J. Austral. Math. Soc. (*Series A*) **60** (1996), 334–342

ALEXANDER POLYNOMIALS OF TWO-BRIDGE KNOTS

YASUTAKA NAKANISHI and MASAKI SUKETA

(Received 22 November 1992; revised 20 April 1993)

Communicated by J. H. Rubinstein

Abstract

For two-bridge knots, the authors give necessary conditions on coefficients of Alexander polynomials.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 57M25.
Keywords and phrases: Two-bridge knot, Alexander polynomial, Seifert matrix.

1. Introduction

A notion of two-bridge knot was introduced by Schubert [8]. In this note, we study Alexander polynomials of two-bridge knots. After the work of Seifert [9], the Alexander polynomial $\Delta(t)$ for a knot is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$ characterized by the following two conditions: $\Delta(t^{-1}) \doteq \Delta(t)$ and $\Delta(1) = \pm 1$. Throughout this note, Alexander polynomials are written as $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ ($a_n \neq 0$). In 1958, Murasugi [6] showed that the signs of coefficients of Alexander polynomials for alternating knots are alternating, and so all a_i's are assumed to be non-negative. In 1979, Hartley [5] showed that the coefficients of Alexander polynomials for two-bridge knots satisfy the descending property: $a_0 = \cdots = a_i > a_{i+1} > \cdots > a_n (> 0)$ for a certain integer i. We give upper and lower bounds for a_i by a_n as follows.

THEOREM 1. $\left(\sum_{k=0}^{j} {}_{2n-2k}C_{j-k} \cdot {}_{2n-k}C_k\right) a_n \geq a_{n-j}$.
Equality holds when the two-bridge knot is equivalent to $C(2, 2, \ldots, 2, 2)$.

THEOREM 2. $(4n - 2)a_n + 1 \geq a_{n-1} \geq 2a_n - 1$.

https://doi.org/10.1017/S1446788700037848 Published online by Cambridge University Press

THEOREM 3. *If* $a_n \neq 1$, *then* $(8n^2 - 15n + 8)a_n + 3n - 3 > a_{n-2}$.

2. Seifert matrices of two-bridge knots

For convenience of calculation, we remove the condition that the a_i's are positive. Therefore, we consider all a_i's to have the same sign (possibly negative).

It is folklore that a two-bridge link can be written in Conway form [4] $(1^*)ijk\cdots lm$ using integers $i, j, k, \ldots, l, m$. In particular, a two-bridge knot can be written as $(1^*)ijk\cdots lm$ for certain even integers $i, j, k, \ldots, l, m$. For convenience, we use the form $C(i, j, k, \ldots, l, m)$ instead of the Conway form $(1^*)ijk\cdots lm$ to present a two-bridge knot. With this convention, a two-bridge knot can be written as $C(2b_1, 2b_2, \ldots, 2b_n)$. It is then easy to see that the knot presented by $C(2b_1, 2b_2, \ldots, 2b_{2n})$ bounds a Seifert surface as in Figure 1, which is a plumbing of a b_1-full-twisted-band, a $(-b_2)$-full-twisted-band, ..., and a $(-b_{2n})$-full-twisted-band.

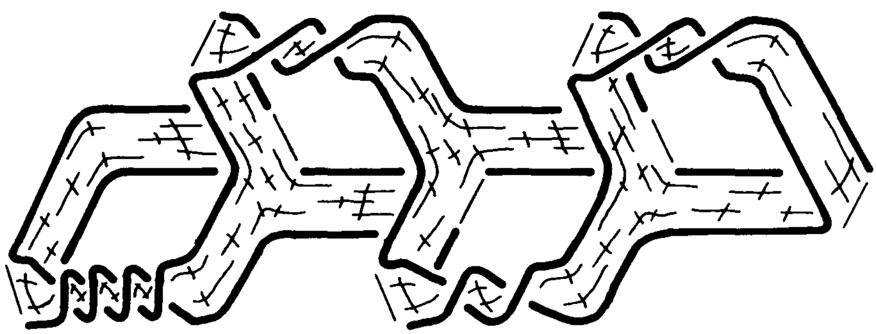

FIGURE 1.

From this surface, we calculate a Seifert matrix as

$$V = \begin{pmatrix} b_1 & 1 & & & & \mathbf{0} \\ & -b_2 & & & & \\ & 1 & b_3 & 1 & & \\ & & & -b_4 & & \\ & & & & \ddots & \\ \mathbf{0} & & & & & -b_{2n} \end{pmatrix}.$$

Again for convenience of calculation, we rewrite b_i as $(-1)^{i+1}c_i$ $(1 \leq i \leq 2n)$. From [9], we calculate the Alexander polynomial as

$$\Delta(t) = \det(tV' - V) = c_1c_2\cdots c_{2n}(t-1)^{2n} + \sum_{k=1}^{n} \gamma_k t^k (t-1)^{2n-2k},$$

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where V' is the transposed matrix of V and

$$\gamma_k = \begin{cases} \sum_* c_1 c_2 \cdots \check{c_{i_{k1}}} \check{c_{i_{k1}+1}} \cdots \check{c_{i_{kk}}} \check{c_{i_{kk}+1}} \cdots c_{2n} & \text{if } k \neq n, \\ 1 & \text{if } k = n. \end{cases}$$

Here $\sum_*$ means the summation over all k-tuples $\{i_{k1}, \ldots, i_{kk}\} \subset \{1, \ldots, 2n-1\}$ satisfying $i_{kl} + 1 < i_{kl+1}$ $(1 \leq l \leq k-1)$.

3. Proof of Theorem 1

We remark that the number of terms in the summation presenting γ_k is ${}_{2n-k}C_k$. On the other hand, $a_{n-j} = {}_{2n}C_j \cdot c_1 c_2 \cdots c_{2n} + \sum_{k=1}^{j} (-1)^k {}_{2n-2k}C_{j-k} \cdot \gamma_k$ when $j \neq 0$, and $a_n = c_1 c_2 \cdots c_{2n}$. Therefore,

$$\Big(\sum_{k=0}^{j} {}_{2n-2k}C_{j-k} \cdot {}_{2n-k}C_k\Big) a_n - a_{n-j} =$$
$$\sum_{k=1}^{j} {}_{2n-2k}C_{j-k} \Big(\sum_{*} c_1 c_2 \cdots \check{c_{i_{k1}}} \check{c_{i_{k1}+1}} \cdots \check{c_{i_{kk}}} \check{c_{i_{kk}+1}} \cdots c_{2n} \Big(\prod_{s=1}^{k} c_{i_{ks}} c_{i_{ks}+1} - (-1)^k\Big)\Big).$$

We remark that the signs of a_n and a_{n-j} are the same. Since $c_{i_{ks}} c_{i_{ks}+1}$ and $c_{i_{ks}} c_{i_{ks}+1} - (-1)^k$ have the same sign, the value of the equation above has the same sign as that of a_n. Furthermore, equality holds when $c_i = (-1)^i$ $(1 \leq i \leq 2n)$ or $c_i = (-1)^{i+1}$ $(1 \leq i \leq 2n)$. In both cases, the given two-bridge knot is equivalent to $C(2, 2, \ldots, 2, 2)$.

4. Proofs of Theorems 2 and 3

A simple proof of Theorem 2 can be given as an analogy of the following fact:

FACT. Let $p_1, \ldots, p_n$ be positive integers with $p_1 \cdots p_n = N$. Then $\sum_{i=1}^{n} p_1 \cdots \check{p}_i \cdots p_n \leq (n-1)N + 1$.

Preparing for a proof of Theorem 3, we give an alternative proof of Theorem 2 as follows.

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$$
\begin{aligned}
&(4n-2)a_n - (-1)^{n-1} - a_{n-1} \\
&\quad = (2n-2)c_1c_2\cdots c_{2n} + \sum_{i_{11}=1}^{2n-1} c_1c_2\cdots \check{c}_{i_{11}}\check{c}_{i_{11}+1}\cdots c_{2n} - (-1)^{n-1} \\
&\quad = \sum_{i_{11}=1}^{2n-2} c_1c_2\cdots \check{c}_{i_{11}}\check{c}_{i_{11}+1}\cdots c_{2n}(c_{i_{11}}c_{i_{11}+1}+1) + c_1c_2\cdots c_{2n-2} - (-1)^{n-1} \\
&\quad = \sum_{k=1}^{n-1} c_1c_2\cdots \check{c}_{2k-1}\check{c}_{2k}\cdots c_{2n}(c_{2k-1}c_{2k}+1) + c_1c_2\cdots c_{2n-2} - (-1)^{n-1} \\
&\qquad + \sum_{k=1}^{n-1} c_1c_2\cdots \check{c}_{2k}\check{c}_{2k+1}\cdots c_{2n}(c_{2k}c_{2k+1}+1) \\
&\quad = \sum_{k=1}^{n-1}(c_1c_2\cdots c_{2k-2}\cdot c_{2n-1}c_{2n} - (-1)^k)(c_{2k-1}c_{2k}+1)c_{2k+1}c_{2k+2}\cdots c_{2n-2} \\
&\qquad + \sum_{k=1}^{n-1} c_1c_2\cdots \check{c}_{2k}\check{c}_{2k+1}\cdots c_{2n}(c_{2k}c_{2k+1}+1).
\end{aligned}
$$

We remark that the signs of a_n and a_{n-1} are the same. Since $c_1c_2\cdots c_{2k-2}\cdot c_{2n-1}c_{2n}$ and $c_1c_2\cdots c_{2k-2}\cdot c_{2n-1}c_{2n} - (-1)^k$ have the same sign, $c_{2k-1}c_{2k}$ and $c_{2k-1}c_{2k}+1$ have the same sign, $c_{2k}c_{2k+1}$ and $c_{2k}c_{2k+1}+1$ have the same sign, and the value of the equation above has the same sign as that of a_n. Therefore $(4n-2)|a_n| + 1 \geq |a_{n-1}|$. Furthermore equality holds when $c_i = (-1)^i$ $(1 \leq i \leq 2n-1$ or $2 \leq i \leq 2n)$ or $c_i = (-1)^{i+1}$ $(1 \leq i \leq 2n-1$ or $2 \leq i \leq 2n)$.

$$
\begin{aligned}
a_{n-1} - 2a_n + 1 &= (2n-2)c_1c_2\cdots c_{2n} - \sum_{i_{11}=1}^{2n-1} c_1c_2\cdots \check{c}_{i_{11}}\check{c}_{i_{11}+1}\cdots c_{2n} + 1 \\
&= \sum_{k=1}^{n-1}(c_1c_2\cdots c_{2k-2}\cdot c_{2n-1}c_{2n} - 1)(c_{2k-1}c_{2k}-1)c_{2k+1}c_{2k+2}\cdots c_{2n-2} \\
&\quad + \sum_{k=1}^{n-1} c_1c_2\cdots \check{c}_{2k}\check{c}_{2k+1}\cdots c_{2n}(c_{2k}c_{2k+1}-1).
\end{aligned}
$$

Again we use the fact that the signs of a_n and a_{n-1} are the same: Since $c_1c_2\cdots c_{2k-2}\cdot c_{2n-1}c_{2n}$ and $c_1c_2\cdots c_{2k-2}\cdot c_{2n-1}c_{2n} - 1$ have the same sign, $c_{2k-1}c_{2k}$ and $c_{2k-1}c_{2k} - 1$ have the same sign, and $c_{2k}c_{2k+1}$ and $c_{2k}c_{2k+1} - 1$ have the same sign, the value of the equation above has the same sign as those of a_n and a_{n-1}. Therefore, we have

$$|a_{n-1}| \geq 2|a_n| - 1.$$

Furthermore, equality holds when $c_i = 1$ $(1 \le i \le 2n-1$ or $2 \le i \le 2n)$ or $c_i = -1$ $(1 \le i \le 2n-1$ or $2 \le i \le 2n)$. The proof of Theorem 2 is complete.

LEMMA 4. *The following value has the same sign as that of* a_n*:*

$$\begin{aligned} E =&(n(2n-1)+(2n-2)^2+(2n-4)(n-1))a_n \\ &-(2n-2)(-1)^{n-1}+\epsilon(n-1)-a_{n-2}, \end{aligned}$$

$$\text{where} \qquad \epsilon = \begin{cases} -1 & \text{if } a_n < 0 \text{ and } n \text{ is odd}, \\ +1 & \text{otherwise}. \end{cases}$$

PROOF. It can be seen that

$$E = (2n-2)((2n-2)a_n + \gamma_1 - (-1)^{n-1}) + (2n-4)(n-1)a_n - \gamma_2 + \epsilon(n-1).$$

Here the first term $(2n-2)a_n + \gamma_1 - (-1)^{n-1}$ has the same sign as that of a_n from the proof of Theorem 2. Therefore, it is sufficient to show that the following value has the same sign as that of a_n: $F = (2n-4)(n-1)a_n - \gamma_2 + \epsilon(n-1)$.

$$\begin{aligned} F =&(2n-4)(n-1)a_n - \sum_{*2} c_1c_2\cdots \check{c}_{i_{21}}\check{c}_{i_{21}+1}\cdots \check{c}_{i_{22}}\check{c}_{i_{22}+1}\cdots c_{2n} + \epsilon(n-1) \\ =&(2n-4)(n-1)a_n - \sum_{**} c_1c_2\cdots \check{c}_{2k_1-1}\check{c}_{2k_1}\cdots \check{c}_{2k_2-1}\check{c}_{2k_2}\cdots c_{2n} + \epsilon(n-1) \\ &- \sum_{***} c_1c_2\cdots \check{c}_{i_{21}}\check{c}_{i_{21}+1}\cdots \check{c}_{i_{22}}\check{c}_{i_{22}+1}\cdots c_{2n} \\ =&(n-2)(n-1)a_n/2 - \sum_{**} c_1c_2\cdots \check{c}_{2k_1-1}\check{c}_{2k_1}\cdots \check{c}_{2k_2-1}\check{c}_{2k_2}\cdots c_{2n} + \epsilon(n-1) \\ &- \sum_{***} c_1c_2\cdots \check{c}_{i_{21}}\check{c}_{i_{21}+1}\cdots \check{c}_{i_{22}}\check{c}_{i_{22}+1}\cdots c_{2n}(c_{i_{21}}c_{i_{21}+1}c_{i_{22}}c_{i_{22}+1} - 1). \end{aligned}$$

Here, $\sum_{*2}$ means the summation over all pairs i_{21} and i_{22} satisfying $i_{21}+1 < i_{22}$; $\sum_{**}$ means the summation over all pairs k_1 and k_2 satisfying $k_1 < k_2$; and $\sum_{***}$ means the summation over all pairs i_{21} and i_{22} satisfying $i_{21}+1 < i_{22}$ with one of i_{21} and i_{22} is even.

It can be seen that the last term

$$\sum_{***} c_1c_2\cdots \check{c}_{i_{21}}\check{c}_{i_{21}+1}\cdots \check{c}_{i_{22}}\check{c}_{i_{22}+1}\cdots c_{2n}(c_{i_{21}}c_{i_{21}+1}c_{i_{22}}c_{i_{22}+1} - 1)$$

has the same sign as that of a_n. Therefore it is sufficient to show the following value has the same sign as that of a_n:

$$G = (n-2)(n-1)a_n/2 - \sum_{**} c_1c_2\cdots \check{c}_{2k_1-1}\check{c}_{2k_1}\cdots \check{c}_{2k_2-1}\check{c}_{2k_2}\cdots c_{2n} + \epsilon(n-1).$$

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For convenience of calculation, we rewrite $c_{2k-1}c_{2k} = d_k$, and then we have $a_n = d_1 d_2 \cdots d_n$, and

$$\sum_{**} c_1 c_2 \cdots \check{c}_{2k_1-1}\check{c}_{2k_1} \cdots \check{c}_{2k_2-1}\check{c}_{2k_2} \cdots c_{2n} = \sum_{**} d_1 d_2 \cdots \check{d}_{k_1} \cdots \check{d}_{k_2} \cdots d_n.$$

From now on, we consider the following two cases: (i) n is even, and (ii) n is odd.

(i) Suppose that n is even. We remark that

$$\text{(A)} \qquad (n/2-1)a_n - \sum_{k=1}^{n/2} d_{i_1} d_{i_2} \cdots \check{d}_{i_{2k-1}} \check{d}_{i_{2k}} \cdots d_{i_n} + 1$$

$$= \sum_{k=1}^{n/2} (d_{i_1} d_{i_2} \cdots d_{2k-2} \cdot d_{i_{n-1}} d_{i_n} - 1)(d_{i_{2k-1}} d_{i_{2k}} - 1) d_{i_{2k+1}} d_{i_{2k+2}} \cdots d_{i_{n-2}}$$

has the same sign as that of $a_n = d_1 d_2 \cdots d_n$, for any permutation i of $\{1, 2, \ldots, n\}$ that is, $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$.

CLAIM 1. *The set of all pairs k_1 and k_2 satisfying $1 \le k_1 < k_2 \le n-1$ can be divided into $n-1$ disjoint families of subsets $S = \{(i_1, i_2), (i_3, i_4), \ldots, (i_{n-1}, i_n)\}$ satisfying $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$.*

The proof of Claim 1 is illustrated in Fig. 2 for the case of $n = 10$. We consider $n-1$ disjoint families of subsets: $\{(1, n), (2, n-1), (3, n-2), \ldots, (n/2, (n+2)/2)\}$, $\{(2, n), (3, 1), (4, n-1), \ldots, ((n+2)/2, (n+4)/2)\}$, $\{(3, n), (4, 2), (5, 1), \ldots, ((n+4)/2, (n+6)/2)\}, \ldots$, and $\{(n-1, n), (1, n-2), (2, n-3), \ldots, ((n-2)/2, n/2)\}$. This division satisfies the condition.

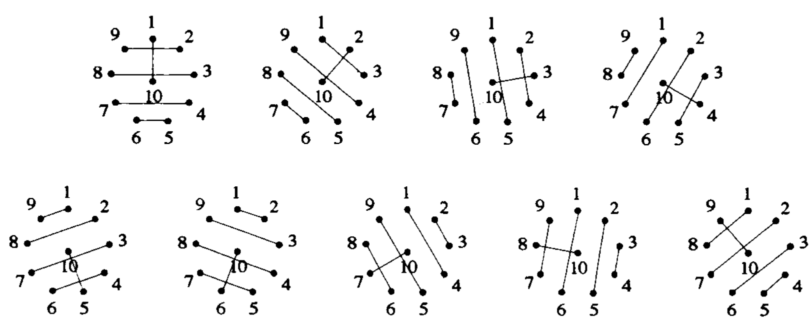

FIGURE 2.

For each subset, we consider an equation as in (A), which has the same sign as that of a_n. From this fact, it is seen that G has the same sign as that of a_n.

(ii) Suppose that n is odd. We remark that

$$\text{(B)} \qquad ((n-1)/2-1)a_n - \sum_{k=1}^{(n-1)/2} d_{i_1}d_{i_2}\cdots \check{d}_{i_{2k-1}}\check{d}_{i_{2k}}\cdots d_{i_n}$$
$$= \sum_{k=1}^{(n-3)/2}(d_{i_1}d_{i_2}\cdots d_{i_{2k-2}}\cdot d_{i_{n-2}}d_{i_{n-1}}-1)(d_{i_{2k-1}}d_{i_{2k}}-1)d_{i_{2k+1}}d_{i_{2k+2}}\cdots d_{i_{n-3}}\cdot d_{i_n} - d_{i_n}$$

for any permutation i of $\{1, 2, \ldots, n\}$. Here,

$$\sum_{k=1}^{(n-3)/2}(d_{i_1}d_{i_2}\cdots d_{i_{2k-2}}\cdot d_{i_{n-2}}d_{i_{n-1}}-1)(d_{i_{2k-1}}d_{i_{2k}}-1)d_{i_{2k+1}}d_{i_{2k+2}}\cdots d_{i_{n-3}}d_{i_n}$$

has the same sign as that of a_n.

CLAIM 2. *The set of all pairs k_1 and k_2 satisfying $1 \leq k_1 < k_2 \leq n-1$ can be divided into n disjoint families of subsets $S = \{(i_1, i_2), (i_3, i_4), \ldots, (i_{n-2}, i_{n-1}), i_n\}$ satisfying $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$.*

The proof of Claim 2 is illustrated in Fig. 3 for the case of $n = 9$. We consider n disjoint families of subsets: $\{1, (2, n), (3, n-1), \ldots, ((n+1)/2, (n+3)/2)\}$, $\{2, (3, 1), (4, n), \ldots, ((n+3)/2, (n+5)/2)\}$, $\{3, (4, 2), (5, 1), \ldots, ((n+5)/2, (n+7)/2)\}, \ldots$, and $\{n, (1, n-1), (2, n-2), \ldots, ((n-1)/2, (n+1)/2)\}$. This division satisfies the condition.

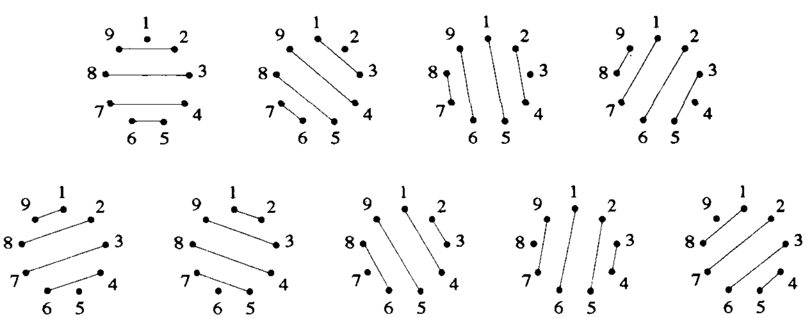

FIGURE 3.

For each subset, we consider an equation as in (B), where

$$\sum_{k=1}^{(n-3)/2}(d_{1_1}d_{i_2}\cdots d_{2k-2}\cdot d_{i_{n-2}}d_{i_{n-1}}-1)(d_{i_{2k-1}}d_{i_{2k}}-1)d_{i_{2k+1}}d_{i_{2k+2}}\cdots d_{i_{n-3}}d_n$$

has the same sign as that of a_n.

Therefore, we can arrange G as follows:

$$G = (\text{terms with the same sign as that of } a_n) - \sum_{i=1}^{n} d_i + a_n + \epsilon(n-1).$$

It is sufficient to show that $a_n - \sum_{1\le i\le n} d_i + \epsilon(n-1)$ has the same sign as that of a_n. Since $a_n = d_1 d_2 \cdots d_n$ and all d_i's are integers, we can see this immediately. Now we see that G has the same sign as that of a_n. The proof of the lemma is complete.

From the lemma, it follows that $(8n^2-15n+8)|a_n|+(3n-3) \ge |a_{n-2}|$. Furthermore, equality holds when $c_i = (-1)^i$ $(1 \le i \le 2n)$ or $c_i = (-1)^{i-1}$ $(2 \le i \le 2n+1)$. Since we suppose $a_n \ne \pm 1$, equality never holds. The proof of Theorem 3 is complete.

5. Remarks

5.1. The conditions in Theorems 1 and 2 are not sufficient. For example, a_{n-1} must be odd if a_n is odd. The condition in Theorem 3 may be improved after some effort. In the proof, we use a result corresponding to a theorem in the theory of 1-factor-decompositions for a complete graph. If we can create a similar result for a complete hyper-graph, we can apply it to estimate a_{n-j}.

5.2. There was a conjecture that the coefficients in the Alexander polynomial for a two-bridged knot have a convex property: $2|a_{n-j}| \ge |a_{n-j-1}| + |a_{n-j+1}|$. But this is false. For the two-bridge knot $S(47, 13)$ in Schubert form (which is the knot 9_{26} in [7]), the Alexander polynomial is $t^6 - 5t^5 + 11t^4 - 13t^3 + 11t^2 - 5t + 1$. Furthermore for the two-bridge knot $S(79, 49)$ in Schubert form ($= 10_{44}$), the Alexander polynomial is $t^6 - 7t^5 + 19t^4 - 25t^3 + 19t^2 - 7t + 1$. Thus we raise the following question:

QUESTION. For an arbitrary pair of integers N and j with $1 \le j \le n-1$, does there exist a two-bridge knot such that $|a_{n-j-1}| + |a_{n-j+1}| - 2|a_{n-j}| \ge N$?

The answer is affirmative when n is sufficiently greater than N and j. For example, we take integers $c_i = (-1)^{i+1}$ $(i = 1, 2, \ldots, 2M)$. If M is sufficiently greater than N and j, then the two-bridge knot corresponding to the above c_i's is as required:

$$|a_n| + |a_{n-2}| - 2|a_{n-1}| = 7M^2 - 16M + 8 > M (M \ge 2),$$
$$|a_{n-1}| + |a_{n-3}| - 2|a_{n-2}| = 32M^3/3 - 54M^2 + 244M/3 - 36 > M (M \ge 3),$$

and so on. But we can see that for this case $2|a_1| \ge |a_0| + |a_2|$.

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Department of Mathematics
Kobe University
Nada-ku
Kobe 657
Japan
e-mail: nakanisi@math.s.kobe-u.ac.jp

https://doi.org/10.1017/S1446788700037848 Published online by Cambridge University Press