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SMALL RANDOM PERTURBATION OF DYNAMICAL SYSTEMS WITH REFLECTING BOUNDARY

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0. Introduction.

Consider a diffusion process in $\boldsymbol{R}^d$ satisfying the stochastic differential equation

$$dX = \varepsilon(\sigma(X_t)dW_t + c(X_t)dt) + b(X_t)dt\ , \qquad X_0 = x\ .$$

Here $x \in \boldsymbol{R}^d$, W is the d-dimensional Wiener process, $\sigma(y)$ is a function with values in $\boldsymbol{R}^d \times \boldsymbol{R}^d$, $c(y)$ and $b(y)$ are $\boldsymbol{R}^d$-valued functions; σ, c, and b are subject to suitable conditions. The solution of the stochastic differential equation depends of course on ε, and in [7] Ventcel and Freidlin study the asymptotic behavior of this solution as ε approaches zero, and relate it to behavior of the dynamical system (non-random) obtained by setting $\varepsilon = 0$. A key role is played be a certain functional $I\colon C_T(\boldsymbol{R}^d) \to \boldsymbol{R}^1_+$. For the case $b = 0$ such results are also in S. R. S. Varadhan [6]. Applications to asymptotic problems in partial differential equations are developed by these authors, and also by Friedman [2].

Our aim here is to study analogous problems when the diffusion is controlled by a stochastic equation as above in the interior of some region of $\boldsymbol{R}^d$, but is subject to reflection on hitting the boundary of the region. As is to be expected, such results have applications to asymptotic questions related to the Neumann problem. For further general remarks about these problems see the beginning of Section 2.

Section 1 is devoted to obtaining the basic estimates. It relies on a new construction for the reflected diffusion. Several constructions for such a process—indeed allowing more general boundary conditions, and under weaker assumptions on the coefficients than we impose—are already known; see [4] and [8] and the bibliography of the latter reference. Our construction, however, provided us with the necessary information for the problems at hand. It is of interest in itself not only because it

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gives a very easy proof of existence and strong uniqueness of the desired process, but also because it provides an interpretation of the local time on the boundary which is intuitive, and useful in obtaining bounds for the growth of this quantity.

1. Basic Asymptotic Theorems.

Let $W = (W_t)$, $0 \leq t < \infty$ be a Wiener process defined on a probability space $(\Omega, \mathscr{F}, P)$ with values in $\boldsymbol{R}^d$. Let $\mathscr{F}_t$ be the P-completion of the least σ-field with respect to which the random variables W_s, $0 \leq s \leq t$ are measurable.

For $D \subseteq \boldsymbol{R}^d$, $C(D)$ is to denote the space of continuous functions from $[0, \infty)$ to D, and for $T \geq 0$ $C_T(D)$ is to denote the space of continuous functions from $[0, T]$ to D. An element η of $C(D)$ or $C_T(D)$ will often be referred to as a *trajectory*, its value at t denoted by η_t and $\dot{\eta}_t$ used for the value of its derivative (if it exists) at t. If π_t are the coordinate functions, $\pi_t(\eta) = \eta_t$, let $\mathscr{C}_t$ be the least σ-field with respect to which the functions π_s are measurable, $0 \leq s \leq t$; let $\mathscr{C}$ be the least σ-field containing $\mathscr{C}_t$, $0 \leq t < \infty$. If for every $t \in [0, \infty)$, α_t is a $\mathscr{C}_t$-measurable map from $C(D)$ into $\boldsymbol{R}^1$ (or into $\boldsymbol{R}^d$ or $\boldsymbol{R}^d \times \boldsymbol{R}^d$), $\alpha = (\alpha_t)$ is called a *non-anticipating functional.*

We shall be dealing largely with continuous stochastic processes (X_t) defined on $(\Omega, \mathscr{F}, P)$ with values in some subset D of $\boldsymbol{R}^d$. Such a process is *adapted* to $\mathscr{F}_t$ if each X_t is measurable with respect to $\mathscr{F}_t$. For $\omega \in \Omega$ the function $t \to X_t(\omega)$, or simply $X(\omega)$ is an element of $C(D)$. As is customary, ω will usually be suppressed. Thus, for example, $P[X \in K]$ must be interpreted as $P[\omega : X(\omega) \in K]$.

Our procedure will be to obtain our results first for the case that D is the half-space $\boldsymbol{R}^d_+ = \{x = (x^1, x^2, \cdots, x^d) : x^1 \geq 0\}$ with normal reflection at the boundary and then to reduce the general case to this by introducing suitable local coordinates, i.e., localization. To treat the half-space problem we begin with a simple generalization of the basic results of Ventcel and Freidlin [7]: this consists of replacing the coefficients $\sigma(X_t), c(X_t), b(X_t)$ by non-anticipating functionals $\sigma_t \circ X, c_t \circ X, b_t \circ X$. Next we observe that the desired reflected process in $\boldsymbol{R}^d_+$ with prescribed coefficients $\sigma(X_t^{x,\varepsilon}), c(X_t^{x,\varepsilon}), b(X_t^{x,\varepsilon})$ can be obtained from an associated unrestricted process $Y^{x,\varepsilon}$ moving in $\boldsymbol{R}^d$ and governed by a stochastic differential equation involving suitable non-anticipating functionals σ_t, c_t,

b_t by a continuous map Γ from $C(\boldsymbol{R}^d)$ to $C(\boldsymbol{R}^d_+)$, i.e., $X^{x,\varepsilon} = \Gamma(Y^{x,\varepsilon})$. Using the explicit form of Γ will allow us to obtain results about $X^{x,\varepsilon}$ from results about $Y^{x,\varepsilon}$.

1.1. The unrestricted process. Let $Y^{x,\varepsilon}$ be the solution of

$$(1) \qquad dY^{x,\varepsilon}_t = \varepsilon(\sigma_t \circ Y^{x,\varepsilon} dW_t + c_t \circ Y^{x,\varepsilon} dt) + b_t \circ Y^{x,\varepsilon} dt \,, \qquad Y^{x,\varepsilon}_0 = x \,.$$

Here σ_t is a $d \times d$ matrix valued non-anticipating functional, and c_t, b_t are $\boldsymbol{R}^d$-valued non-anticipating functionals. Let $a_t = \sigma_t \sigma_t^*$. We also write $a_t = (a_t^{ij}), \sigma_t = (\sigma_t^{ij})$; likewise, c_t^i and b_t^i are the ith coordinates of c_t and b_t, respectively. The following assumptions will be made:

There exist positive constants m_1, m_2, m_3 such that for $0 \leq t < \infty$ and $1 \leq i \leq d$, $1 \leq j \leq d$, and η and $\xi \in C(\boldsymbol{R}_d)$:

$$|\sigma_t^{ij}(\eta)| \leq m_1 \,, \quad |c_t^i(\eta)| \leq m_1 \,, \quad |b_t^i(\eta)| \leq m_1$$
$$|\sigma_t^{ij}(\eta) - \sigma_t^{ij}(\xi)| \leq m_2 \max_{0 \leq s \leq t} |\eta_s - \xi_s|$$
$$|b_t^i(\eta) - b_t^i(\xi)| \leq m_2 \max_{0 \leq s \leq t} |\eta_s - \xi_s|$$
$$|c_t^i(\eta) - c_t^i(\xi)| \leq m_2 \max_{0 \leq s \leq t} |\eta_s - \xi_s|$$

$$m_3 |\theta|^2 \geq \sum_{i=1}^d \sum_{j=1}^d a^{ij} \theta^i \theta^j \geq m_3^{-1} |\theta|^2 \qquad \text{for all } \theta = (\theta^1, \theta^2, \cdots, \theta^d) \,.$$

These conditions allow one to apply the usual successive approximation arguments to obtain the existence of a unique solution to (1). Only in the special case that σ_t, c_t, b_t are given by functions (i.e., $\sigma_t(\eta) = \sigma(\eta(t))$, etc.) can we expect $Y^{x,\varepsilon}$ to be a Markov process. We refer to $Y^{x,\varepsilon}$ as the *unrestricted process* because it moves freely in $\boldsymbol{R}^d$; there are no boundaries.

We come to the results of [7], stated in the more general setting in which we require them. The notation (x, y) will always denote the Euclidean inner product in $\boldsymbol{R}^d$, $|x| = (x, x)^{1/2}$. In addition, if $a = (a^{ij})$, $1 \leq i \leq d$, $1 \leq j \leq d$ is a positive definite matrix, let

$$(x, y)_a = \sum_{i=1}^d \sum_{j=1}^d a^{ij} x^i y^j \,, \qquad \|x\|_a = (x, x)_a^{1/2} \,.$$

With the process determined by (1) associate the functional I_T, defined as follows: for $\varphi \in C_T(\boldsymbol{R}^d)$, let $I_T(\varphi) = \infty$ if φ is not absolutely continuous, otherwise

$$(2) \qquad I_T(\varphi) = \int_0^T \|\dot{\varphi}_s - b_s(\varphi)\|^2_{a^{-1}(\varphi_s)} ds .$$

THEOREM A. *Let $Y^{x,\varepsilon}$ be the solution of* (1) *with t restricted to* $[0, T]$.

a) *$I_T: C(\mathbf{R}^d) \to \mathbf{R}^1$ is lower semi-continuous.*

b) *For $x \in \mathbf{R}^d$, $\varphi \in C_T(\mathbf{R}^d)$, $\varphi_0 = x$, $h > 0$, $\delta > 0$,*

$$P[\sup_{0 \le t \le T} |Y_t^{x,\varepsilon} - \varphi_t| < \delta] > \exp\left\{-\frac{1}{2\varepsilon^2}(I_T(\varphi) + h)\right\}$$

provided only that $\varepsilon \le \varepsilon_0 \equiv (h \wedge \delta)/C(T, K)$, where $K = I_T(\varphi) + T$ and $C(T, K)$ is a constant depending only on T and K and the underlying parameters m_1, m_2, m_3.

b′) *For any open subset G of $C_T(\mathbf{R}^d)$,*

$$\underline{\lim}_{\varepsilon \to 0} 2\varepsilon^2 \log P[Y^{x,\varepsilon} \in G] \ge -\inf\{I_T(\varphi): \varphi_0 = x, \varphi \in G\} .$$

c) *For $\varphi \in C_T(\mathbf{R}^d)$, $x \in \mathbf{R}^d$, $\delta > 0$, let*

$$i(\varphi, \delta, x) = \inf\{I_T(\psi): \psi \in C_T(\mathbf{R}^d) , \quad \psi_0 = x , \quad \sup_{0 \le t \le T} |\psi_t - \varphi_t| < \delta\} .$$

For every $\delta > 0$, and $\alpha > 0$ there exists a $\beta > 0$ such that for $\delta' \le \beta$, and $\varepsilon \le \beta$ and every $\lambda \ge 0$, $x \in D$

$$P[\sup_{\{y: |y - x| < \delta'\}} i(Y^{x,\varepsilon}, \delta, y) > \lambda] \le \exp\left\{-\frac{1}{2\varepsilon^2}(\lambda - \alpha(\lambda + T + 1))\right\} .$$

c′) *For any closed subset F of $C_T(\mathbf{R}^d)$*

$$\overline{\lim}_{\varepsilon \to 0} 2\varepsilon^2 \log P[Y^{x,\varepsilon} \in F] \le -\inf\{I_T(\varphi): \varphi_0 = x, \varphi \in F\} .$$

Proof. Not much is required, since the results are known when the coefficients are functions rather than non-anticipating functionals, and the proofs go over to the more general case. Thus a) follows from the proof of the corresponding assertion given in [7], Lemma 2.1. In Theorem 1.1 of [7] b) is proved, though the dependence of ε_0 on h and δ is not given. However, as remarked by Friedman [2] Theorem 1.1, by following the proof of [7] one obtains constants n_1 and n_2, depending only on the parameters m_1, m_2, m_3 such that

$$P[\sup_{0 \le t \le T} |Y_t^{x,\varepsilon} - \varphi_t| < \delta] \ge \frac{1}{2} \exp\left\{-\frac{1}{2\varepsilon^2}(I_T(\varphi) + n_1\delta K + 4(n_1K)^{1/2}\varepsilon)\right\}$$

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provided $\delta \geq \delta_0(\varepsilon, t) = n_2\varepsilon\sqrt{T}$. But then

$$P[\sup_{0\leq t\leq T}|Y_t^{x,\varepsilon}-\varphi_t| < \delta] \geq P[\sup_{0\leq t\leq T}|Y_t^{x,\varepsilon} - \varphi_t| < \delta_0(\varepsilon, T)]$$
$$\geq \frac{1}{2}\exp\left\{-\frac{1}{2\varepsilon^2}(I_T(\varphi) + n_1\delta_0(\varepsilon, T)K + 4(n_1K)^{1/2}\varepsilon)\right\}$$

and b) follows. Assertion b′) follows from b) immediately. Again, in the Markovian case c) is essentially Theorem 1.2 of [7]. Actually, only $i(Y^{x,\varepsilon}, \delta, x)$ is discussed, but the indicated extension is trivial. Assertion c′) follows from c) together with a).

We note that in case σ_t is a function, and $c_t \equiv b_t \equiv 0$, assertions a), b′), c′) were given by S.R.S. Varadhan [6].

1.2. The half-space. We deal now with the case of the half-space with normal reflection. Let $\boldsymbol{R}_+^d = \{x = (x^1, x^2, \cdots, x^d) : x^1 \geq 0\}$ and let e_1 be the unit vector $(1, 0, \cdots, 0)$. We wish to obtain the diffusion $X^{x,\varepsilon}$ which satisfies

$$dX_t^{x,\varepsilon} = \varepsilon(\sigma(X_t^{x,\varepsilon})dW_t + c(X_t^{x,\varepsilon})dt) + b(X_t^{x,\varepsilon})dt \tag{3}$$

on $\{x : x^1 > 0\}$ and which is instantaneously reflected according to e_1 on reaching the boundary $\{x : x^1 = 0\}$, and such that $X_0^{x,\varepsilon} = x$. We assume that the coefficients $\sigma(x), c(x), b(x)$ are defined on the closed half-space $\boldsymbol{R}_+^d$, and let $a(x) = \sigma(x)\sigma^*(x)$. The coefficients are to satisfy the conditions of boundedness, uniform Lipschitz continuity, and uniform positive definiteness corresponding to those listed after (1). We will utilize a characterization of the desired process due to S. Watanabe [8]. Introduce the following notation: $D = \boldsymbol{R}_+^d$, $\partial D = \{x : x^1 = 0\}$, γ is to be the vector valued function defined on ∂D with $\gamma(x) \equiv e_1$. Then $X^{x,\varepsilon}$ can be characterized as satisfying

$$\begin{aligned} dX_t^{x,\varepsilon} &= \varepsilon(\sigma(X_t^{x,\varepsilon})dW_t + c(X_t^{x,\varepsilon})dt) + b(X_t^{x,\varepsilon})dt \\ &\quad + \chi_{\partial D}(X_t^{x,\varepsilon})\gamma(X_t^{x,\varepsilon})d\xi_t^{x,\varepsilon}, \quad X_0^{x,\varepsilon} = 0, \quad \xi_0^{x,\varepsilon} = 0 \end{aligned} \tag{4}$$

where $(X_t^{x,\varepsilon})$ and $(\xi_t^{x,\varepsilon})$ are continuous stochastic processes, adapted to the underlying σ-fields $(\mathscr{F}_t)$ and satisfying the following conditions with probability one: $X_t^{x,\varepsilon} \in D$; $\xi_t^{x,\varepsilon}$ is non-decreasing in t and increases only during $\Delta \equiv \{t : X_t^{x,\varepsilon} \in \partial D\}$; Δ has Lebesgue measure zero. These requirements in fact determine the pair $(X^{x,\varepsilon}, \xi^{x,\varepsilon})$ uniquely, see Proposition 1 below. We present a new method of constructing the desired pair, which will be most useful for our purposes.

Define a transformation $\Gamma: C(\boldsymbol{R}^d) \to C(\boldsymbol{R}^d_+)$ (the same symbol Γ may be used when the parameter set is $[0, T]$) as follows: for $\zeta = (\zeta^1, \zeta^2, \cdots, \zeta^d) \in C(\boldsymbol{R}^d), \eta = \Gamma(\zeta)$ is defined by $\eta = (\eta^1, \eta^2, \cdots, \eta^d)$, where $\eta^i = \zeta^i$, $i = 2, 3, \cdots, d$, and $\eta^1_t = \zeta^1_t - ((\inf_{0 \le s \le t} \zeta^1_s) \wedge 0)$. Write $\Gamma_t(\zeta)$ for $(\Gamma(\zeta))_t$. Now define the transformation $\xi: C(\boldsymbol{R}^d) \to C(\boldsymbol{R})$ by $\Gamma(\zeta) - \zeta = (\xi(\zeta), 0, \cdots, 0)$ and write $\xi_t(\zeta)$ for $(\xi(\zeta))_t$.

We note some immediate consequences of the definitions. First (Γ_t) and (ξ_t) are respectively $C(\boldsymbol{R}^d_+)$ valued and non-negative real valued non-anticipating functionals. Secondly Γ and ξ are continuous, and in fact the supremum norm of $\Gamma(\zeta) - \Gamma(\eta)$ is bounded by twice the supremum norm of $\zeta - \eta$. The third, and final observation now is that always $|\Gamma_s(\eta) - \Gamma_t(\eta)| \le |\eta_s - \eta_t|$.

PROPOSITION 1. *Let $Y^{x,\varepsilon}$ be the solution to* (1), *where $\varepsilon > 0$, $x \in \boldsymbol{R}^d_+$, and σ_t, c_t, b_t are related to the coefficients of* (4) *by $\sigma_t = \sigma \circ \Gamma_t, c_t = c \circ \Gamma_t$, $b_t = b \circ \Gamma_t$. Then $X^{x,\varepsilon} = \Gamma \circ Y^{x,\varepsilon}$ and $\xi^{x,\varepsilon} = \xi \circ Y^{x,\varepsilon}$ solves* (4) *and $X^{x,\varepsilon}$ and $\xi^{x,\varepsilon}$ satisfy the conditions imposed in connection with* (4). *The pair $(X^{x,\varepsilon}, \xi^{x,\varepsilon})$ is uniquely determined by* (4) *and the associated conditions, i.e., any other pair satisfying* (4) *and the associated conditions is equal to $(X^{x,\varepsilon}, \xi^{x,\varepsilon})$ with probability one.*

Proof. From the fact that $Y^{x,\varepsilon}$ satisfies (1), it follows immediately that $(X^{x,\varepsilon}, \xi^{x,\varepsilon})$ satisfies (4). Evidently $X^{x,\varepsilon}_t \in \boldsymbol{R}^d_+$. It is clear that $\xi^{x,\varepsilon}_t$ is non-decreasing, increasing only on the set $\Delta \equiv \{t: X^{x,\varepsilon}_t \in \partial D\}$. It must be shown that Δ is a Lebesgue null set, with probability one. Note that if Y^1 is the first component of $Y^{x,\varepsilon}, \Delta = \{t: Y^1_t = (\inf_{0 \le s \le t} Y^1_s) \wedge 0\}$. Let $\hat{Y}$ be the process satisfying an equation like (1), but with $b = c = 0$. Restricting $Y^{x,\varepsilon}$ and $\hat{Y}$ to $0 \le t \le T$ these processes induce measures in $C_T(\boldsymbol{R}^d)$, namely $\mu(A) = P[Y^{x,\varepsilon} \in A]$ and $\hat{\mu}(A) = P[\hat{Y} \in A]$ and these measures are mutually absolutely continuous. Hence it suffices to prove that for the first component $\hat{Y}^1$ of $\hat{Y}$ the set $\{t: \hat{Y}^1_t = (\inf_{0 \le s \le t} \hat{Y}^1_s) \wedge 0\}$ is a Lebesgue null set. If σ is the identity matrices and $\varepsilon = 1$, $\hat{Y}^1$ is Brownian motion, and the result is known. For general $\sigma, \varepsilon > 0$, $\hat{Y}^1$ is obtained from Brownian motion by a strictly increasing time change, and so the property is preserved.

In order to show uniqueness, suppose $(\hat{X}_t, \hat{\xi}_t)$ also satisfy (4) and the associated conditions. Write $X^{x,\epsilon} = (X^1, X^2, \cdots, X^d)$, $\hat{X} = (\hat{X}^1, \hat{X}^2, \cdots, \hat{X}^d)$ and define $Y = (Y^1, Y^2, \cdots, Y^d)$, $\hat{Y} = (\hat{Y}^1, \hat{Y}^2, \cdots, \hat{Y}^d)$ by $Y^1 = X^1 - \xi^{x,\epsilon}$, $Y^2 = X^2, \cdots, Y^d = X^d$ and $\hat{Y}^1 = \hat{X}^1 - \hat{\xi}$, $\hat{Y}^2 = \hat{X}^2, \cdots, \hat{Y}^d = \hat{X}^d$. Then both Y and $\hat{Y}$ satisfy (1), and since uniqueness holds for this equation $Y = \hat{Y}$. So $X^2 = \hat{X}^2, \cdots, X^d = \hat{X}^d$ and $X^1 - \xi^{x,\epsilon} = \hat{X}^1 - \hat{\xi}$. That is to say

$$X_t^1 - \hat{X}_t^1 = \xi_t^{x,\epsilon} - \hat{\xi}_t$$

and this implies that both sides must vanish (a fact originally observed by Skorokhod [3]) because the left side can increase only when $X_t^1 = 0$, that is when $X_t^1 \leq \hat{X}_t^1$ so that since $X_0^1 - \hat{X}_0^1 = 0$ and the functions are continuous, always $X_t^1 \leq \hat{X}_t^1$; the same argument proves that always $\hat{X}_t^1 \leq X_t^1$. Uniqueness has now been proved.

Now define for $\varphi \in C_T(D)$

$$I_T^+(\varphi) = \inf \{I_T(\psi) : \varphi \in C_T(\boldsymbol{R}^d) , \qquad \Gamma(\psi) = \varphi\} .$$

Observe that if $\varphi \in C_T(D)$ and $\psi \in C_T(\boldsymbol{R}^d)$ are absolutely continuous, then $\Gamma(\psi) = \varphi$ will hold if and only if

$$\psi_t = \varphi_t - \int_0^t \chi_{\partial D}(\varphi_s) w(s) \gamma(\varphi_s) ds \tag{5}$$

for some measurable, non-negative function w. Then $I_T^+(\psi)$ is given by

$$\int_0^T \| \dot{\varphi}_s - \chi_{\partial D}(\varphi_s) w(s) \gamma(\varphi_s) - b(\varphi_s) \|_{a^{-1}(\varphi_s)}^2 ds \tag{6}$$

and this is minimized, under the restriction $w(s) \geq 0$ by taking

$$w(s) = \frac{(\dot{\varphi}_s - b(\varphi_s)\gamma, (\varphi_s))_{a^{-1}(\varphi_s)}}{\| \gamma(\varphi_s) \|_{a^{-1}(\varphi_s)}} \vee 0 \tag{7}$$

and inserting this in (6) gives an explicit expression for $I_T^+(\varphi)$.

PROPOSITION 2. *The assertions of Theorem A hold for the solution $X^{x,\epsilon}$ of (4) if I_T^+ is used in place of I_T and D and $C_T(D)$ take the place of $\boldsymbol{R}^d$ and $C_T(\boldsymbol{R}^d)$ respectively.*

Proof. To prove a) consider a sequence $\varphi^{(n)}$ of trajectories in $C_T(D)$ converging uniformly to φ, and satisfying $I_T^+(\varphi^{(n)}) \leq \beta < \infty$, $n = 1, 2, \cdots$. It must be shown that $I_T^+(\varphi) \leq \beta$. It follows from (5), (6), and (7) that there exist $\psi^{(n)} \in C_T(\boldsymbol{R}^d)$ such that $\Gamma(\psi^{(n)}) = \varphi^{(n)}$ and $I_T(\psi^{(n)}) = I_T^+(\varphi^{(n)}) \leq \beta$. This implies that there exists $\beta' < \infty$ such that $\int_0^T |\dot{\psi}_s^{(n)}|^2 \, dx \leq \beta'$ and it follows that $(\psi^{(n)})$ must be an equicontinuous, uniformly bounded sequence of functions. There exists then a subsequence converging uniformly to a limit function ψ. By the lower semi-continuity of $I_T, I_T(\psi) \leq \beta$. By the continuity of Γ, $\Gamma(\psi) = \varphi$, so that $I_T^+(\varphi) \leq I_T(\psi) \leq \beta$.

All the remaining parts of Theorem A can be handled without any complications by using the definition of I_T^+ and recalling the simple properties of Γ noted just before Proposition 1.

1.3. Localization. Let D_0 be a connected open subset of $\boldsymbol{R}^d$ with closure D, and suppose that D has smooth boundary ∂D on which is defined a vector field γ pointing into the interior D_0. We want to construct $X^{x,\epsilon}$ and $\xi^{x,\epsilon}$ so that (4) holds, and the conditions stated after (4) are satisfied. $X^{x,\epsilon}$ is a diffusion with instataneous oblique reflection determined by γ at the boundary. As far as $X^{x,\epsilon}$ is concerned only the direction of γ, and not its magnitude is relevant; however, as is clear from (4), the magnitude of γ will affect $\xi^{x,\epsilon}$. The process $\xi^{x,\epsilon}$ is called the *local time on the boundary*.

The construction problem is reduced to the corresponding problem in the half-space, with normal reflection, by means of local coordinate systems. We will assume that coordinate systems satisfying certain conditions exist; this will be the case if ∂D is smooth, $\gamma(x)$ varies smoothly, and $\gamma(x)$ is uniformly bounded away from vectors tangant to the boundary at x.

We turn to details. We consider $\boldsymbol{R}^d$ endowed with a fixed Euclidean coordinate system, so that every point in $\boldsymbol{R}^d$, hence every point in D can be identified with its Euclidean coordinates, $x = (x^1, x^2, \cdots, x^d)$. Now let $\mathscr{U} = \{U^0, U^1, \cdots\}$ be a countable or finite family of relatively open subsets of D which cover D_0 and such that each $U \in \mathscr{U}$ is associated with a coordinate system, that is a map $u\colon U \to \boldsymbol{R}^d$, giving each point $x \in U$ coordinates $u(x) = (u^1(x), u^2(x), \cdots, u^d(x))$. The following assumptions are to hold:

(i) $U^0 \subseteq D_0$ and the coordinates corresponding to U^0 are just the original Euclidean coordinates. If $U = U^k$, $k > 0$, the coordinate mapping $u\colon U \to \boldsymbol{R}^d$ is one-one and twice continuously differentiable. In this case U intersects ∂D and

$$U \cap \partial D = \{x \in U : u^1(x) = 0\}\,, \qquad U \cap D_0 = \{x \in U : u^1(x) > 0\}\,.$$

(ii) There exists a positive constant ρ_0, and for each $x \in U$ a non-negative integer $k(x)$ such that all points in D within a distance of ρ_0 or less from x belong to $U^{k(x)}$.

Suppose (4) holds. If $X_t^{x,\epsilon} \in U^k$, and if u is the coordinate mapping associated with U^k, then $u(X_t^{x,\epsilon})$ will, according to Itô's formula satisfy an equation like (4), with new coefficients σ_k, c_k, b_k in place of σ, c, b and a vector field γ_k on the boundary of $\boldsymbol{R}_+^d$ in place of γ; $\xi^{x,\epsilon}$ will be unchanged.

(iii) There exist positive constants m_1, m_2, m_3 such that for all k the coefficients σ_k, c_k, b_k satisfy the conditions after (1).

(iv) $\gamma_k(0, y^2, \cdots, y^k) = e_1$, for $(0, y^2, \cdots, y^d)$ in the range of u. (This assumption is equivalent to $(\nabla u^i, \gamma) = \delta_{1i}$, where ∇ is the gradient.)

(v) There exists a positive constant m_5 so that for every U^k the associated coordinate mapping u satisfies

$$\frac{1}{m_5}|u(x) - u(y)| \le |x - y| \le m_5|u(x) - u(y)|$$

for all x and y in U^k.

If U is a coordinate neighborhood with associated coordinate mapping u, then $u: U \to \boldsymbol{R}_+^d$. Also if $x \in U$, and $\beta = (\beta^1, \beta^2, \cdots, \beta^d)$ is a tangent vector acting on x, u makes β correspond in an obvious way to a tangent vector $\hat\beta = (\hat\beta^1, \cdots, \hat\beta^d)$ acting at $u(x)$, and β and $\hat\beta$ are related by $\hat\beta^i = \sum_{j=1}^d (du^i/dx^j)\beta^j$. For $x \in \partial D$ our assumptions guarantee that $u(x) \in \partial\boldsymbol{R}_+^d$; if β is tangent to ∂D, $\hat\beta^1 = 0$, and if $\beta = \gamma, \hat\beta = e_1$.

The construction of the desired process is now straight forward. The process will first be constructed up to the time S_1 of leaving the open ball of radius ρ_0 around x_0 then from S_1 up to the least time bigger than S_2 of leaving the open ball around $X_{S_1}^{x,\epsilon}$, etc. At the first stage use the coordinate patch $U^{k(x)}$. If $k(x) = 0$, the boundary plays no role, and the process is immediately constructed by the usual technique of successive iteration. If $k(x) > 0$, the coordinate mapping u associated with $U^{k(x)}$ changes the problem to a problem in the half-space with normal reflection. This problem was solved in the previous section, and we obtain a process in the half-space with normal reflection; applying the mapping inverse to u gives us $X_t^{x,\epsilon}$, $0 \le t \le S_1$. For this process (4) will hold, with $\xi_t^{x,\epsilon}$ being identical to the local time on the boundary for the process in the half-space. Iterating this procedure leads to a construction of $(X_t^{x,\varepsilon}, \xi_t^{x,\varepsilon})$ on $0 \leq t < \infty$.

PROPOSITION 3. *There is a unique pair of processes* $(X^{x,\varepsilon}, \xi^{x,\varepsilon}))$ *satisfying* (4) *and the associated conditions, in the sense that any other such pair equals* $(X^{x,\varepsilon}, \xi^{x,\varepsilon})$ *with probability one.*

Proof. An explicit construction of the solution has just been given. Since uniqueness is a local property, this follows from Proposition 2 by using appropriate local coordinates.

For $\eta \in C(D)$, $t \geq 0$, let $\theta_t \eta \in C(D)$ be defined by $(\theta_t \eta)(s) = \eta_{t+s}$. Since $X^{x,\varepsilon}(\omega) \in C(D)$, the notation $\theta_t X^{x,\varepsilon}(\omega)$ is defined; and naturally $\theta_t X^{x,\varepsilon}$ is a function $\Omega \to C(D)$ with value $\theta_t X^{x,\varepsilon}(\omega)$ for $\omega \in \Omega$. More generally if S is a random variable $\theta_S X^{x,\varepsilon}$ is the function which for $\omega \in \Omega$ assumes the value $\theta_{S(\omega)} X^{x,\varepsilon}(\omega)$ in $C(D)$; the notation will be used only when S is a stopping time with respect to $(\mathscr{F}_t)$.

It is well known that the processes $X^{x,\varepsilon}$ constitute, for fixed ε, a diffusion process. In particular they have the *strong Markov property*, a convenient version of which we now state: let $\alpha(\eta, t)$ be a bounded measurable function defined on the product of $(C(D), \mathscr{C})$ with $([0, \infty)$, Borel sets), with values in $\boldsymbol{R}^1$. Let $g(x, t) = E[\alpha(X^{x,\varepsilon}, t)]$. Then for any stopping time T with respect to $\mathscr{F}_t, E[\alpha(\theta_T X^{x,\varepsilon}, T|_T^{\sigma})] = g(X_T)$.

1.4. Asymptotic inequalities. The reflecting diffusion process has been constructed by localization, which reduced the problem to the half-space problem. The results in the half-space also lead to the correct form of I^+. For $\varphi \in C_T(D), I_T^+(\varphi)$ is to be $+\infty$ if φ is not absolutely continuous, and otherwise equal to the expression (6), with $w(s)$ given by (7). We now wish to extend the results proved in sub-section 1.2 for $D = \boldsymbol{R}_+^d, \gamma \equiv e_1$, to general D and γ.

THEOREM 1. *Under the assumptions of sub-section* 1.3, *the assertions of Theorem A hold for the* $X^{x,\varepsilon}$ *of* (4) *if* I_T^+ *is used in place of* I_T, *and* D *and* $C_T(D)$ *take the place of* $\boldsymbol{R}^d$ *and* $C_T(\boldsymbol{R}^d)$ *respectively.*

Proof. By introducing appropriate local coordinates, the problems are transformed locally to the half-space with normal reflection, and Proposition 2 applies. It is then a matter of patching together local results. For this purpose the following *additive property* of I^+ is crucial: if $0 \leq S \leq T$ and $\varphi \in C_T(D)$, then $I_T^+(\varphi) = I_S^+(\varphi) + I_{T-S}^+(\theta_S \varphi)$.

Let $S_0(\varphi) = 0$ and let $S_1(\varphi) = \inf\{t: |\varphi_t - \varphi_0| \geq \rho_0\}$, $S_2(\varphi) = \inf\{t \geq S_1: |\varphi_t - \varphi_{S_1(\varphi)}| \geq \rho_0\}$, etc. with $S_i(\varphi) = \infty$, if not otherwise defined. Here ρ_0 is as in (ii) of sub-section 1.3.

To prove a), let $\varphi \in C_T(D)$ with $S_{m-1}(\varphi) \leq T < S_m(\varphi)$ and suppose $\varphi^{(n)} \in C_T(D)$ so that $\varphi^{(n)} \to \varphi$ uniformly. For convenience, temporarily redefine $S_m(\varphi)$ so as to be equal to T. We can assume without loss of generality that $\varphi_u^{(n)} \in U^{k(\varphi_{S_{j-1}(\varphi)})}$, $S_{j-1}(\varphi) \leq u \leq S_j(\varphi)$, for $j = 1, 2, \cdots, m$. Thus by the half-space result

$$\underline{\lim}_n I^+_{S_1(\theta_{S_{j-1}}(\varphi))}(\theta_{S_{j-1}}(\varphi^{(n)})) \geq I^+_{S_1(\theta_{S_{j-1}}(\varphi))}(\theta_{S_{j-i}}(\varphi)) .$$

which by the additive property of I_T^+ and the fact that $m < \infty$ is enough to conclude

$$\underline{\lim}_n I_T^+(\varphi) \geq I_T^+(\varphi) .$$

For b) consider $\varphi \in C_T(D)$, with $S_{m-1}(\varphi) \leq T < S_m(\varphi)$. For $m = 1$, the result follows immediately from b) of Proposition 2, by using suitable local coordinates. For $m > 1$ one must patch local results. For details see the proof of Theorem 1.1 in [7], where all the work is done on a Riemann manifold. Assertion b′) follows from b).

We turn to the proof of c). The first step will be to prove by induction on m that for every positive integer m and positive δ and α, there exists a positive β so that for $\delta' \leq \beta$ and $\varepsilon \leq \beta$ and any $t > 0$

$$(8)\qquad \begin{aligned} &P(\sup_{\{y:|y-x|<\delta'\}} i(X^{x,\varepsilon}, \delta, y, S_m(X^{x,\varepsilon}) \wedge T) > \lambda) \\ &\qquad \leqq \exp\left\{-\frac{1}{2\varepsilon^2}(\lambda - \alpha(\lambda + m(T+1)))\right\} \end{aligned}$$

where

$$i(\varphi, \delta, y, t) = \inf\{I_t^+(\psi): \psi \in C_t(D)\ ,\quad \psi_0 = y \quad \text{and} \quad \sup_{0 \leq u \leq t} |\psi_u - \phi_u| < \delta\} .$$

For $m = 1$, this is again immediate by Proposition 2. For the induction step from m to $m + 1$, consider $\varphi \in C_T(D)$ and let $S = S_m(\varphi) \wedge T$. It follows from the definition of i and the additive property of I^+ that for $0 \leq \delta' \leq \delta$, $0 \leq \delta'' \leq \delta'$,

$$\begin{aligned} \sup_{\{y:|\varphi_0 - y| \leq \delta''\}} i(\varphi, \delta, y, S_{m+1}(\varphi) \wedge T) &\leq \sup_{\{y:|\varphi_0 - y| \leq \delta''\}} i(\varphi, \delta', y, S_m(\varphi) \wedge T) \\ &+ \sup_{\{z:|(\theta_S\varphi)_0 - z| \leq \delta'\}} i(\theta_S\varphi, \delta, z, S_1(\theta_S\varphi) \wedge T) . \end{aligned}$$

Now in this last relation, replace φ by $X^{x,\varepsilon}(\omega)$, and write the resulting inequality as

$$Z(X^{x,\varepsilon}) \leq Z_1(X^{x,\varepsilon}) + Z_2(\theta_S X^{x,\varepsilon}) .$$

Thus by the induction hypothesis, for n any positive integer

$$\begin{aligned} P[Z(X^{x,\varepsilon}) > \lambda] &\leq \sum_{k=1}^{n} \int_{((k-1/n)\lambda \leq Z_1(X^{x,\varepsilon}) \leq (k/n)\lambda)} P\Big[Z_2(\theta_S X^{x,\varepsilon}) \geq \frac{n-k}{n}\lambda \,|\, \mathscr{F}_S\Big] dP \\ &\quad + \int_{(Z_1(X^{x,\varepsilon}) > \lambda)} P[Z_2(\theta_S X^{x,\varepsilon}) > 0 \,|\, \mathscr{F}_S] dP \\ &\leq \sum_{k=1}^{n} \exp\Big\{-\frac{1}{2\varepsilon^2}\Big(\frac{k-1}{n}\lambda + \frac{n-k}{n}\lambda - \alpha(\lambda + (m+1)(T+1))\Big)\Big\} \\ &\quad + \exp\Big\{-\frac{1}{2\varepsilon^2}(\lambda - \alpha(\lambda + m(T+1)))\Big\} \\ &\leq (n+1)\exp\Big\{-\frac{1}{2\varepsilon^2}\Big(\lambda - \alpha(\lambda + (m+1)(T+1)) - \frac{\lambda}{n}\Big)\Big\} , \end{aligned}$$

the inequality between the extremes holding provided δ'' and ε are sufficiently small, depending on δ and α, but independent of λ and n. Now (8) follows easily.

Let $T_j^{x,\varepsilon} = S_j(X^{x,\varepsilon}) - S_{j-1}(X^{x,\varepsilon})$, $j = 1, 2, \cdots$. We will show that there exists a positive constant ε_0 and a positive integer m_0 depending upon T and a positive constant c_0 so that for $\varepsilon \leq \varepsilon_0$ and $m \geq m_0$.

$$(9) \qquad P[T_i^{x,\varepsilon} < T/m \text{ for some } i \leq m] \leq \exp\Big\{-\frac{c_0}{2\varepsilon^2}\frac{m}{T}\Big\}$$

Since $P[S_m < T] \leq P[T_i^{x,\varepsilon} < T/m$ for some $i \leq m]$ c) follows from (8) and (9).

First, we show there is an integer m_0 depending upon T and a positive constant c_1 so that for $m \geq m_0$ and $\varepsilon \leq 1$,

$$(10) \qquad P[T_i^{x,\varepsilon} \leq T/m \,|\, \mathscr{F}_{S_{i-1}(X^{x,\varepsilon})}] \leq \exp\Big\{-\frac{c_1}{2\varepsilon^2}\frac{m}{T}\Big\} .$$

Because of the strong Markov property, (10) need only be shown for $i = 1$. Again introduce local coordinates u so that $u(X^{x,\varepsilon})$ is the process in the half-space with normal reflection. Let $\rho_1 = \rho_0/m_5$ with m_5 as in v) of sub-section 1.3. It suffices to prove that for $T' = \inf\{t : |u(X^{x,\varepsilon}) - u(x)| \geq \rho_1\}$, if $m \geq m_0$ and $\varepsilon \leq 1$

$$P[T' \leq T/m] \leq \exp\left\{-\frac{c_1}{2\varepsilon^2}\frac{m}{T}\right\}.$$

Now by our construction $u(X^{x,\varepsilon}) = \Gamma(Y^{x,\varepsilon})$, where $Y^{x,\varepsilon}$ is the associated unrestricted process. Let

$$T'' = \inf\{t: |Y^{x,\varepsilon} - u(x)| \geq \rho_1\}.$$

It now is enough, keeping in mind the third property of the Γ-transformation noted before Proposition 1, to show for $m \geq m_0$ and $\varepsilon \leq 1$

(11) $$P[T'' \leq T/m] \leq \exp\left\{-\frac{c_1}{2\varepsilon^2}\frac{m}{T}\right\}.$$

From (1) we have

$$Y_t^{u(x),\varepsilon} = \varepsilon\int_0^t \sigma_s \circ Y^{u(x),\varepsilon} dW(s) + \int_0^t (\varepsilon c_s \circ Y^{u(x),\varepsilon} + b_s \circ Y^{u(x),\varepsilon}) ds .$$

For $\varepsilon \leq 1$, the second integral on the right is bounded, uniformly for $t \leq T/m$, in absolute value by $2m_1T/m$. Selecting $m_0 > 4m_1T/\rho_1$, the left hand side can not exceed ρ_1 in absolute value for $t < T/m$, $m > m_0$ unless the first term on the right exceeds $\rho_1/2$. For that term, we can apply a known bound for stochastic integrals:

$$P\left[\sup_{0\leq t\leq T/m}\left|\varepsilon\int_0^t \sigma_s \circ Y^{u(x),\varepsilon} dW_s\right| > \lambda\right] \leq \exp\left\{-\frac{\lambda^2 m}{2\varepsilon^2 m_3 c_4 T}\right\}$$

where c_4 is a constant depending only on the dimension d of the range space; see for example, Theorem 2.1 of [4]. Now putting $\lambda = \rho_0/2$ leads to (11), hence (10). From (10) follows

$$P[T_i \geq T/m \text{ for all } i \leq m] \geq \left(1 - \exp\left\{-\frac{c_1}{2\varepsilon^2}\frac{m}{T}\right\}\right)^m$$

if $m \geq m_0$ and $\varepsilon \leq 1$. Since

$$\left(1 - \exp\left\{-\frac{c_1}{2\varepsilon^2}\frac{m}{T}\right\}\right)^m \geq 1 - m\exp\left\{-\frac{c_1 m}{2\varepsilon^2 T}\right\},$$

one has

$$P[T_i^{x,\varepsilon} < T/m \text{ for some } i \leq m] \leq \exp\left\{-\frac{m}{2\varepsilon^2}\left(\frac{c_1}{T} - 2\varepsilon^2\right)\right\}$$

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and (9) follow for $m \geq m_0$ and $\varepsilon < \varepsilon_0$; m_0 and ε_0 are positive constants depending upon T.

Finally, c′) follows from c) together with a).

1.5. The deterministic limit process. Setting $\varepsilon = 0$ in (4) leads to the equation

$$dX_t^{x,0} = b(X_t^{x,0})dt + \chi_{\partial D}(X_t^{x,0})\gamma(X_t^{x,0})d\xi_t^{x,0}\,, \qquad X_t^{x,0} = x\,. \tag{12}$$

Again we require that $X^{x,0}$ and $\xi^{x,0}$ be continuous stochastic processes adapted to $(\mathscr{F}_t)$, $X_t^{x,0} \in D$, and $\xi^{x,0}$ is an increasing process, with $\xi_0^{x,0} = 0$, and increasing only on $\Delta \equiv \{t\colon X_t^{x,0} \in \partial D\}$; however, we do not now demand that Δ have Lebesgue measure zero. The proof of Proposition 1 serves also to prove the existence of a unique pair $(X^{x,0}, \xi^{x,0})$ satisfying (12) and the associated conditions.

Consider for a moment $(X^{x,0}, \xi^{x,0})$ in the special case of the half-space with normal reflection. Let X^1 denote the first component of $X^{x,0}$, b^1 the first component of b. Let $T_0 = \inf\{t\colon X_t^1 = 0\}$. Then (12) gives for $t \geq T_0$

$$X_t^1 - X_{T_0}^1 = \int_{T_0}^t \chi_{D_0}(X_s^{x,0})b^1(X_s^{x,0})ds + \int_{T_0}^t \chi_{\partial D}(X_s^{x,0})b^1(X_s^{x,0})ds + \xi_t^{x,0}$$

If $t \in \Delta$, the left side vanishes. On the other hand the first term on the right must also vanish: this becomes clear if one notes first that the integral is not changed if one integrates over $[T_0, t]\backslash\Delta$ instead of $[T_0, t]$, and $[T_0, t]\backslash\Delta$ is the union of a finite or countable number of open intervals, and integrating over any one of these yields zero. We obtain then

$$\xi_t^{x,0} = -\int_{T_0}^t \chi_{\partial D}(X_s^{x,0})b^1(X_s^{x,0})ds = -\int_0^t \chi_{\partial D}(X_s^{x,0})b^1(X_s^{x,0})ds\,.$$

This has been justified for $t \in \Delta, t \geq T_0$. For $t \leq T_0$ all term vanish, while for $t > T$ each term remains unchanged if t is replaced by $\max\{s\colon s \leq t, s \in \Delta\}$. Finally, since $\xi_t^{x,0}$ is non-decreasing we may write for $t \geq 0$,

$$\xi_t^{x,0} = -\int_0^t \chi_{\partial D}(X_s^{x,0})(b^1(X_s^{x,0}) \wedge 0)ds\,. \tag{13}$$

Return now to the case of general D and γ. For $x \in \partial D$, denote by

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$\text{comp}_{\gamma,D}\, b(x)$ the unique number such that $b(x) - \text{comp}_{\gamma,D}\, b(x)\gamma(x)$ is a vector tangent to ∂D at x. Then

$$\xi_t^{x,0} = -\int_0^t \chi_{\partial D}(X_s^{x,0})\,(\text{comp}_{\gamma,D}\, b(X_s^{x,0}) \wedge 0)ds\ . \tag{14}$$

For in the case of the half-space with normal reflection (13) and (14) coincide, and in general (14) is reduced to (13) by introducing suitable local coordinates. Substituting (14) in (12) gives

$$\begin{aligned} dX_t^{x,0} &= (b(X_t^{x,0}) - \chi_{\partial D}(X_t^{x,0})\,(\text{comp}_{\gamma,D}\,(b(X_t^{x,0}) \wedge 0)\gamma(X_t^{x,0}))dt\ , \\ &\qquad X_0^{x,0} = x\ . \end{aligned} \tag{15}$$

PROPOSITION 4. *If* $(X^{x,0}, \xi^{x,0})$ *is the unique solution of* (12) *satisfying the associated condition, then* (14) *holds, and* $X^{x,0}$ *restricted to* $0 \le t \le T$ *is the unique* $\varphi \in C_T(D)$ *satisfying* $\varphi_0 = x, I_T^+(\varphi) = 0$.

Proof. Relation (14) has already been proved. Now let $\varphi \in C_T(D)$, $\varphi_0 = x$. From the formulas (6) and (7) for $I_T^+(\varphi)$ we see that a necessary and sufficient condition for $I_T^+(\varphi) = 0$ is that φ is absolutely continuous with derivative $\dot\varphi_t$ satisfying for almost every t.

$$\dot\varphi_t = b(\varphi_t) - \chi_{\partial D}(\varphi_t) w(t)\gamma(\varphi_t)\ .$$

Then letting $\hat\xi_t$ be the integral with respect to t from zero to t of the second term on the right, $(\varphi, \hat\xi)$ given a solution to (12) and the associated conditions. So $\varphi = X^{x,0}$ is the only possibility; and $I_T^+(X^{x,0}) = 0$ by (15).

We will obtain the convergence of $(X^{x,\varepsilon}, \xi^{x,\varepsilon})$ to $(X^{x,0}, \xi^{x,0})$ as $\varepsilon \to 0$. First we require a lemma.

LEMMA. *For* $\delta > 0, T > 0$ *there exists* $k(\delta, T) > 0$ *such that for all* $\varphi \in C_T(D)$, *if* $\varphi_0 = x$ *and* $\sup_{0 \le t \le T} |X_t^{x,0} - \varphi_t| \ge \delta$ *then* $I_T^+(\varphi) \ge k(\delta, T)$.

Proof. First note that the dependence of $X_t^{x,0}$ on the initial point x is continuous, in fact Lipschitz continuous. This can be seen by reducing the situation to the half-space with normal reflection by localization, and this case in turn is reduced to the unresticted system $Y^{x,0}$ by the Γ-transformation construction; for the $Y^{x,0}$ system the assertion is immediate from the uniform Lipschitz continuity of b_t, and Gronwall's inequality.

Next we note that if the lemma is proved for $0 \le T \le T_1$ for some positive T_1, it follows in general. Let us show how to extend the lemma to the range $0 \leq T \leq 2T_1$. We may of course assume that $k(\delta, T)$ is non-increasing in T, non-decreasing in δ. Consider then $\varphi \in C_T(D), \varphi_0 = x$ where $T_1 < T \leq 2T_1$. Suppose $\varphi_{t_0} - X_{t_0}^{x,0} \geq \delta, T_1 < t_0 \leq T$. By the initial paragraph of this proof we can find $\alpha(\delta) > 0$ so that $|X_t^{y,0} - X_t^{z,0}| < \delta/2$ for $|y - z| < \alpha(\delta)$, $0 \leq t \leq T_1$. Let us apply this with $y = X_{T_1}^{x,0}, z = \varphi_{T_1}$. If $|y - z| \geq \alpha(\delta)$ then, $I_T^+(\varphi) \geq k(\alpha(\delta), T_1)$. On the other hand if $|y - z| < \alpha(\delta)$, then $|\theta_{T_1}\varphi_{t_0} - X_{t_0}^{z,0}| > \delta/2$ and so $I_T^+(\varphi) \geq I_{T-T_1}^+(\theta_{T_1}\varphi) \geq k(\delta/2, T - T_1) \geq k(\delta/2, T_1)$. So we may set $k(\delta, T) = k(\alpha(\delta) \wedge \delta, T_1)$. By iterating the result is extended to $0 \leq T \leq mT_1$, $m = 1, 2, \cdots$.

Finally we prove the result for $0 \leq T \leq T_1$, where $T_1 = 1/(2m_2)$. Again, by introducing local coordinates the problem can be reduced to the half-space with normal reflection, and then, by the construction of this process it is reduced to the unrestricted process $Y^{x,\varepsilon}$. Suppose then that $\varphi \in C_T(\boldsymbol{R}^d), \varphi_0 = x$, $T \leq T_1$, and that $|\varphi_{t_0} - Y_{t_0}^{x,0}| = \delta$, $t_0 \leq T$. Define $\dot{\psi}_t$ by $\dot{\varphi}_t = b_t(\varphi) + \dot{\psi}_t$ and obtain

$$\int_0^{t_0} \dot{\varphi}_t dt = \int_0^{t_0} b_s(Y^{x,0}) ds + \int_0^{t_0} (b_s(\varphi) - b_s(Y^{x,0})) ds + \int_0^{t_0} \dot{\psi}_s ds .$$

The difference between the left side and the first term on the right is δ. The second term on the right is bounded by $m_2 t_0 \delta$ so that

$$\left| \int_0^{t_0} \dot{\psi}_a ds \right| \geq \delta[1 - t_0 m_2] \geq \delta/2 .$$

Finally

$$I_T(\varphi) = \int_0^T \|\dot{\psi}_s\|_{a_s^{-1}(\varphi)} ds \geq \frac{1}{m_3} \int_0^{t_0} |\dot{\psi}_s|^2 ds \geq \frac{1}{m_3 t_0} \left(\int_0^t |\dot{\psi}_s| ds \right)^2$$
$$\geq \frac{1}{m_3 t_0} \left| \int_0^{t_0} \dot{\psi}_s ds \right|^2$$

so we can take $k(\delta, T) = \delta^2/(m_3 T_1 4)$.

Now we obtain the convergence of the process $X^{x,\varepsilon}$ to $X^{x,0}$ as $\varepsilon \to 0$, and some consequences of this fact.

THEOREM 2. a) *For every $\delta > 0$ there exists an $\alpha > 0$ such that*

$$\overline{\lim_{\varepsilon \to 0}}\, 2\varepsilon^2 \log P[\sup_{0 \leq s \leq T} |X_s^{x,\varepsilon} - X_s^{x,0}| > \delta] \leq \alpha .$$

b) $\sup_{0 \leq s \leq T} |X_s^{x,\varepsilon} - X_s^{x,0}| \to 0$ *in* $L_p(P)$ *as* $\varepsilon \to 0$, $1 \leq p < \infty$, *uniformly for* $x \in D$.

c) $\sup_{0\le s\le T}|\xi_s^{x,\varepsilon}-\xi_s^{x,0}|\to 0$ *in* $L_p(P)$ *as* $\varepsilon\to 0$, $1\le p<\infty$, *uniformly for* $x\in D$.

d) *For* $g\in D\times \boldsymbol{R}_+^1\times \boldsymbol{R}_+^1\to \boldsymbol{R}^1$ *bounded and uniformly continuous* $\int_0^T g(X_s^{x,\varepsilon},\xi_s^{x,\varepsilon},s)d\xi_s^{x,\varepsilon}\to\int_0^T g(X_s^{x,0},\xi_s^{x,0},s)d\xi_s^{x,0}$ *in* $L_p(P)$ *as* $\varepsilon\to 0$. $1\le p<\infty$, *uniformly for* $x\in D$.

Proof. We will be considering $X_t^{x,\varepsilon}$ on $0\le t\le T$ only. Let $F^{x,\delta} = \{\varphi\in C_T(D)\colon \varphi_0=x,\ \sup|\varphi_s-X_s^{x,0}|\ge\delta\}$. It follows from the lemma that for $\delta>0$ there exists $\alpha(\delta)>0$ such that $\varphi\in F^{x,\delta}$ implies $I_T^+(\varphi)\ge\alpha(\delta)$. Note that if $\varphi\in F^{x,\delta}$ and $\psi\in C_T(D)$ satisfies $\psi_0=x$, $\sup_{0\le t\le T}|\psi_t-\varphi_t|<\delta/2$ then $\psi\in F^{x,\delta/2}$, hence $I_T^+(\psi)\ge\alpha(\delta/2)$. So Theorem 1, part c) is available to show that $P[X^{x,\varepsilon}\in F^{x,\delta}]\to 0$ as $\varepsilon\to 0$, exponentially fast, and uniformly in x.

Since a) implies $\sup_{0\le s\le T}|X_s^{x,\varepsilon}-X_s^{x,0}|\to 0$ in probability, to conclude b) it is only necessary to prove that the p^{th} power of this quantity is integrable uniformly with respect to ε, $0\le\varepsilon\le\varepsilon_0$, and $x\in D$, the integration being with respect to dP. Since $X^{x,0}$ is deterministic it will suffice to show that

$$E[(\sup_{0\le s\le T}|X_s^{x,\varepsilon}-x|)^p]<k_p\,,\qquad 0\le\varepsilon\le\varepsilon_0$$

for some finite k_p. Let $S_i(X^{x,\varepsilon})$ have the same meaning as in the proof of c) of Theorem 1, and set $S_i=S_i(X^{x,\varepsilon})\wedge T$. Let $\nu=\nu(X^{x,\varepsilon}) = \min\{i\colon S_i=T\}$. It follows from (9), and the sentence succeeding it, that for $m\ge m_0$, $\varepsilon\le\varepsilon_0$

$$P[\nu(X^{x,\varepsilon})>m]\le\exp\left\{-\frac{c_0}{2\varepsilon^2}\frac{m}{T}\right\}. \tag{16}$$

By construction

$$\sup_{S_{i-1}\le s\le S_i}|X_s^{x,\varepsilon}-X_{S_{i-1}}^{x,\varepsilon}|=|X_{S_i}^{x,\varepsilon}-X_{S_{i-1}}^{x,\varepsilon}|=\rho_0\,,\qquad 1\le i<\nu$$

and still

$$\sup_{S_{i-1}\le s\le S_i}|X_s^{x,\varepsilon}-X_{S_{i-1}}^{x,\varepsilon}|\le\rho_0\qquad\text{for } i=\nu\,.$$

Thus

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$$\sup_{0\le s\le T}|X_s^{x,\varepsilon}-x|\le\sum_{i=1}^{\nu}\sup_{S_{i-1}\le s\le S_i}|X_s^{x,\varepsilon}-X_{S_i}^{x,\varepsilon}|\le\nu(X^{x,\varepsilon})\rho_0\ .$$

It follows, using (16) that the p^{th} moment of the left number is bounded, uniformly for $0\le\varepsilon\le\varepsilon_0$. So b) is established.

For part c) consider the half-space with normal reflection. Since, for $\varepsilon>0, \xi^{x,\varepsilon}$ is given explicitly as $\xi(Y^{x,\varepsilon})$, where $Y^{x,\varepsilon}$ solves (1), and since b) is also true for $Y^{x,\varepsilon}$ and $Y^{x,0}$ in place of $X^{x,\varepsilon}$ and $X^{x,0}$, it follows by the continuity of ξ that c) hold in this case. Defining S_i as in the preceding proof of b), introduction of local coordinates will serve to prove $\sup_{0\le u\le S_i}|\xi_u^{x,\varepsilon}-\xi_u^{x,0}|\to 0$ in probability as $\varepsilon\to 0$. By (16), $\sup_{0\le t\le T}|\xi_u^{x,\varepsilon}-\xi_u^{x,0}|\to 0$ in probability as $\varepsilon\to 0$, and so as in part b) it is enough to show that $E[|\sup_{0\le s\le T}\xi_s^{x,\varepsilon}|^p]=E[|\xi_T^{x,\varepsilon}|^p]$ can be bounded uniformly in $\varepsilon, 0\le\varepsilon\le\varepsilon_0$, and $x\in D$. From the half-space case and the strong Markov property, we easily obtain

$$E[(\xi_{S_i}^{x,\varepsilon}-\xi_{S_{i-1}}^{x,\varepsilon})^p|\mathscr{F}_{S_{i-1}}]\le k_p$$

for some finite k_p, $0\le\varepsilon\le 1$. Now

$$\xi_T^{x,\varepsilon}=\sum_{n=1}^{\infty}\chi_{[\nu=n]}\sum_{i=1}^{n}(\xi_{S_i}^{x,\varepsilon}-\xi_{S_{i-1}}^{x,\varepsilon})$$

and so for p any positive integer,

$$\begin{aligned}E[(\xi_T^{x,\varepsilon})^p]&=E\left[\sum_{n=1}^{\infty}\chi_{[\nu=n]}\left(\sum_{i=1}^{n}(\xi_{S_i}^{x,\varepsilon}-\xi_{S_{i-1}}^{x,\varepsilon})\right)^p\right]\\&\le\sum_{n=1}^{\infty}(P[\nu=n])^{1/2}\left(E\left[\left(\sum_{i=1}^{n}(\xi_{S_i}^{x,\varepsilon}-\xi_{S_{i-1}}^{x,\varepsilon})\right)^{2p}\right]\right)^{1/2}\\&\le\sum_{n=1}^{\infty}(P[\nu=n])^{1/2}\left(n^{2p}\sum_{i=1}^{n}E\left[\left((\xi_{S_i}^{x,\varepsilon}-\xi_{S_{i-1}}^{x,\varepsilon})^{2p}\right]\right)^{1/2}\\&\le\sum_{n=1}^{\infty}(P[\nu=n])^{1/2}(n^{2p}nk_{2p})^{1/2}\end{aligned}$$

and again by (16) this converges uniformly for $0\le\varepsilon\le\varepsilon_0, x\in D$.

Lastly for part d), consider g bounded and uniformly continuous, and

$$\begin{aligned}&\int_0^T g(X_s^{x,\varepsilon},\xi_s^{x,\varepsilon},s)d\xi_s^{x,\varepsilon}-\int_0^T g(X_s^{x,0},\xi_s^{x,0},s)d\xi_s^{x,0}\\&\quad=\int_0^T(g(X_s^{x,\varepsilon}\xi_s^{x,\varepsilon},s)-g(X_s^{x,0},\xi_s^{x,0},s))d\xi_s^{x,\varepsilon}\end{aligned}$$

$$+\left(\int_0^T g(X_s^{x,0}, \xi_s^{x,0}, s)d\xi_s^{x,\varepsilon} - \int_0^T g(X_s^{x,0}, \xi_s^{x,0}, s)d\xi_s^{x,0}\right) \equiv I + II .$$

By the uniform continuity, for $\beta > 0$ there is a δ so that on the set

$$A \equiv [\sup_{0\le s\le T} |X_s^{x,\varepsilon} - X_s^{x,0}| < \delta, \sup_{0\le s\le T} |\xi_s^{x,\varepsilon} - \xi_s^{x,0}| < \delta] ,$$

one has

$$|g(X_s^{x,\varepsilon}, \xi_s^{x,\varepsilon}, s) - g(X_s^{x,0}, \xi_s^{x,0}, s)| < \beta .$$

Hence, since $|g(X, \xi, s)| \le K$,

$$\begin{aligned} E[|I|^p] &\le \beta E[\chi_A(\xi_T^{x,\varepsilon})^p] + (2K)^p E(\chi_{\Omega\backslash A}(\xi_T^{x,\varepsilon})^p] \\ &\le \beta E[[(\xi_T^{x,\varepsilon})^p] + (2K)^p (P[\Omega\backslash A])^{1/2}(E[(\xi_T^{x,\varepsilon})^{2p}])^{1/2} , \end{aligned}$$

and since

$$P[\Omega\backslash A] \le P[\sup_{0\le s\le T} |X_s^{x,\varepsilon} - \chi_s^{x,0}| > \delta] + P[\sup_{0\le s\le T} |\xi_s^{x,\varepsilon} - \xi_s^{x,0}| > \delta]$$

we have by part c), $E[|I|^p] \to 0$ as $\varepsilon \to 0$.

For II, we use the fact that if F is an increasing continuous function of bounded variation and $\int_0^1 f(s)dF(s) < \infty$, then if

$$|f(s + u) - f(s)| < \delta , \quad 0 \le s \le 1 , \quad 0 \le u \le 1/n ,$$

then

$$\left|\int_0^1 f(s)dF(s) - \sum_{k=1}^n f\left(\frac{k-1}{n}\right)\left(F\left(\frac{k}{n}\right) - F\left(\frac{k-1}{n}\right)\right)\right| < \delta(F(1) - F(0)) . \tag{17}$$

Select n then so that for $0 \le n \le T/n$ and all $0 \le s \le T$

$$|g(X_{s+u}^{x,0}, \xi_{s+u}^{x,0}, s + u) - g(X_s^{x,0}, \xi_s^{x,0}, s)| < \delta .$$

Hence by (17) for p an integer,

$$\begin{aligned} E[|II|^p] &\le 3^p\delta^p(E[(\xi_T^{x,\varepsilon})^p] + E[(\xi_T^{x,0})^p]) \\ &\quad + 3^p E\Bigg[\Bigg|\sum_{k=1}^n g\left(X_{((k-1)/n)T}^{x,0}, \xi_{((k-1)/n)T}^{x,0}, \frac{k-1}{n}T\right)((\xi_{(k/n)T}^{x,\varepsilon} - \xi_{((k-1)/n)T}^{x,\varepsilon}) \\ &\quad - (\xi_{(k/n)T}^{x,0} - \xi_{((k-1)/n)T}^{x,0}))\Bigg|^p\Bigg] \\ &\le 3^p\delta^p(E[(\xi_T^{x,\varepsilon})^p] + E[(\xi_T^{x,\varepsilon})^p]) \end{aligned}$$

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$$+ 3^p K^p n^p \sum_{k=1}^{n} E[|((\xi^{x,\varepsilon}_{(k/n)T} - \xi^{x,\varepsilon}_{((k-1)/n)T}) - \xi^{x,0}_{(k/n)T} - \xi^{x,0}_{((k-1)/n)T})|^p]$$
$$\leq 3^p \delta^p (E[(\xi^{x,\varepsilon}_T)^p] + E[(\xi^{x,0}_T)^p])$$
$$+ 3^p K^p n^p 2^p n E[(\sup_{0 \leq s \leq T} |\xi^{x,\varepsilon}_s - \xi^{x,0}_s|)^p]$$

and therefore by part c). $E[|II|^p] \to 0$ as $\varepsilon \to 0$.

2. Applications.

We proceed to give a few applications to asymptotic problems in partical differential equations connected with the operators

$$L^\varepsilon = \varepsilon^2 \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{d} b^j(x) \frac{\partial}{\partial x^j}$$

as ε tends to zero. In subsections 2.1 and 2.2 we investigate the behavior of solutions to the Neumann problem and certain mixed boundary value problems as $\varepsilon \to 0$ under various conditions. The methods are those of [7], where the corresponding questions for the Dirichlet problem were investigated. In subsection 2.3 we treat the asymptotic behavior of the Neumann function $r^\varepsilon(t, x, y)$ for the operator $\partial/\partial t + L^\varepsilon$ and a given region in $\boldsymbol{R}^d$. The corresponding problem for the fundamental solution $p^\varepsilon(t, x, y)$ as well as for the Green's functions $q^\varepsilon(t, x, y)$ was originally solved by S.R.S. Varadhan [5]; the case of general b was treated by A. Friedman [2].

2.1. Neumann problem. Let D be the closure of an open bounded set $D_0 \subseteq \boldsymbol{R}^d$, simply connected, and with smooth boundary ∂D and let γ be an oblique vector field on the boundary such that Theorems 1 and 2 apply. For $\lambda > 0$ and g bounded and uniformly continuous on the ∂D, the problem

$$L^\varepsilon u - \lambda u = 0 \quad \text{in } D_0 \text{ with } \quad (\nabla u, \gamma) = g \quad \text{on } \partial D$$

has a unique solution u^ε. Using the representation for $X^{x,\varepsilon}$ and applying Itô's formula to $f^\varepsilon(t, X^{x,\varepsilon}_t)$, with $f^\varepsilon(t, x) = e^{-\lambda t} u^\varepsilon(x)$, gives

$$E[f^\varepsilon(t, X^{x,\varepsilon}_t) - f^\varepsilon(0, X^{x,\varepsilon}_0)] = E\left[\int_0^t e^{-\lambda s} (\nabla u^\varepsilon, \gamma)(X^{x,\varepsilon}_t) d\xi^{x,\varepsilon}_s\right].$$

Now u^ε is bounded, and so letting $t \to \infty$, one obtains

$$u^\varepsilon(x) = -E\left[\int_0^\infty e^{-\lambda s} g(X^{x,\varepsilon}_s) d\xi^{x,\varepsilon}_s\right].$$

To conclude

$$u^\varepsilon(x) \to -E\left[\int_0^\infty e^{-\lambda s} g(X_s^{x,0}) d\xi_s^{x,0}\right]$$

write

$$\int_0^\infty e^{-\lambda s} g(X_s^{x,\varepsilon}) d\xi_s^{x,\varepsilon} = \int_0^T e^{-\lambda s} g(X_s^{x,\varepsilon}) d\xi_s^{x,\varepsilon} + \int_T^\infty e^{-\lambda s} g(X_s^{x,\varepsilon}) d\xi_s^{x,\varepsilon} .$$

By Theorem 2d,

$$E\left[\int_0^T e^{-\lambda s} g(X_s^{x,\varepsilon}) d\xi_s^{x,\varepsilon}\right] \to E\left[\int_0^T e^{-\lambda s} g(X_s^{x,0}) d\xi_s^{x,0}\right] .$$

To finish, it need only be shown that for large T, $E\left[\int_T^\infty e^{-\lambda s} g(X_s^{x,\varepsilon}) d\xi_s^{x,\varepsilon}\right]$ is small uniformly in ε. Since

$$\left| E\left[\int_T^\infty e^{-\lambda s} g(X_s^{x,\varepsilon}) d\xi_s^{x,\varepsilon}\right]\right| \leq C \sum_{j=0}^\infty e^{-\lambda(T+j)} E[\xi_{T+j+1}^{x,\varepsilon} - \xi_{T+j}^{x,\varepsilon}] ,$$

where C is a bound for $|g|$, it suffices to note that $E[\xi_{t+1}^{x,\varepsilon} - \xi_t^{x,\varepsilon}]$ is bounded independent of ε and t, see Theorem 2.

2.2. Mixed boundary value problem. Let D and γ be as in subsection 2.1, but consider the problem

$$L^\varepsilon u = 0 \quad \text{in } D_0 \text{ and } -\lambda u + (\nabla u, \gamma) = h \text{ on } \partial D ,$$

$\lambda > 0$, with h bounded and uniformly continuous. Let u^ε be the unique solution and define

$$Z_t^\varepsilon = u^\varepsilon(X_t^{x,\varepsilon}) - \int_0^t (\nabla u^\varepsilon, \gamma)(X_t^{x,\varepsilon}) d\xi_s^{x,\varepsilon} .$$

Now Z_t^ε is a martingale, and hence from Itô's formula applied to the product $Z_t^\varepsilon \exp\{-\lambda \xi_t^{x,\varepsilon}\}$,

$$u^\varepsilon(X^{x,\varepsilon}) \exp\{-\lambda\xi_s^{x,\varepsilon}\} - \int_0^t -\lambda u^\varepsilon(X_s^{x,\varepsilon}) + (\nabla u^\varepsilon, \gamma)(X_s^{x,\varepsilon}) \exp\{-\lambda\xi_s^{x,\varepsilon}\} d\xi_s^{x,\varepsilon} \tag{18}$$

is also seen to be a matringale.

Let $a = \sigma\sigma^*$, which by assumption is uniformly positive definite, and this guarantees $\xi_t^{x,\varepsilon} \to \infty$ as $t \to \infty$, (c.f. the argument in the second part of this subsection). So by (18), letting $t \to \infty$,

(19) $$u^\varepsilon(x) = E\left[\int_0^\infty h(X_s^{x,\varepsilon}) \exp\{-\lambda\xi_s^{x,\varepsilon}\} d\xi_s^{x,\varepsilon}\right].$$

If $\xi_t^{x,0} \to 0$ as $t \to \infty$ Theorem 2d) shows u^ε goes to the right side of (19) with $\varepsilon = 0$, as $\varepsilon \downarrow 0$.

If $\xi_t^{x,0}$ does not tend to infinity (e.g., $\xi_t^{x,0} \equiv 0$ if $X^{x,0}$ remains in D_0) things are more difficult. We now consider a special case of this situation. We impose the following assumptions:

i) There exists $w_0 \in D_0$ such that $X_t^{x,0} \to w_0$ without leaving D_0 as $t \to \infty$ for every $x \in D$.

ii) $(\gamma(z), b(z)) \geq c_1 > 0$ for all $z \in \partial D$.

iii) Let $V(y) = \inf\{I_T^+(\varphi) : \varphi \in C_T(D),\ \varphi_0 = w_0,\ \varphi_T = y,\quad 0 \leq T < \infty\}$. There is to be a unique point $y_0 \in \partial D$ satisfying $V(y_0) = \min_{y \in \partial D} V(y)$.

Evidently there exists a $T > 0$, a $\varphi \in C_T(D)$ with $\varphi_0 = w_0$ and $\varphi_T = y_0$, $I_T^+(\varphi) = V(y_0)$ and $\varphi_t \in D_0$ for $0 \leq t < T$. So $I_T^+(\varphi) = I_T(\varphi)$.

Conditions i), ii), and iii) were imposed by Ventcel and Freidlin [7] in Theorem 3.1, which we will use below.

The representation (19) of $u^\varepsilon(x)$ and Theorem 3.1 [7] suggests that as $\varepsilon \to 0$, $u^\varepsilon(x)$ tends to $\lambda^{-1}h(y_0)$. Indeed this will follows under our present assumption if $x \in D_0$ from the representation together with the following assertion:

(20) For every positive M and ε there exists a stopping time $T^{M,\varepsilon}$ such that for every neighborhood U of y_0,
$$\lim_{\varepsilon \to 0} P[\xi_{T^{M,\varepsilon}}^{x,\varepsilon} > M,\ X_s^{x,\varepsilon} \notin \partial D \backslash U \text{ for any } s \in [0, T^{M,\varepsilon}]] = 1.$$

We proceed to prove (20). Let U_0 be a small neighborhood of w_0, and define: $S_0 = 0$, $S_1 = \inf\{t : X_t^{x,\varepsilon} \in U_0\}$, $T_1 = \inf\{t > S_1 : X_t^{x,\varepsilon} \in \partial D\}$, $S_n = \inf\{t \geq T_{n-1} : X_t^{x,\varepsilon} \in U_0\}$, $T_n = \inf\{t \geq S_n : X_t^{x,\varepsilon} \in \partial D\}$. Let $Z_n = \xi_{S_{n+1}}^{x,\varepsilon} - \xi_{S_n}^{x,\varepsilon}$, $n = 1, 2, \cdots$. Let U be a neighborhood of y_0. From Theorem 3.1 of [7], we know the probable behavior of $X^{x,\varepsilon}$ between times S_0 and S_1, and also between time S_k and S_{k+1}. Between time S_0 and S_1, $X^{x,\varepsilon}$ will in all likelihood remain within a δ-neighborhood of the trajectory $X^{x,0}$. Starting at time S_k the process will for a long time remain near w_0, eventually, it will, however, hit ∂D at time T_k, most likely in U, and then return to U_0 without intersecting $\partial D \backslash U$. During the time interval $S_{k+1} - S_k$, other behavior is of course possible, we will refer to it as *exceptional behavior*, but Theorem 3.1 of [7] implies that there is a constant $c(U) > 0$ such that the conditional probability given $\mathscr{F}_{S_k}$ of exceptional

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behavior during the interval $S_{k+1} - S_k$ is bounded by $\exp\{-c(U)/(2\varepsilon^2)\}$. Thus on a typical interval $[S_k, S_{k+1})$ the local time will increase only while $X^{x,\varepsilon}$ is in U. However, when ε is small the increment in local time during $[S_k, S_{k+1})$ will also be small with high probability. We will show that there exists a positive constant c_0 such that for $k \geq 1$,

$$P[\xi^{x,\varepsilon}_{S_{k+1}} - \xi^{x,\varepsilon}_{S_k} > c_0\varepsilon^2 | \mathscr{F}_{S_k}] > 1/4 \,. \tag{21}$$

Before deriving (21), let us show how it implies (20). Let for $k = 1, 2, \cdots$,

$$X_k = \begin{cases} 1 & \text{if} \quad \xi^{x,\varepsilon}_{S_{k+1}} - \xi^{x,\varepsilon}_{S_k} > c_0\varepsilon^2 \\ 0 & \text{if} \quad \xi^{x,\varepsilon}_{S_{k+1}} - \xi^{x,\varepsilon}_{S_k} \leq c_0\varepsilon^2 \end{cases}$$

and let $X'_1, X'_2, \cdots$ be a sequence of independent identically distributed random variables, $P[X'_1 = 1] = 1/4$, $P[X'_1 = 0] = 3/4$. Then for all n

$$P\left[\sum_{k=1}^{n} X_k \leq n/8\right] \leq P\left[\sum_{k=1}^{n} X'_k \leq n/8\right]$$

and the right side approaches 0 as n approaches infinity by the law of large numbers. Let M be given and choose $n \geq 8M/(c_0\varepsilon^2)$. We then conclude that $P[\sum_{K=1}^{n} (\xi^{x,\varepsilon}_{k+1} - \xi^{x,\varepsilon}_{k}) < M] \to 0$. Since n increases like ε^{-2} while the conditional probability of exceptional behavior during $[S_k, S_{k+1})$ decrease like $\exp\{-c(U)/(2\varepsilon^2)\}$, we may conclude (20) with $T^{M,\varepsilon} = S_n$.

It remains to prove (21). Clearly it suffices to consider $k = 1$. In order to study the increase in $\xi^{x,\varepsilon}$ just after T_1, introduce appropriate coordinates, so that $\hat{X}_t = u(X^{x,\varepsilon}_{T_1+t})$ is the desired diffusion in $\boldsymbol{R}^d_+$ with normal reflection, and $\hat{\xi}_t$ is the associated local time. Furthermore, $\hat{X}_t = \Gamma(Y_t)$, where Y_t is the associated unrestricted process. Let Y^1 be the first component of Y. Recall by our construction, $\hat{\xi}_t = \inf_{0 \leq s \leq t} Y^1_s$, and as long as $X^{x,\varepsilon}_t$ remains in the coordinate patch $\xi^{x,\varepsilon}_{T_1+t} - \xi^{x,\varepsilon}_{T_1} = \hat{\xi}_t$. Now Y^1 satisfies

$$Y^1_t = \varepsilon\int_0^t \sum_{k=1}^{d} \hat{\sigma}^{1,k} \circ \Gamma_s(Y)dW^k + \int_0^t \hat{\beta} \circ \Gamma_s(Y)ds$$

where $\hat{\sigma}^{i,k}(x)$ and $\hat{\beta}(x)$ are the coefficients in the equations for $d\hat{X}$; by assumption $\hat{a}^{1,1}(x) = \sum_{k=1}^{d} \hat{\sigma}^{1,k}(\hat{\sigma}^*)^{k,1}(x) \geq m_3^{-1}$ and for $\varepsilon \leq 1$, $\hat{\beta}(x) \leq 2m_1$. Now the first term on the right in the representation of Y^1 is a martingale M_t with square variation $A_t = \varepsilon^2 \int_0^t \hat{a}^{1,1} \circ \Gamma_s(Y)ds$ and by the above $A_t \geq \varepsilon^2 m_3^{-1} t$. As is well known $V_t = M_{A_t^{-1}}$ is a one-dimensional Brownian motion. Write then

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$$Y_t^1 = V_{A_t} + \int_0^t \beta \circ \Gamma_s(Y) ds$$

and deduce for $\varepsilon < 1$,

$$\inf_{0 \le s \le t} Y_s^1 \le \inf_{s \le \varepsilon^2 m_3^{-1} t} V_s + 2m_1 t \le V_{\varepsilon^2 m_3^{-1} t} + 2m_1 t .$$

Let c be a positive constant such that $P[V_t < -c\sqrt{t}\,] > 1/4$. Thus

$$P[V_{\varepsilon^2 m_3^{-1} t} + 2m_1 t < -c\varepsilon\sqrt{m_3^{-1} t} + 2m_1 t] > 1/4 .$$

Now set $t^* = c^2\varepsilon^2 m_3^{-1}/(4m_1)^2$ and find

$$P[\inf_{0 \le s \le t^*} Y_s^1 \le -\varepsilon^2 c^2/(8m_1 m_2)] \ge P[V_{m_3^{-1}\varepsilon^2 t^*} + 2m_1 t^* \le -\varepsilon^2 c^2/(8m_1 m_2)] > 1/4 .$$

This gives the local time increment for $X_t^{x,\varepsilon}$ only as long as this process remains in a coordinate patch about $X_{T_1}^{x,\varepsilon}$; however, since it has to transverse a distance in excess of ρ_0 before leaving such a patch, and t^* is of order ε^2, it follows from Theorem 1 that the probability of getting out by time t^* is negligible, and (21) is proved.

2.3 The Neumann function. Again let D and γ be as in subsection 2.1. In addition to our underlying assumptions about the coefficients in (4), we assume in this subsection that b is differentiable and that there exists $\alpha > 0$ such that

$$\left|\frac{\partial b}{\partial x^i}(x) - \frac{\partial b}{\partial x^i}(y)\right| \le |x - y|^\alpha , \quad 1 \le i \le d , \quad x, y \in D . \tag{22}$$

Setting $r^\varepsilon(t, x, dy) = P[X_t^{x,\varepsilon} \in dy]$ it is known that $r^\varepsilon(t, x, dy) = r^\varepsilon(t, x, y)dy$, with $r^\varepsilon(t, x, y)$ a continuous function, the Neumann function for the operator $\partial u/\partial t + L^\varepsilon u$ for the region $D_0 \times [0, \infty)$.

We will obtain the asymptotic behavior of $r^\varepsilon(t, x, y)$. To do this we use information about two related functions. One of these is the Green's function $q^\varepsilon(t, x, y)$ for the operator $\partial u/\partial t + L^\varepsilon u$ for the region $D_0 \times [0, \infty)$. It is familiar that

$$q^\varepsilon(t, s, y)dy = P[X_t^{x,\varepsilon} \in dy, X_s^{x,\varepsilon} \in D_0 , \qquad 0 \le s \le t] .$$

Next, extending the coefficients $\sigma(x), c(x), b(x)$ to all of $\boldsymbol{R}^d$ so that our boundedness and regularity hypotheses are preserved, we obtain operators $\partial u/\partial t + \tilde{L}^\varepsilon u$, with u defined on $\boldsymbol{R}^d \times [0, \infty)$. Let $p^\varepsilon(t, x, y)$ be the corresponding fundamental solution. Naturally associated are the diffusion processes

$$d\tilde{X}_t^{x,\varepsilon} = \varepsilon\sigma(\tilde{X}_t^{x,\varepsilon})dW_t + (\varepsilon c(\tilde{X}_t^{x,\varepsilon}) + b(\tilde{X}_t^{x,\varepsilon}))dt\,, \qquad \tilde{X}_0^{x,0} = x$$

in $\boldsymbol{R}^d$. In fact

$$p^\varepsilon(t,x,y)dy = P[\tilde{X}_t^{x,\varepsilon} \in dy]\,.$$

Evidently

(23) $p^\varepsilon(t,x,y) \geq q^\varepsilon(t,x,y)\,, \quad r^\varepsilon(t,x,y) \geq q^\varepsilon(t,x,y)\,, \quad t \geq 0\,, \quad x,y \in D_0\,.$

Define

$$I_t^+(x,y) = \inf\{I_t^+(\varphi)\colon \varphi \in C_t(D), \varphi_0 = x, \varphi_t = y\}\,.$$

We will prove

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon^2}\log r^\varepsilon(t,x,y) = I_t^+(x,y)\,, \qquad x,y \in D_0\,. \tag{24}$$

Observe that $\tilde{X}^{x,0}$ is simply the solution of the dynamical system

$$d\tilde{X}_t^{x,0} = b(\tilde{X}_t^{x,0})dt\,, \qquad \tilde{X}^{x,0} = x\,.$$

Evidently if $x \in D_0$ and $\varepsilon \geq 0, \tilde{X}_t^{x,\varepsilon}$ coincides with $\tilde{X}_t^{x,\varepsilon}$ up to the first time that ∂D is hit. Also, for $\varphi \in C_T(\boldsymbol{R}^d)$, it is clear that if $\varphi_t \in D_0$ for $0 \leq t \leq T$ then $I_T(\varphi) = I_T^+(\varphi)$.

We will utilize, as did Friedman [2], the following useful extimates of Aronson [1]:

$$p^\varepsilon(t,x,y) \leq \frac{A_0}{\varepsilon^d t^{d/2}}\exp\left\{-\frac{c\,|\tilde{X}_t^{x,0} - y|^2}{\varepsilon^2 t}\right\} \qquad \text{if } t < T^*\,, \tag{25}$$

$$\begin{aligned} p^\varepsilon(t,x,y) \geq{}& \frac{A_1}{\varepsilon^d t^{d/2}}\exp\left\{-\frac{c_1|\tilde{X}_t^{x,0} - y|^2}{\varepsilon^2 t}\right\} \\ &- \frac{A_2}{\varepsilon^{d-2\alpha}t^{d/2-\alpha}}\exp\left\{-\frac{c^2|\tilde{X}_t^{x,0} - y|^2}{\varepsilon^2 t}\right\} \qquad \text{if } t < t^*\,, \end{aligned} \tag{26}$$

where T^* is any positive number, t^* is a sufficiently small positive number, $A_0, A_1, A_2, c, c_1, c_2$ are positive constants. Now fix y. For $\delta \geq 0$, $s \geq 0$, let

$$C_s^\delta = \{z \in D\colon |X_s^{z,0} - y| \leq \delta\}$$

Let $\psi^{(s)}$ be the trajectory determined by

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$$d\psi_u^{(s)} = b(\psi_u^{(s)})dt\ , \quad 0 \le u \le t\ , \quad \psi_s^{(s)} = y\ ,$$

It follows from our assumptions that one can choose positive numbers δ_0 and s_0 sufficiently small so that the following hold for all $s \le s_0, \delta \le \delta_0$: if $z \in C_s^\delta, X_u^{z,0} \in D_0$, for $0 \le u \le s$; $\sup_{0 \le u \le s} |\psi_u^{(s)} - \psi_0^{(s)}| \le \rho_0$; there exists a positive number m such that the distance between $\psi_0^{(s)}$ and $D \backslash C_s^\delta$ exceeds δ/m.

Now we choose a suitably big constant n, the exact requirements on its size will be made evident below. For ε sufficiently small $C_s^{2nm\varepsilon} \subseteq C_s^{\delta_0}$. Also there exists $d_1 > 0$ such that $C_s^{\delta_0}$ is at a distance at least d_1 from ∂D, with d_1 not depending on $s, 0 < s \le s_0$.

Observe now that (26) provides a positive $c(s)$ (it will go to zero with s) such that

$$p^\varepsilon(s, z, y) \ge \frac{c(s)}{\varepsilon^d}\ , \qquad z \in C_s^{2nm\varepsilon}\ .$$

Noting that one can go from z to y in time s either without or by hitting the boundary of D, it follows as usual from the strong Markov property that

$$p^\varepsilon(s, z, y) \le q^\varepsilon(s, z, y) + P[\tilde{X}_u^{z,\varepsilon} \in \partial D \qquad \text{for some } u, 0 \le u \le s]\ .$$
$$\sup\{p^\varepsilon(u, w, y) : 0 \le u \le s, w \in \partial D\}\ .$$

Since $|w - y| > d_1$ for $w \in \partial D$ the second factor in the last term can be bounded by (25), and we obtain for ε sufficiently small

$$r^\varepsilon(s, z, y) \ge q^\varepsilon(s, z, y) \ge \frac{c_1(s)}{2\varepsilon^d}\ , \qquad z \in C_s^{2nm\varepsilon}\ . \tag{27}$$

Let $h > 0$ and choose $\psi \in C_t(D)$ so that $\psi_0 = x, \psi_t = y, I_t^+(\psi) \le I_t^+(x, y) + h$, and in addition there exists an $s, 0 < s \le s_0$, such that $\psi_{t-s+u} = \psi_u^{(s)}$. Then

$$\begin{aligned} r^\varepsilon(t, x, y) &\ge \int_{C_s^{2nm\varepsilon}} r^\varepsilon((t-s), x, z) r^\varepsilon(s, z, y) dz \\ &\ge P[X_{t-s}^{x,\varepsilon} \in C_s^{2nm\varepsilon}] \frac{c_1(s)}{2\varepsilon^d} \end{aligned}$$

since the distance between ψ_{t-s} and $D \backslash C_s^{2nm\varepsilon}$ is at least $2n\varepsilon$,

$$[|X_{t-s}^{x,\varepsilon} - \psi_{t-s} < n\varepsilon] \subseteq [X_{t-s}^{x,\varepsilon} \in C_s^{2nm\varepsilon}]$$

and the probability of the set on the left can be estimated by Theorem 1, part b), provided only n is chosen big enough. It follows that

$$\varliminf_{\varepsilon\to 0} \frac{1}{2\varepsilon^2} \log r^\varepsilon(t, x, y) \geq I_t^+(x, y) + h$$

and since h is arbitrary we have obtained the desired lower estimate.

For th upper estimate define

$$B_{s_0}^\delta = \{(t-s, z) : 0 \leq s \leq s_0 , \qquad z \in D , \quad |X_s^{z,0} - y| < \delta\}$$

Fix $\delta > 0$ and s_0, $0 < s_0 < t$ so that $B_{s_0}^\delta \subseteq [t - s_0, t] \times D_0$. Observe that this implies the existence of a positive k_1 such that for $(t-s, z) \in B_{s_0}^\delta$, $\varphi \in C_{t-s}(D)$, $\varphi_0 = z$, and $\varphi_u \in \partial D$ for some u, $0 \leq u \leq t - s$ always

$$I_{t-s}^+(\varphi) \geq k_1 . \tag{28}$$

It follows from (23), and (25) that there exists $\beta > 0$ so that for $N_\beta(y) = \{w : |w - y| \leq \beta\}$, there exists $A > 0$ such that

$$q^\varepsilon(s, z, w) \leq \frac{A}{\varepsilon^d s_0^{d/2}} , \qquad \text{for } (t-s, z) \in \partial B_{s_0}^\delta, w \in N_\beta(y) . \tag{29}$$

Define the following sequences of stopping times:

$$T_1 = \inf\{u : (t-u, X_u^{x,\varepsilon}) \in B_{s_0}^\delta\} \wedge t, \quad S_1 = \inf\{u > T_1 : X_u^{x,\varepsilon} \in \partial D\} \wedge t$$
$$T_n = \inf\{u \geq S_{n-1} : (t-u, X_u^{x,\varepsilon}) \in B_{s_0}^\delta\} \wedge t ,$$
$$S_n = \inf\{u > T_n : X_u^{x,\varepsilon} \in \partial D\} \wedge t .$$

Let $|N_\beta(y)|$ stand for the Lebesgue measure of $N_\beta(y)$. It is evident, using the strong Markov property, that

$$\begin{aligned} P[X_t^{x,\varepsilon} \in N_\beta(y)] &= \sum_{n=1}^\infty P[X_t^{x,\varepsilon} \in N_\beta(y), T_n < t \leq T_{n+1}] \\ &\leq \left(\sum_{n=1}^\infty P[T_n < t]\right) |N_\beta(y)| \cdot \sup\{q^\varepsilon(s, z, w) : (t-s, z) \in \partial B_{s_0}^\delta, \\ &\qquad\qquad w \in N_\beta(y)\} . \end{aligned} \tag{30}$$

The last factor is bounded by (29). On the other hand it follows from (28) and Theorem 1, part c) that

$$P[T_n < t \mid \mathscr{F}_{T_{n-1}}] \leq \exp\left\{-\frac{1}{2\varepsilon^2} \cdot \frac{k_1}{2}\right\}$$

for ε sufficiently small. Hence the series in the last member converges exponentially fast and we obtain a constant A' such that

$$P[X_t^{x,\varepsilon} \in N_\beta(y)] \leq P[T_1 < t] \cdot \frac{A'}{\varepsilon^d s_0^{d/2}} |N_\beta(y)| \,. \tag{31}$$

Let $i_\delta = \inf \{I_t^+(\varphi) : \varphi \in C_t(D),\ \varphi_0 = x, \varphi_s \in B_{s_0}^\delta$ for some $s \in [0, t]\}$. It is an easy consequence of Theorem 1, parts c) and a), that for any $u > 0$, the first factor on the right of (31) is bounded by $\exp\{-(i_\delta - h)/2\varepsilon^2\}$ for ε sufficiently small. Using this in (31), dividing through by $(N_\beta(y))$ and letting $\beta \to 0$ gives

$$r^\varepsilon(t, x, y) \leq \frac{A'}{\varepsilon^\delta s^{\delta/2}} \exp\left\{-\frac{1}{2\varepsilon^2}(i_\delta - h)\right\} . \tag{32}$$

Finally, observe that i_δ increases to $I_t^+(x, y)$, and since $\delta > 0$ and $h > 0$ are arbitrary the desired upper estimate is proved.

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