(x))]\eta(x,y))} \\
&\quad \leqslant \frac{(\alpha g(y))/(\alpha g(y)+(1-\alpha)g(x))f(x) + ((1-\alpha)g(x))/(\alpha g(y)+(1-\alpha)g(x))f(y)}{(\alpha g(y))/(\alpha g(y)+(1-\alpha)g(x))g(x) + ((1-\alpha)g(x))/(\alpha g(y)+(1-\alpha)g(x))g(y)} \\
&\quad = \frac{\alpha g(y)f(x)+(1-\alpha)g(x)f(y)}{\alpha g(y)g(x)+(1-\alpha)g(x)g(y)} \\
&\quad = \frac{\alpha g(y)f(x)+(1-\alpha)g(x)f(y)}{g(x)g(y)} \\
&\quad = \alpha\frac{f(x)}{g(x)} + (1-\alpha)\frac{f(y)}{g(y)}
\end{aligned}
$$

$$= \alpha\Big(\frac{f}{g}\Big)(x) + (1-\alpha)\Big(\frac{f}{g}\Big)(y)$$

That is, $f(x)/g(x)$ is a semipreinvex function with respect to $\eta^*(x,y,\alpha)$. $\square$

The following result gives a positive answer to the open question in [1].

Theorem 2.2. *Let $X_0 \subset R^n$ and let f and g be real-valued differential functions defined on X_0. If $f(x) \geqslant 0$, $g(x) > 0$, $f(x)$ and $-g(x)$ are preinvex with respect to a same $\eta(x,y)$ on X_0, then $f(x)/g(x)$ is an invex function with respect to $\overline{\eta}(x,y) = \big[g(y)/g(x)\big]\eta(x,y)$.*

Proof: By Theorem 2.1, we know that $f(x)/g(x)$ is a semipreinvex function with respect to $\eta^*(x,y,\alpha) = \Big[\big(g(y)\big)/\big(\alpha g(y) + (1-\alpha)g(x)\big)\Big]\eta(x,y)$. That is, for all $x, y \in X_0$ and $\alpha \in [0,1]$,

$$\Big(\frac{f}{g}\Big)\big(y + \alpha\eta^*(x,y,\alpha)\big) \leqslant \alpha\Big(\frac{f}{g}\Big)(x) + (1-\alpha)\Big(\frac{f}{g}\Big)(y).$$

Then,

$$\frac{(f/g)(y + \alpha\eta^*(x,y,\alpha)) - (f/g)(y)}{\alpha} \leqslant \Big(\frac{f}{g}\Big)(x) - \Big(\frac{f}{g}\Big)(y).$$

Let $\alpha \to 0$, and note that $\lim\limits_{\alpha\to 0} \eta^*(x,y,\alpha) = \overline{\eta}(x,y)$, we have

$$\nabla\Big(\frac{f}{g}\Big)(y)\overline{\eta}(x,y) \leqslant \Big(\frac{f}{g}\Big)(x) - \Big(\frac{f}{g}\Big)(y).$$

Hence, $f(x)/g(x)$ is an invex function with respect to $\overline{\eta}(x,y)$. $\square$

References

[1] X.M. Yang, X.Q. Yang and K.L. Teo, 'On properties of semipreinvex function', *Bull. Austral. Math. Soc.* **68** (2003), 449–459.

[2] M.A. Hanson, 'On sufficiency of the Kuhn-Tuker conditions', *Math. Anal. Appl.* **80** (1981), 544–550.

[3] B.D. Craven, 'Invex functions and constrained local minima', *Bull. Austral. Math. Soc.* **24** (1981), 357–366.

[4] T. Weir and B. Mond, 'Pre-invex functions in multiple objective optimizaton', *J. Math. Anal. Appl.* **136** (1988), 29–38.

[5] T. Weir and V. Jeyakumar, 'A class of nonconvex functions and mathematical programming', *Bull. Austral. Math. Soc.* **38** (1988), 177–189.

[6] X.Q. Yang and G.Y. Chen, 'A class of nonconvex functions and pre-variational inequalities', *J. Math. Anal. Appl.* **169** (1992), 359–373.

[7] Z.A. Khan and M.A. Hanson, 'On retio of invexity in mathematical programming', *J. Math. Anal. Appl.* **205** (1997), 330–336.

[8] B.D. Craven and B. Mond, 'Fractional programming with invexity', in *Progress in optimization*, Appl. Optim. **30** (Kluwer Acad. Publ., Dordrecht, 1999), pp. 79–89.

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