THE EQUIVALENCE OF ASYMPTOTIC DISTRIBUTIONS UNDER RANDOMISATION AND NORMAL THEORIES

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§ 1. *Introduction.* A problem of some interest in mathematical statistics is that of determining conditions under which the randomisation distribution of a statistic and its normal theory distribution are asymptotically equivalent, as these two distributions are used in alternative approaches to the same inference problem.

Let $\{\xi_n\}$ be a sequence of independent random variables such that, for each n, the distribution of ξ_n is $N(\mu, \sigma^2)$ (*i.e.*, is normal with mean μ and variance σ^2).

Let $\{a_n\}$ be a given sequence of real numbers with $a_{n_1} \neq a_{n_2}$ for some n_1, n_2.

Let $\{X_n\}$ be a sequence of random variables, the joint probability distribution of X_1, $X_2, \ldots, X_n$ being defined for each n as follows :

$$P\{X_1 = a_{\rho_1}, X_2 = a_{\rho_2}, \ldots, X_n = a_{\rho_n}\} = \frac{1}{n!},$$

for each permutation $(\rho_1, \rho_2, \ldots, \rho_n)$ of the integers $(1, 2, \ldots, n)$, where $P\{R\}$ denotes the probability of a relation R.

Let $t_n(x_1, x_2, \ldots, x_n)$, denoted by $t_n(x)$, be a function of n variables $x_1, x_2, \ldots, x_n$, defined for each n.

Then $\{t_n(\xi)\}$ and $\{t_n(X)\}$ are sequences of random variables, and the problem stated above is that of determining conditions subject to which, for all c,

$$\lim_{n\to\infty} P\{t_n(\xi) < c\} = \lim_{n\to\infty} P\{t_n(X) < c\}.$$

Of particular interest is the case where t_n has, for each n, the properties :

(i) $t_n(kx_1, kx_2, \ldots, kx_n) = t_n(x_1, x_2, \ldots, x_n)$ for any positive number k,

(ii) $t_n(x_1 + c, x_2 + c, \ldots, x_n + c) = t_n(x_1, x_2, \ldots, x_n)$ for any number c.

Many statistics in common use have these properties.

For such sequences $\{t_n\}$ the distribution of $t_n(\xi)$ is independent of μ and σ^2, since

$$P\{t_n(\xi) < c\} = \iint\limits_{t_n(\xi)<c} \cdots \int (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n} (\xi_i - \mu)^2\right] d\xi_1\, d\xi_2 \ldots d\xi_n$$

$$= \iint\limits_{t_n(\eta)<c} \cdots \int (2\pi)^{-\frac{n}{2}} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{n} \eta_i^2\right] d\eta_1\, d\eta_2 \ldots d\eta_n,$$

where $\eta_i = \dfrac{\xi_i - \mu}{\sigma}$, $i = 1, 2, \ldots, n$, since the region $t_n(\xi) < c$ corresponds to the region

$$t_n(\sigma\eta_1 + \mu, \sigma\eta_2 + \mu, \ldots, \sigma\eta_n + \mu) < c,$$

which by the properties (i), (ii) of t_n is the region $t_n(\eta) < c$.

For such sequences $\{t_n\}$ the distribution of $t_n(\xi)$ will be called the normal theory distribution of t_n, that of $t_n(X)$ the randomisation distribution of t_n. In discussing the normal theory distribution of t_n we can, without loss of generality, take $\mu = 0$ and $\sigma^2 = 1$.

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§ 2. *Geometrical Interpretation.* The properties (i) and (ii) imply that the distribution of $t_n(\xi)$ is the same as the conditional distribution of $t_n(\xi)$ given

$$\bar{\xi}_n = \frac{1}{n} \sum_{i=1}^{n} \xi_i \quad \text{and} \quad m_{2,n}(\xi) = \frac{1}{n} \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2,$$

this conditional distribution in turn being independent of $\bar{\xi}_n$ and $m_{2,n}(\xi)$. For, taking $\mu = 0$, $\sigma^2 = 1$, the probability density element of $\xi_1, \xi_2, \ldots, \xi_n$ is

$$(2\pi)^{-\frac{n}{2}} \exp\left(-\tfrac{1}{2} \sum_{i=1}^{n} \xi_i^2\right) d\xi_1\, d\xi_2 \ldots d\xi_n,$$

i.e.,

$$(2\pi)^{-\frac{n}{2}} \exp\left[-\tfrac{1}{2} \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2\right] \exp\left(-\frac{n}{2} \bar{\xi}_n^2\right) d\xi_1\, d\xi_2 \ldots d\xi_n.$$

Applying an orthogonal linear transformation from $\xi_1, \xi_2, \ldots, \xi_n$ to $\eta_1, \eta_2, \ldots, \eta_n$ in which $\eta_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i$, we get the probability density element of $\eta_1, \eta_2, \ldots, \eta_n$ as

$$(2\pi)^{-\frac{n}{2}} \exp\left(-\tfrac{1}{2} \sum_{i=2}^{n} \eta_i^2\right) \exp\left(-\tfrac{1}{2}\eta_1^2\right) d\eta_1\, d\eta_2 \ldots d\eta_n.$$

Under this transformation $t_n(\xi)$ becomes $t_n'(\eta)$, say, where the property (ii) of t_n implies that $t_n'(\eta)$ is functionally independent of η_1. Clearly $t_n'(\eta)$ is also a homogeneous function of $\eta_2, \eta_3, \ldots, \eta_n$ of degree zero.

It follows that if the change is made from Cartesian coordinates $(\eta_2, \eta_3, \ldots, \eta_n)$ to polar coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{n-2})$, $t_n'(\eta)$ becomes a function $t_n''(\theta_1, \theta_2, \ldots, \theta_{n-2})$ of $\theta_1, \theta_2, \ldots, \theta_{n-2}$ only. Also the probability density element of the random variables $\eta_1, r, \theta_1, \ldots, \theta_{n-2}$ can be expressed in the form

$$K \exp\left(-\tfrac{1}{2}\eta_1^2\right) d\eta_1 \,.\, r^{n-2} \exp\left(-\tfrac{1}{2}r^2\right) dr \,.\, J(\theta_1, \theta_2, \ldots, \theta_{n-2})\, d\theta_1\, d\theta_2 \ldots d\theta_{n-2},$$

where $J(\theta_1, \theta_2, \ldots, \theta_{n-2})$ is a function derived from the Jacobian $\left|\frac{\partial(\eta_2, \eta_3, \ldots, \eta_n)}{\partial(r, \theta_2, \ldots, \theta_{n-2})}\right|$ and K is a constant.

Hence the random variable $t_n(\xi)$ is independent of the random variables η_1 and r, *i.e.*, of the random variables $\bar{\xi}_n$ and $m_{2,n}(\xi)$.

Furthermore, from the above it is clear that the conditional distribution of $t_n(\xi)$, given $\bar{\xi}_n$ and $m_{2,n}(\xi)$, depends only on the " volume " element $d\xi_1\, d\xi_2 \ldots d\xi_n$.

Hence if the random variables $\xi_1, \xi_2, \ldots, \xi_n$ are represented by a n-dimensional Euclidean space W_n, if Q_{n-2} denotes the hypersphere $\xi_1 + \xi_2 + \ldots + \xi_n = nq_1$, $\sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2 = nq_2$, if $Q_{n-2,c}$ denotes the subset of Q_{n-2} in which $t_n(\xi) < c$, and if l_n denotes n-dimensional Lebesgue measure, then

$$P\{t_n(\xi) < c\} = P\{t_n(\xi) < c \mid \bar{\xi}_n = q_1, m_{2,n}(\xi) = q_2\}$$

$$= \frac{l_{n-2}(Q_{n-2,c})}{l_{n-2}(Q_{n-2})}.$$

Again the space of variation $\mathfrak{X}_n$ of the random variables $X_1, X_2, \ldots, X_n$ is a set of $n!$ points (not necessarily all distinct) in a n-dimensional Euclidean space W_n', with mutually

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perpendicular axes $OX_1, OX_2, \ldots, OX_n$. If W_n' is superimposed on W_n with $OX_i \longleftrightarrow O\,\xi_i$ $i=1, 2, \ldots, n$, then $\mathfrak{X}_n$ is contained in the hypersphere A_{n-2} with equations

$$\xi_1+\xi_2+\ldots+\xi_n=n\bar{a}_n, \quad \sum_{i=1}^{n}(\xi_i-\bar{\xi}_n)^2=nm_{2,n}(a),$$

where

$$\bar{a}_n=\frac{1}{n}\sum_{i=1}^{n}a_i, \quad m_{2,n}(a)=\frac{1}{n}\sum_{i=1}^{n}(a_i-\bar{a}_n)^2.$$

If ν_c denotes the number of points of $\mathfrak{X}_n$ for which $t_n(X)<c$, then

$$P\{t_n(X)<c\}=\frac{\nu_c}{n!}\cdot$$

Hence in order that the limiting distribution functions of $t_n(\xi)$ and $t_n(X)$ should be the same, it is necessary and sufficient that

$$\lim_{n\to\infty}\frac{l_{n-2}(A_{n-2,c})}{l_{n-2}(A_{n-2})}=\lim_{n\to\infty}\frac{\nu_c}{n!} \text{ for all } c,$$

i.e., that the set of points $\mathfrak{X}_n$ should tend to be distributed uniformly throughout A_{n-2}, relative to the class $\mathfrak{C}$ of subsets $A_{n-2,c}$, when W_n' is superimposed on W_n as above.

§ 3. *Linear Combinations.* The discussion is now particularised from the general class of sequences $\{t_n\}$ to a subset of this class.

For each n, let $y_{n1}, y_{n2}, \ldots, y_{nn}$ be an assigned set of real numbers with $y_{ni_1}\neq y_{ni_2}$ for some i_1, i_2.

Let

$$\bar{y}_n=\frac{1}{n}\sum_{i=1}^{n}y_{ni},$$

$$m_{j,n}(y)=\frac{1}{n}\sum_{i=1}^{n}(y_{ni}-\bar{y}_n)^j, \quad j=2, 3, \ldots,$$

and

$$b_{j,n}(y)=\frac{m_{j,n}(y)}{[m_{2,n}(y)]^{j/2}}, \; j=2, 3, \ldots .$$

Also let

$$y'_{ni}=\frac{y_{ni}-\bar{y}_n}{[m_{2,n}(y)]^{1/2}}, \; i=1, 2, \ldots, n,$$

so that $\bar{y}_n'=0$, $m_{2,n}(y')=1$, and $m_{j,n}(y')=b_{j,n}(y)$.

Similarly let

$$a'_{ni}=\frac{a_i-\bar{a}_n}{[m_{2,n}(a)]^{1/2}}, \; i=1, 2, \ldots, n.$$

We consider sequences $\{r_n\}$, where $r_n(\xi)$ is of the form

$$r_n(\xi)=\frac{(n-1)^{1/2}}{n}\sum_{i=1}^{n}y'_{ni}\xi'_{ni},$$

where

$$\xi'_{ni}=\frac{\xi_i-\bar{\xi}_n}{[m_{2,n}(\xi)]^{\frac{1}{2}}}, \quad i=1, 2, \ldots, n.$$

A sequence $\{r_n(X)\}$ can be regarded as a sequence of standardised linear combinations of the random variables $X_1, X_2, \ldots$. We discuss conditions subject to which the limiting distributions of $r_n(\xi)$ and $r_n(X)$ are equivalent.

The following lemmas are required.

3.1. LEMMA. *Every $r_n(\xi)$ has the same distribution which tends to the $N(0, 1)$ form as $n\to\infty$.*

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We have
$$\frac{[r_n(\xi)]^2}{n-1}=\frac{\left(\sum_{i=1}^{n}\frac{y'_{ni}}{n^{1/2}}\xi_i\right)^2}{\sum_{i=1}^{n}\xi_i^2-n\bar{\xi}_n^2}.$$

Applying an orthogonal linear transformation from $\xi_1, \xi_2, \ldots, \xi_n$ to $\eta_1, \eta_2, \ldots, \eta_n$ in which
$$n^{\frac{1}{2}}\eta_1=\sum_{i=1}^{n}\xi_i,$$
and
$$n^{\frac{1}{2}}\eta_2=\sum_{i=1}^{n}y'_{ni}\xi_i,$$
(these being orthogonal since $\sum_{i=1}^{n}y'_{ni}=0$), we get
$$\frac{[r_n(\xi)]^2}{n-1}=\frac{\eta_2^2}{\sum_{i=2}^{n}\eta_i^2}. \qquad \text{(3.1.1)}$$

Since $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables each with a $N(0, 1)$ distribution, $\eta_1, \eta_2, \ldots, \eta_n$ have the same property.

It follows that the distribution of $r_n(\xi)$ does not depend on a particular set of values $y_{n1}, y_{n2}, \ldots, y_{nn}$.

Further, it is easily shown from (3.1.1) that if $-(n-1)^{\frac{1}{2}}\leqslant c_1<c_2\leqslant(n-1)^{\frac{1}{2}}$, then
$$P\{c_1\leqslant r_n(\xi)<c_2\}=\left(\frac{n}{n-1}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\left(\frac{n}{2}\right)^{\frac{1}{2}}\Gamma\left(\frac{n-2}{2}\right)}(2\pi)^{-\frac{1}{2}}\int_{c_1}^{c_2}\left(1-\frac{x^2}{n-1}\right)^{\frac{n-4}{2}}dx.$$

From this it is clear that
$$\lim_{n\to\infty}P\{r_n(\xi)<c\}=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{c}\exp\left(-\tfrac{1}{2}x^2\right)dx.$$

3.2. Lemma. *Let $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ be a partition of an integer s. Let $S_n(a', \alpha)$ denote the symmetric polynomial $\Sigma\, a'^{\alpha_1}_{i_1}a'^{\alpha_2}_{i_2}\ldots a'^{\alpha_h}_{i_h}$, where summation extends over all ordered sets $i_1, i_2, \ldots, i_h$ of h distinct integers from $1, 2, \ldots, n$. Then $S_n(a', \alpha)$ can be expressed in the form*
$$S_n(a', \alpha)=n^h\prod_{i=1}^{h}m_{\alpha_i, n}(a')+R_n[m(a')],$$
where $R_n[m(a')]$ is a sum of terms of the form $C_{\alpha,\beta}\, n^k\prod_{i=1}^{k}m_{\beta_i, n}(a')$, in which

(i) *$(\beta_1, \beta_2, \ldots, \beta_k)$ is a partition of s in which each β is either an α or a sum of more than one α.*

(ii) *$k<h$,*

(iii) *$C_{\alpha,\beta}$ is a constant independent of n,*

(iv) *the number of terms is independent of n.*

This follows directly from the well-known expression for symmetric polynomials of the type $S_n(a', \alpha)$ in terms of sums of powers $\sum_{i=1}^{n}a_i'^p$, since $\sum_{i=1}^{n}a_i'^p=nm_{p,n}(a')$.

(i), (ii), (iii), and (iv) are properties of this expression.

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3.3. LEMMA. (i) $m_{2s, n}(a') \geqslant 1$, $s = 1, 2, 3, \ldots$.

(ii) $| m_{s, n}(a') | \leqslant n^{\frac{s-2}{2}}$, $s = 2, 3, 4, \ldots$.

(i) follows from well-known inequalities on absolute moments [2].

$$\text{(ii)} \; | m_{s, n}(a') | = | \frac{1}{n}(a_1'^s + a_2'^s + \ldots + a_n'^s) |$$
$$\leqslant \frac{1}{n}(a_1'^2 + a_2'^2 + \ldots + a_n'^2)^{s/2}$$
$$= n^{\frac{s-2}{2}}.$$

These inequalities hold also when a' is replaced by y'.

3.4. THEOREM. *Randomisation distributions and normal theory distributions of all statistics r_n are asymptotically equivalent if and only if the distribution of the set of measures $a_1, a_2, \ldots, a_n$ tends to the normal form as $n \to \infty$.*

Necessity. Let the sequence $\{r_n^0\}$ be defined by $y_{n1} = 1$, $y_{ni} = 0$, $i = 2, 3, \ldots, n$, $n = 2, 3, 4 \ldots$, so that

$$r_n^0(X) = \frac{X_1 - \bar{X}_n}{[m_{2, n}(X)]^{\frac{1}{2}}} = \frac{X_1 - \bar{a}_n}{[m_{2, n}(a)]^{\frac{1}{2}}}.$$

If $F(c)$ denotes the proportion of the numbers $a_1, a_2, \ldots, a_n$ with the property that

$$\frac{a_i - \bar{a}_n}{[m_{2, n}(a)]^{\frac{1}{2}}} < c,$$

then the proportion of the points of $\mathfrak{X}_n$ for which $r_n^0(X) < c$ is $F(c)$, since corresponding to each such number a_i there are $(n-1)!$ points of $\mathfrak{X}_n$ for which $r_n^0(X) < c$.

Hence $$P\{r_n^0(X) < c\} = F(c).$$

Also, by (3.1), $P\{r_n^0(\xi) < c\} \to (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{c} \exp(-\tfrac{1}{2}x^2)\, dx$, as $n \to \infty$.

Hence for equivalence of the asymptotic distributions of $r_n^0(X)$ and $r_n^0(\xi)$ it is necessary that

$$F(c) \to (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{c} \exp(-\tfrac{1}{2}x^2)\, dx,$$

i.e., that the set of numbers $a_1, a_2, \ldots, a_n$ should tend to be normally distributed.

Sufficiency. As a consequence of (3.1) it has to be shown that, if the set $a_1, a_2, \ldots, a_n$ tends to be normally distributed, then the limiting form of the distribution of $r_n(X)$ in any sequence $\{r_n(X)\}$ of linear combinations is $N(0, 1)$. Now the set $a_1, a_2, \ldots, a_n$ tends to be normally distributed if and only if

$$b_{j, n}(a) \to 0, \qquad j = 3, 5, 7, \ldots,$$

and

$$b_{j, n}(a) \to \frac{j!}{(\frac{1}{2}j)!\, 2^{j/2}}, \qquad j = 2, 4, 6, \ldots,$$

as $n \to \infty$.

Hence it has to be shown that, subject to

$$b_{j, n}(a) = \begin{cases} o(1), & j = 3, 5, 7, \ldots \\ \dfrac{j!}{(\frac{1}{2}j)!\, 2^{j/2}} + o(1), & j = 2, 4, 6, \ldots, \end{cases}$$

the distribution of any statistic $r_n(X)$ is asymptotically $N(0, 1)$.

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Let
$$X'_{ni} = \frac{X_i - \overline{X}_n}{[m_{2,n}(X)]^{\frac{1}{2}}} = \frac{X_i - \bar{a}_n}{[m_{2,n}(a)]^{\frac{1}{2}}}.$$

Then for any integers $\alpha_1, \alpha_2, \ldots, \alpha_h$, if E denotes the expected value of a random variable,

$$E\left\{X'^{\alpha_1}_{ni_1} X'^{\alpha_2}_{ni_2} \ldots X'^{\alpha_h}_{ni_h}\right\} = \frac{1}{n^{[h]}} S(a', \alpha),$$

where $n^{[h]} = n(n-1) \ldots (n-h+1)$.

Also
$$r_n(X) = \frac{(n-1)^{\frac{1}{2}}}{n} \sum_{i=1}^{n} y'_{ni} X'_{ni}.$$

Let t be a positive integer, $t \geqslant 2$.

Expanding $\left(\sum_{i=1}^{n} y'_{ni} X'_{ni}\right)^t$, taking expected values term by term and collecting terms, we get

$$E[r_n(X)]^t = \frac{(n-1)^{t/2}}{n^t} \Sigma \frac{1}{n^{[h]}} C_\alpha \,.\, S_n(y', \alpha) S_n(a', \alpha), \qquad (3.4.1)$$

where summation extends over all partitions $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ of t. Also

$$C_\alpha = \frac{t!}{\alpha_1!\, \alpha_2! \ldots \alpha_h!} \frac{1}{\pi_1!\, \pi_2! \ldots \pi_\rho!},$$

where, when the α's are chosen from ρ different integers $i_1, i_2, \ldots, i_\rho$, π_j of the α's are equal to i_j, $j = 1, 2, \ldots, \rho$.

By (3.2)
$$S_n(a', \alpha) = n^h \prod_{i=1}^{h} m_{\alpha_i, n}(a') + R_n(a', \alpha),$$

where
$$R_n(a', \alpha) = O(n^{h-1}),$$

and so
$$\frac{1}{n^{[h]}} S_n(a', \alpha) = \prod_{i=1}^{h} m_{\alpha_i, n}(a') + o(1).$$

If any α_i is odd, then $\prod_{i=1}^{h} m_{\alpha_i, n}(a') = o(1)$.

If t is odd, then for each partition $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ of t at least one α_i is odd, and so for every partition of t

$$\frac{1}{n^{[h]}} S_n(a', \alpha) = o(1).$$

By applying (3.2) to $S_n(y', \alpha)$ and using (3.3) it is easily shown that $n^{-t/2} S_n(y', \alpha)$ is bounded.

Then, since the number of terms on the right side of (3.4.1) is independent of n,

$$E[r_n(X)]^t = o(1), \text{ if } t \text{ is odd}.$$

Also if t is even, $t = 2u$, say, those terms on the right-hand side of (3.4.1) corresponding to partitions $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ of $2u$ in which some α_i is odd are $o(1)$.

Hence $E[r_n(X)]^{2u} = \frac{(n-1)^u}{n^{2u}} \Sigma \frac{1}{n^{[h]}} C_{2\beta} S_n(y', 2\beta) S_n(a', 2\beta) + o(1)$, summation extending over all partitions $(2\beta_1, 2\beta_2, \ldots, 2\beta_h)$ of $2u$, $\beta_1, \beta_2, \ldots, \beta_h$ being integers.

Now
$$S_n(y', 2\beta) = S_n(y'^2, \beta),$$

while
$$\frac{1}{n^{[h]}} S_n(a', 2\beta) = \frac{1}{2^u} \prod_{i=1}^{h} \frac{(2\beta_i)!}{\beta_i!} + o(1).$$

Hence
$$E[r_n(X)]^{2u} = \frac{1}{2^u n^u} \Sigma \left[C_{2\beta}\, S_n(y'^2, \beta) \prod_{i=1}^{h} \frac{(2\beta_i)!}{\beta_i!} \right] + o(1),$$
since, as above, $n^{-u} S_n(y'^2, \beta)$ is bounded.

Also
$$C_{2\beta} \prod_{i=1}^{h} \frac{(2\beta_i)!}{\beta_i!} = \frac{(2u)!}{u!} C_\beta, \quad \text{and so}$$
$$\begin{aligned} E[r_n(X)]^{2u} &= \frac{(2u)!}{2^u u!} \frac{1}{n^u} \Sigma\, C_\beta S_n(y'^2, \beta) + o(1) \\ &= \frac{(2u)!}{2^u u!} \left(\frac{y_1'^2 + y_2'^2 + \ldots + y_n'^2}{n} \right)^u + o(1) \\ &= \frac{(2u)!}{2^u u!} + o(1). \end{aligned}$$

Finally $E[r_n(X)] = 0$.

Hence the moments of the distribution of $r_n(X)$ tend to those of a $N(0, 1)$ distribution, and since this distribution is completely determined by its moments, this completes the proof.

While the very stringent conditions on $\{a_n\}$ of (3.4) are necessary for equivalence of the asymptotic distributions of $r_n(X)$ and $r_n(\xi)$ for all sequences $\{r_n\}$, for " most " such sequences much less restrictive conditions are sufficient. This is brought out by the following theorem, which is a more general form of a result proved by Wald and Wolfowitz [**5**], and partially extended by Noether [**4**].

3.6. Theorem. *If $b_{j,n}(y) = O[n^{\theta(j-2)}]$, $j = 2, 3, 4, \ldots$, where θ is a given real number such that $0 \leqslant \theta < \frac{1}{2}$, then the distribution of $r_n(X)$ is asymptotically $N(0, 1)$ provided $b_{j,n}(a) = o[n^{\phi(j-2)}]$, $j = 3, 4, \ldots$, where $\phi = \frac{1}{2} - \theta$.*

Application of lemma 3.2 to $S_n(y', \alpha)$ and $S_n(a', \alpha)$ in each term of the right side of (3.4.1) shows that $E[r_n(X)]^t$ can be expressed as a sum of terms of the form
$$C_\alpha K_{\alpha,\beta,\gamma} \left(\frac{n-1}{n} \right)^{t/2} \frac{1}{n^{t/2}} \frac{1}{n^{[h]}} \left\{ n^{h_1} \prod_{i=1}^{h_1} m_{\beta_i, n}(y') \right\} \left\{ n^{h_2} \prod_{j=1}^{h_2} m_{\gamma_j, n}(a') \right\} = B, \text{ say,}$$
where

(i) the number of terms is independent of n,

(ii) $\begin{matrix}(\beta_1, \beta_2, \ldots, \beta_{h_1}) \\ (\gamma_1, \gamma_2, \ldots, \gamma_{h_2})\end{matrix}\Big\}$ is a partition of t in which $\begin{Bmatrix} h_1 \leqslant h \\ h_2 \leqslant h \end{Bmatrix}$ and each $\begin{Bmatrix} \beta_i \\ \gamma_j \end{Bmatrix}$ is either an α or a sum of α's.

(iii) $K_{\alpha,\beta,\gamma}$ is independent of n and equals 1 if $h_1 = h_2 = h$, *i.e.*, if (α), (β), and (γ) are all the same partition of t.

We consider the order of the term B.

If any $\beta_i = 1$, or any $\gamma_j = 1$, then $B = 0$.

If some $\gamma_j > 2$ and every $\beta_i \geqslant 2$, then $B = o(n^p)$, where
$$\begin{aligned} p &= h_1 + h_2 - h - \tfrac{1}{2}t + \theta(t - 2h_1) + \phi(t - 2h_2) \\ &= 2\phi h_1 + 2\theta h_2 - h, \quad \text{since } \theta + \phi = \tfrac{1}{2}, \\ &\leqslant 0 \qquad\qquad\quad, \quad \text{since } h_1 \leqslant h \text{ and } h_2 \leqslant h, \end{aligned}$$
i.e., $B = o(1)$, if any $\gamma_j > 2$.

Hence, if t is odd,
$$E\{r_n(X)\}^t = o(1),$$
for then at least one γ_j in each partition is odd, *i.e.*, is either 1 or is greater than 2.

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Furthermore, if t is even, $t=2u$, say, then $B=o(1)$ unless possibly when $h_2=u$, and $\gamma_1=\gamma_2=\ldots=\gamma_u=2$.

If $t=2u$ and $h_2=u$ and $\gamma_j=2, j=1, 2, \ldots, u$, then

(i) $B=0$, if any $\beta_i=1$,

(ii) $B=O[n^{2\phi h_1+2\theta h_2-h}]$, if each $\beta_i \geqslant 2$.

In case (ii) $B=o(1)$, unless $h_1=h_2=h$, since $2\phi h_1+2\theta h_2-h<0$ except when $h_1=h_2=h$, *i.e.*, $B=o(1)$, unless possibly when $h=h_1=h_2=u$, and $(\alpha)=(\beta)=(\gamma)=(2, 2, \ldots, 2)$.

For the only term for which this is true $K_{\alpha,\beta,\gamma}=1$, by (iii), while $C_\alpha=\dfrac{(2u)!}{u!\,2^u}$, as in (3.4).

The term itself is, then, $\dfrac{(2u)!}{u!\,2^u}+o(1)$.

It follows as in (3.4) that $r_n(X)$ is asymptotically distributed in the $N(0, 1)$ form.

3.7 *Applications.* (1). Asymptotic normality of the distribution of the product-moment rank correlation coefficient was originally proved by Hotelling and Pabst [1]. Derivation of this result from Theorem 3.6 illustrates to some extent the width of the conditions there established.

If
$$y_{ni}=i, \quad i=1, 2, \ldots, n, \quad n=2, 3, \ldots$$
and
$$a_i=i, \quad i=1, 2, \ldots,$$
then the corresponding sequence $\{r_n(X)]$ is a sequence of product-moment rank correlation coefficients. It is easily shown that, in this case, $b_{j,n}(y)=b_{j,n}(a)=O(1), j=2, 3, \ldots$, so that, for this sequence, the conditions of Theorem 3.6 are more than satisfied. In fact, for $b_{j,n}(y)=O(1)$, $j=2, 3, \ldots$, it is sufficient for asymptotic normality of $r_n(X)$ to have

$$b_{j,n}(a)=o\left(n^{\frac{j-2}{2}}\right) j=3, 4, \ldots.$$

(2). Madow [4] has established conditions subject to which linear combinations of the measures of a random sample drawn without replacement from a finite population are approximately normal. Such sampling results in an actual situation to which the above theory can be applied, the connection being as follows.

Let $a_1, a_2, \ldots, a_n$ be considered as the measures of a population P_n of n individuals in a sequence $\{P_n\}$ of populations. Then the random variables $X_1, X_2, \ldots, X_n$ can be considered as arising from a random ordering of P_n, and $X_1, X_2, \ldots, X_k$, $k<n$, can be considered as the measures of a random sample of k individuals drawn without replacement from P_n.

The following is a particular application.

Let f be a rational number with $0<f<1$.

Let $\{P_{n_i}\}$ be a subsequence of $\{P_n\}$ for which fn_i is integral and equal to p_i, say, for $i=1, 2, \ldots$.

Let
$$y_{n_ij}=\frac{1}{p_i}-\frac{1}{n_i}, \quad j=1, 2, \ldots, p_i,$$
$$=-\frac{1}{n_i}, j=p_i+1, \ldots, n_i.$$

Then it is easily shown that $b_{j,n_i}(y)=O(1), j=2, 3, \ldots$.

Let
$$\bar{x}_i=\frac{1}{p_i}(X_1+X_2+\ldots+X_{p_i}).$$

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Then $r_n(X)$ corresponding to these values of y is given by

$$r_{n_i}(X) = \left(\frac{fn_i}{1-f}\right)^{\frac{1}{2}} \frac{\bar{x}_i - \bar{a}_{n_i}}{[m_{2,n_i}(a)]^{\frac{1}{2}}},$$

and the distribution of $r_{n_i}(X)$ is asymptotically $N(0, 1)$ provided

$$b_{j, n_i}(a) = o\left(n_i^{\frac{j-2}{2}}\right), j = 3, 4, \ldots .$$

Tying this up with more usual terminology we have the result that if a random sample, with sampling fraction f, is drawn without replacement from a large finite population of N individuals with mean $\mu(=\bar{a}_N)$ and variance $\sigma^2(=m_{2,N}(a))$ then the distribution of the sample mean is approximately normal with mean μ and variance $\frac{\sigma^2}{N}\left(\frac{1}{f}-1\right)$, unless the population is very unusual.

Proofs establishing the equivalence of the normal theory approach and the randomisation approach to a wider field of practical problems depend on an extension of Theorem 3.6 to the joint distribution of more than one linear combination. It is hoped to publish this extension in a later paper.

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