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# On Cabled Knots and Vassiliev Invariants (Not) Contained in Knot Polynomials

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*Abstract.* It is known that the Brandt–Lickorish–Millett–Ho polynomial $Q$ contains Casson's knot invariant. Whether there are (essentially) other Vassiliev knot invariants obtainable from $Q$ is an open problem. We show that this is not so up to degree 9. We also give the (apparently) first examples of knots not distinguished by 2-cable HOMFLY polynomials which are not mutants. Our calculations provide evidence of a negative answer to the question whether Vassiliev knot invariants of degree $d \leq 10$ are determined by the HOMFLY and Kauffman polynomials and their 2-cables, and for the existence of algebras of such Vassiliev invariants not isomorphic to the algebras of their weight systems.

## 1 Introduction and Historical Motivation

The standard definition of a Vassiliev invariant [BL, BN, BS, Va, Vo] of degree at most $d$ is to be an invariant vanishing on $d+1$-singular knots. Vassiliev invariants are a class of knot invariants, which can be associated in many ways with polynomials. One such analogy is to think of singularity resolutions as a way to differentiate a knot invariant, and in this setting the Vassiliev invariants are (as polynomials) functions with a vanishing derivative. An extension of this idea is the approach of braiding sequences and braiding polynomials, which was initiated in a special case in [Tr] and later developed in [St]. It provides a method of studying Vassiliev invariants via their polynomial behaviour on certain sequences of knots. This approach works directly on knots and so it is a counterpart to the classical approach of chord diagrams. Another relation to polynomials was conjectured by Lin and Wang [LW], asserting that (the values of) Vassiliev invariants are polynomially bounded in the crossing number of knots. The first substantial application of the approach of braiding sequences [St7] was to give a new proof of the statement conjectured by Lin and Wang. (It was proved previously by Bar-Natan [BN2], and also by Stanford [S].) Later [St2] this proof was extended to Vassiliev invariants of links of arbitrary number of components. Recently a paper by Eisermann [Ei2] appeared which, apart from the application to $S^1 \times S^2$, covers some initial part of our braiding sequence theory [St, St7, St2]. This also illustrates how braiding sequences are a natural concept.

Birman and Lin explained how to obtain Vassiliev invariants from the link polynomials (or polynomials of cables) [BL]. Since this procedure is *a priori* not exhaustive, it is not straightforward to prove that some Vassiliev invariant $v$ is actually *not* obtainable from the link polynomials (or cables). The only way is to find knots not

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distinguished by the polynomials (or cables), but by $v$, as in [K4, St6]. Unfortunately, in particular for cables, coincidences of polynomials are rare, and this makes the task difficult. It was known [LL] that mutants [Co] have equal 2-cable skein (or HOMFLY) $P$ [FY, LM] and Kauffman $F$ [Ka2] polynomials, and that they are not distinguished by Vassiliev invariants of degree $d \leq 10$ [Mr]. This led to the question whether all such invariants are determined by the skein and Kauffman polynomials and their 2-cables.

A different suggestive problem with Vassiliev invariants is to decide for a given invariant whether it is such or not. Usually, a knot invariant is a Vassiliev invariant or can be excluded from being such by rather elementary means (as far as the Vassiliev invariant part of the argument goes) [De, Tr, Bi, Ei]. However, we introduced a certain type of invariants that satisfy similar polynomial behaviour, but in some weaker sense than Vassiliev invariants [St2]. We called such invariants extended Vassiliev invariants. As an extended Vassiliev invariant behaves polynomially on braiding sequences, it becomes difficult to recognize it as not of finite degree. The first class of examples of such invariants [St2] are the derivatives of the Brandt–Lickorish–Millett–Ho polynomial $Q$ [BLM, Ho] evaluated at $-2$. Kanenobu had been studying the values $Q^{(k)}(-2)$ earlier. For knots $Q(-2) \equiv 1$, and by his result [K] we have $Q'(-2) = V''(1)$, with $V$ the Jones polynomial [J], which is the Vassiliev invariant of degree 2. (A similar statement holds for links, which we do not discuss here, since in this case the further terms occurring are products of linking numbers, which are Vassiliev invariants of degree 1.) Kanenobu [K2, Theorem 1] found a formula expressing the $Q$ polynomial of a rational (2-bridge) knot in terms of its Jones polynomial. A consequence of this formula is that $Q^{(k)}(-2)$ on rational knots equals a polynomial of degree $\leq 2k$ in the derivatives of $V(t)$ at $t = 1$ (where the $n$-th derivative is taken to be of degree $n$). Hence the restriction of $Q^{(k)}(-2)$ to rational knots is a Vassiliev invariant of degree $\leq 2k$.

It turns out to be rather difficult to examine the finite degree property for $Q^{(k)}(-2)$ on arbitrary knots. Apparently they are not Vassiliev invariants (see §3.3). However, as also independently observed by Kanenobu, the previous easy arguments will not suffice to show this. Whether $Q^{(k)}(-2)$ are Vassiliev invariants (and of which degree, in the unlikely event that they are) remains an open problem.

The actual origin for the considerations in [St2] was the search for a way to obtain Vassiliev invariants out of the $Q$ polynomial. The polynomials $V$, $P$ and $F$, and the Alexander–Conway polynomial $\nabla$ [Al, Co] have been treated in [BL, BN], but apparently $Q$ received little attention. Unfortunately, as the previous remarks already suggest, beyond degree 2 the question whether (or how) one can obtain Vassiliev invariants from $Q$ seems rather difficult. Our aim here will be to provide a negative answer up to degree 9. This problem was investigated independently in a recent paper by Choi, Jeong, and Park [CJP].

This paper has two main parts. In Section 3, we explain how to show that $Q$ determines no low degree Vassiliev invariants, and settle degree up to 7. To that extent the problem is treated with a more detailed argument and mainly in its own right. Then in Section 4 we are led to consider invariants of 2-cable knots and links for degrees 8 and 9. Here the application to the problem requires more of an explanation of our computation. This computation has other noteworthy implications. In particular, it provides some evidence that not all Vassiliev knot invariants of degree $\leq$ 10 are determined by the HOMFLY and Kauffman polynomials and their 2-cables. It also turns up the (apparently) first examples of knots not distinguished by 2-cable HOMFLY polynomials which are not mutants (because distinguished by 2-cable Kauffman polynomials and by hyperbolic volume), and determines the braid index of prime knots up to 12 crossings.

We should mention that some of our calculations are related to work by Meng [Me] and Lieberum [Li], and extend similar previous calculations in degree up to 6 due to Kanenobu [K4]. We will make some remarks that put these and other results into our context. For the computations, various programs written in C++ and Mathematica™ were used, as well as some tools included in the program KnotScape [HT].

## 2 Notations and Basic Terminology

### 2.1 General Notations

**Z**, **N**, $\mathbf{N}_+$, **Q**, **R** and **C** denote the integer, natural, positive natural, rational, real and complex numbers, respectively.

For a set $S$, the expressions $|S|$ and $\#S$ are equivalent and both denote the cardinality of $S$. In the sequel the symbol $\subset$ denotes a not necessarily proper inclusion.

An expression containing an asterisk $*$ *subscript* is meant to denote the union of all expressions in which the asterisk is replaced by all values that make sense, including omission. Contrarily, an asterisk as *superscript* is meant to denote the dual of a space.

Let $[Y]_{t^a} = [Y]_a$ be the coefficient of $t^a$ in a polynomial $Y \in \mathbf{Z}[t^{\pm 1}]$. For $Y \neq 0$, let $\mathcal{C}_Y = \{\, a \in \mathbf{Z} : [Y]_a \neq 0 \,\}$ and

$$\min\deg Y = \min \mathcal{C}_Y, \quad \text{and} \quad \max\deg Y = \max \mathcal{C}_Y,$$

be the minimal and maximal degree of $Y$, respectively. Similarly one defines for $Y \in \mathbf{Z}[x_1, \dots, x_n]$ the coefficient $[Y]_X$ for some monomial $X$ in the $x_i$, and $\min\deg_{x_i} Y$, etc.

We use the following encoded notation for 1-variable polynomials: if the absolute term occurs between the minimal and maximal degrees, then it is bracketed, else the minimal degree is recorded in braces before the coefficient list [St3]. It is the same as the notation used by Adams [Ad, appendix], or the one used by Lickorish–Millett [LM, appendix] for the $m$-coefficients of $P$, whichever is shorter.

### 2.2 Knots and Knot Diagrams

The *crossing number* $c(L)$ of a link $L$ is the minimal number of crossings $c(D)$ of all diagrams $D$ of $L$, *cf.* [Ka]. The *braid index* $b(L)$ of $L$ is the minimal number of strands of a braid whose closure is $L$, *cf.* [Mo, FW].

The diagram on the right of Figure 1 is called the *connected sum* $A\#B$ of the diagrams $A$ and $B$. If a diagram $D$ can be represented as the connected sum of diagrams $A$ and $B$, such that both $A$ and $B$ have at least one crossing, then $D$ is called *composite*, else it is called *prime*. A knot $K$ is *prime* if, whenever $A\#B$ is a composite diagram of $K$, one of $A$ or $B$ (but not both) represents an unknotted arc. Otherwise, if $K$ is not the unknot, $K$ is called *composite*. $K$ is the connected sum $K = K_1\#K_2$ of $K_1$ and $K_2$, with $K_1$ represented by $A$ and $K_2$ by $B$.

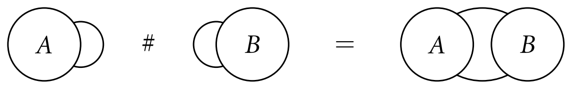


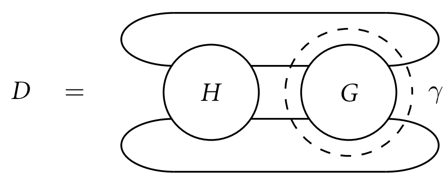

*Figure 1*

Prime knots are denoted according to [Ro, appendix] for up to 10 crossings and according to [HT] for $\geq 11$ crossings. We number non-alternating knots after alternating ones. So for example $11_{216} = 11_{a216}$ and $11_{484} = 11_{n117}$.

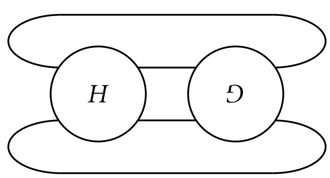
The *obverse* (mirror image) of $K$ is denoted $!K$. If $K =!K$, then $K$ is called *achiral*. For a knot invariant $v$, define the invariant $v!$ by $v!(K) = v(!K)$. If $v = v!$ (resp., $v = -v!$), $v$ is called *symmetric* (resp., *antisymmetric*).

$K$ is called *rational* (2-*bridge*) if it has a diagram on which the one (planar) coordinate has exactly two local minima (or two local maxima) [Sh].

Given a knot diagram $D$ and a closed curve $\gamma$ intersecting $D$ in exactly four points, $\gamma$ defines a *tangle decomposition* of $D$.




A *mutation* of $D$ is obtained by removing one of the tangles in some tangle decomposition of $D$ and replacing it by a version rotated 180° along the axis vertical to the projection plane, or horizontal or vertical in the projection plane. For example:




(To make the orientations compatible, possibly the orientation of either $H$ or $G$ must be altered.) Then $\gamma$ is called the *Conway circle* for this mutation. If some knots $K_{1,2}$ have diagrams differing by a mutation, then $K_{1,2}$ are called *mutants* [Co]. We call $K$ an iterated mutant of $K'$, if there are knots $K = K_1, K_2, \ldots, K_n = K'$ with $K_i$ and $K_{i+1}$ being mutants. In the following, we will abuse the word "iterated" when referring to mutants but assume it implicitly.

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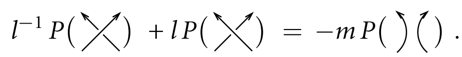
### 2.3 Link polynomials

As for knot invariants, our notation is also the usual one: $\Delta(t)$ denotes the Alexander [Al], $\nabla(z)$ the Conway [Co], $V(t)$ the Jones [J] (see also [Ka]), $P(l,m)$ the HOMFLY (skein) [LM, FY], $F(a,z)$ the Kauffman [Ka2], and $Q(z)$ the Brandt–Lickorish–Millett–Ho polynomial [BLM, Ho]. In our convention the skein and Kauffman polynomials are conjugate (that is, obtained by replacing $a$ by $a^{-1}$ in $F$ and $l$ by $l^{-1}$ in $P$) to those in [LM, Ka2]. The local relations in this convention will be given below. We assume $\Delta$ is normalized so that $\Delta(1) = 1$ and $\Delta(t^{-1}) = \Delta(t)$. For $V$ and $Q$ the conventions (also used here) are fairly standard.

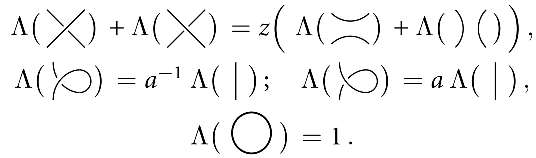
The *skein HOMFLY polynomial* $P(l,m)$ is a Laurent polynomial in two variables $l$ and $m$ of oriented knots and links and can be defined by being 1 on the unknot with the (skein) relation

(1) 

We call the crossings in the first two fragments *positive* and *negative*, respectively. The sum of the signs ($\pm 1$) of the crossings of a diagram $D$ is called *writhe* of $D$ and written $w(D)$. The writhe is invariant under simultaneous reversal of orientation of *all* components of the diagram, so is in particular well defined for unoriented *knot* diagrams.

The *Conway polynomial* $\nabla$ [Co], given by $\nabla(z) = P(\sqrt{-1}, \sqrt{-1}z)$, is well known to be equivalent to the (1-variable) *Alexander polynomial* $\Delta$ by a variable substitution: $\Delta(t) = \nabla(t^{1/2} - t^{-1/2})$. Another well-known property of $\nabla$ is that for any link $L$ we have $[\nabla_L(z)]_{z^i} = 0$, if $i$ has the same parity as the number $n(L)$ of components of $L$, and that $z^{n(L)-1} \mid \nabla_L(z)$. For a knot $K$ we always have $[\nabla_K(z)]_{z^0} = 1$.

For the *Kauffman polynomial* $F$, we have (in our convention) the relation $F(D)(a,z) = a^{w(D)}\Lambda(D)(a,z)$, where $w(D)$ is the writhe of $D$, and $\Lambda(D)$ is the writhe-unnormalized version of $F$. Then $\Lambda$ is given in our convention by the properties



Thus the positive (right-hand) trefoil has $\min\deg_a F = 2$.

The *Brandt–Lickorish–Millett–Ho polynomial* is given by $Q(z) = F(1,z)$, and the *Jones polynomial* by

$$V(t) = F(-t^{3/4}, t^{1/4} + t^{-1/4}) = P(-\sqrt{-1}t, \sqrt{-1}(t^{-1/2} - t^{1/2}))\,.$$

(See [Ka2, §III] and [LM].)

$Q$ and $\nabla$ (and hence $\Delta$) are symmetric *knot* invariants, *i.e.*, coincide on $K$ and $!K$ for any knot $K$. ($Q$ is symmetric also for links, while $\nabla$ is symmetric or antisymmetric depending on the parity of the number of components.) $V$, $P$ and $F$ differ on mirror

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images under conjugation of a variable:

$$V!(t) = V(t^{-1}), \tag{2}$$

$$F!(a, z) = F(a^{-1}, z), \tag{3}$$

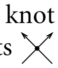
$$P!(l, m) = P(l^{-1}, m). \tag{4}$$

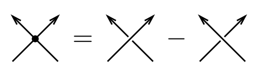
All polynomials $X \in \{F, P, Q, V, \Delta\}$ are multiplicative under the connected sum: $X(K_1 \# K_2) = X(K_1)X(K_2)$.

By $\operatorname{vol}(L)$ we denote the (finite) volume of the (unique if it exists) hyperbolic structure on the complement $S^3 \setminus L$ of a link $L$ in $S^3$ (that is, a representation $S^3 \setminus L = H^3/\Gamma$, where $H^3$ is the 3-dimensional hyperbolic space, and $\Gamma$ is a properly discontinuously acting discrete group of isometries of $H^3$). We write $\operatorname{vol}(L) = 0$ if $S^3 \setminus L$ has no hyperbolic structure.

## 3 Vassiliev Invariants

### 3.1 Generalities

Consider the linear space $\mathcal{V}$, (freely) generated by all the (isotopy classes of) knot embeddings. Let $\mathcal{V}^d$ be the space of singular knots with exactly $d$ double points  (up to isotopy). $\mathcal{V}^d$ can be identified with a linear subspace of $\mathcal{V}$ by resolving the singularities into the difference of an overcrossing and an undercrossing via the rule

(5) 

where all the rest of the knot projections are assumed to be equal. This yields a filtration of $\mathcal{V}$

$$\mathcal{V} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \mathcal{V}^2 \supset \mathcal{V}^3 \supset \cdots . \tag{6}$$

There is a combinatorial description of the graded vector space,

$$\bigoplus_{d=0}^{\infty} \left( \mathcal{V}^d \middle/ \mathcal{V}^{d+1} \right) \tag{7}$$

associated with this filtration, namely

$$\mathcal{A}_d := \mathcal{V}^d \middle/ \mathcal{V}^{d+1} \simeq \mathcal{L}in\,\{\text{chord diagrams of degree } d\} \middle/ \begin{matrix}\text{4T relation}\\ \text{FI relation}\end{matrix}, \tag{8}$$

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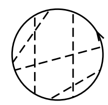
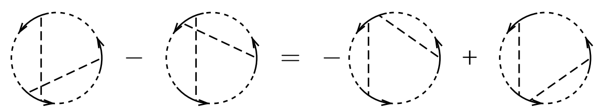
where $\mathcal{L}in$ denotes linear span, and the *chord diagrams* (CDs) are objects like this (an oriented circle with finitely many dashed chords in it, up to isotopy)



and are graded by the number of chords. The 4T *(4 term) relations* have the form

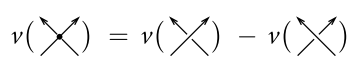


and the FI (*framing independence) relation* requires that each CD with an *isolated* chord, *i.e.,* a chord not crossed by any other one, is zero.

The map which yields the isomorphism (8) is a simple way to assign a CD $D_K$ to a singular knot $K$. In the parameter space of $K$ (which is an oriented $S^1$), connect pairs of points with the same image by a chord. When adding arrows for the crossings of $K$ oriented from the preimage of the undercrossing to the preimage of the overcrossing, we obtain a (singular) *Gauss diagram*; see [PV, St5].

The *connected sum* of chord diagrams is defined by $D_{K_1}\#D_{K_2} = D_{K_1\#K_2}$ (well up to the 4T relation).

We define a knot invariant $v$ to be a *Vassiliev invariant of degree* $\leq d$ if, when extended to singular knots via



it vanishes on $(d+1)$-singular knots. The *degree* $\deg v$ of $v$ is (suggestively) the smallest integer $d$ such that $v$ is of degree $\leq d$. Several properties and constructions of Vassiliev (finite degree) invariants were known from [BL, BN]. In particular, introducing $\mathcal{V}_d$ to be the linear space of Vassiliev invariants of degree $\leq d$, the space $\mathcal{V}_d/\mathcal{V}_{d-1}$ is isomorphic to the dual $\mathcal{A}_d^*$ of the linear space $\mathcal{A}_d$ of chord diagrams of $d$ chords modulo the 4T relation. Elements in $\mathcal{A}_d^*$ are called *weight systems* (of degree $d$). Each $v \in \mathcal{V}_d$ gives rise to a weight system $W_v \in \mathcal{A}_d^*$ by evaluating it on a $d$-singular knot representing the chord diagram, $W_v(D_K) := v(K)$. The bijectivity of this assignment is dual to the isomorphism (8), and is established using a *universal Vassiliev invariant*, such as the *Kontsevich-integral* $Z$ [Ko]. The application $W_v \circ Z$ of the weight system of $v \in \mathcal{V}_d$ on the Kontsevich-integral gives back $v$ modulo lower degree invariants:

$$v(K) \equiv (W_v \circ Z)(K) = W_v(Z(K)) \mod \mathcal{V}_{d-1}\,.$$

If in fact $v = W_v \circ Z$, we call $v$ *canonical*; see [BG].

Vassiliev invariants are easily seen to form an algebra with usual addition and multiplication, and the structure of this algebra was known to be the free symmetric (polynomial) graded algebra generated by *primitive* Vassiliev invariants. Such invariants $v$ are given by the additional property that $v(K_1\#K_2) = v(K_1) + v(K_2)$ for any knots $K_{1,2}$.

### 3.2 Deterministic Sets for Vassiliev Invariants

Since the space of Vassiliev invariants of given degree is finite-dimensional, there exist finite sets $\mathcal{K}_d$ of knots, the values on which determine uniquely a Vassiliev invariant of degree $\leq d$. Equivalently, we say

***Definition 3.1*** A set $\mathcal{K}_d$ of knots is *d-deterministic*, if any Vassiliev invariant of degree at most $d$ vanishing on $\mathcal{K}_d$ vanishes identically. It is called *d-primitive deterministic* if this property holds for primitive Vassiliev invariants of degree at most $d$.

In practice, it is desirable to choose a $d$-deterministic set as small as possible. The minimal size is clearly $\dim \mathcal{V}_d$, and many such sets of this cardinality exist, but no one knows how to find any of them except by computation for a few small values of $d$. Thus we may try to find larger sets which are provable to be $d$-deterministic. This problem has been considered (including for links) in several previous papers of the author (see [St7]), and estimates on the crossing number of knots in one particular $d$-deterministic set $\mathcal{K}_d$ were given. The estimates, however, are not optimal. For our subsequent purposes, we will derive a more efficient estimate for knots. It is formulated in the following lemma, which is needed to make the later arguments more rigorous.

***Lemma 3.2*** *For any $d > 0$, the set of knots with (prime) diagrams of at most $d + 1 + \frac{d(d-2)}{4}$ crossings is d-(primitive) deterministic.*

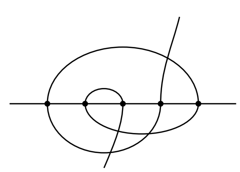
***Remark*** Note that knots with prime diagrams may well be composite, and so we do not make any claim as to the primeness of the knots represented by our diagrams.

**Proof** We use the result of [CD] that chord diagrams modulo the 4T-relation and composite chord diagrams are generated by such with a *special chord* (that is, a chord intersecting all the others). Note (as in [CD]) that such a chord diagram is described by a permutation of the endpoints of the non-special chords.

Thus it suffices to consider chord diagrams with a special chord or connected sums of such diagrams. To realize a prime chord diagram with $d$ chords, including a special one, by a singular knot diagram, put $d - 1$ singular crossings on a straight line.

(9) 

First assume $d$ is even. The other strand must pass through these singular crossings in some (arbitrary) permuted order. Its part above and below the line in (9) consists of $\frac{d}{2} - 1$ arcs joining two singular crossings and one arc connected to the remaining singular crossing with a "loose end":

(10) 

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Clearly any two of these $\frac{d}{2}$ arcs can be made to have at most one intersection. Thus the strand can be made to have at most $2\binom{d/2}{2}$ self-intersections. There remain the $d$ singular crossings. Call the one of the special chord the *special singular crossing*. One additional (non-singular) crossing is needed for the second (generally self-intersecting) strand in (10) to exit the loop made up of the first strand between the two passes of the special singular crossing.

In case $d$ is odd, one side of (9) contains $\frac{d-1}{2}$ arcs joining two singular crossings, and the other $\frac{d-3}{2}$ such arcs, and two arcs with a loose end. Then one has at most

$$d+1+\binom{(d-1)/2}{2}+\binom{(d-3)/2}{2}+2\cdot\frac{d-3}{2} = \frac{(d-1)^2}{4}+d \leq d+1+\frac{d(d-2)}{4}$$

crossings.

Now with $f(d) := d+1+\frac{d(d-2)}{4}$, we have $f(d) \geq \sum_{i=1}^{k} f(d_i)$, when $d_i \geq 2$ and $\sum_{i=1}^{k} d_i = d$. This establishes the assertion of the lemma for arbitrary Vassiliev invariants. Now considering primitive Vassiliev invariants, we can restrict ourselves to chord diagrams which are not connected sums. Thus, we must argue why the (singular) knot diagrams representing prime chord diagrams with a special chord are prime. It is easy to see that each arrow of a non-singular crossing intersects a chord of a singular crossing. Then the intersection graph of the (singular) Gauß diagram is connected, which (see [St5]) is equivalent to the knot diagram being prime. ■

***Corollary 3.3*** *A primitive Vassiliev invariant of degree* $\leq 4$ *is determined by its values on rational knots. (See also* [K4].*)*

**Proof** Knots with prime $\leq 7$ crossing diagrams are all rational. ■

### 3.3 Vassiliev Invariants Derived from the Polynomials

From [BN, BL] we know that the Conway, Jones and Kauffman polynomials give rise to Vassiliev invariants. We recall that there is a relation between the Conway–Vassiliev invariants $\nabla_i = [\nabla]_{z^i}$ and the Kauffman–Vassiliev invariants; see [K3], given by

$$(11) \qquad F_{i,j}(K) := \sqrt{-1}^{\,i+j}\, \frac{d^j}{da^j}\Big|_{a=\sqrt{-1}} [F(K)]_{z^i}\,.$$

By [BL], this is a Vassiliev invariant of degree $\leq i+j$. (Since $F_{i,0} \equiv \delta_{i,0}$ is constantly 1 or 0, we can assume $j > 0$.)

We have the identity $F_{1,1} = -2\nabla_2$ coming from the uniqueness of the (symmetric) Vassiliev invariant of degree 2. For higher degree, the evident problem is that the dimension of the space of Vassiliev invariants grows rapidly. The only further relation to the Conway Vassiliev invariants is (see [K3, p. 422])

$$(12) \qquad \frac{F_{2,1}+F_{2,2}}{2} - 6F_{3,1} = \nabla_2 - 7\nabla_2^2 + 18\nabla_4\,.$$

With Lemma 3.2 in hand, the verification of such identities (at least in not too high a degree) is straightforward.

For $i > 4$, $\nabla_i$ cannot be expected to be related to the $F_{i',j'}$. Indeed, $\nabla_6$ is not contained in $F$, as shown in [K4, St6]. That is, there are two distinct knots $K_1$ and $K_2$ with $F(K_1) = F(K_2)$, but $\nabla_6(K_1) \neq \nabla_6(K_2)$. For instance, $K_1$ and $K_2$ can be taken to be the two 11 crossing knots $11_{30}$ and $!11_{189}$ with equal Kauffman polynomial, but different Conway polynomial, as pointed out by Lickorish [L]. As observed by Kanenobu, for the higher $\nabla_i$ the same property then follows by taking the connected sum of the $K_{1,2}$ with trefoils.

The Jones polynomial gives rise to a series of Vassiliev invariants by its values $V^{(n)}(1)$. The skein polynomial $P$ yields Vassiliev invariants in the same way as $F$. For a link $L$,

$$P_{i,j}(L) := \sqrt{-1}^{\,i+j} \left.\frac{d^j}{dl^j}\right|_{l=\sqrt{-1}} [P(L)]_{m^i}, \tag{13}$$

is a Vassiliev invariant of degree $\leq i + j$. However, here rather than $j > 0$ we must pose $j \geq 0$ and $i$ of the opposite parity to the number of components $n(L)$ of $L$, and $i \geq 1 - n(L)$. (Remark that for $j = 0$ we obtain, up to sign, the $\nabla_i$.)

As for $Q$, the results of Kanenobu, explained in §1, suggest that we consider the values $Q^{(k)}(-2)$ for $k \geq 2$. Here we are less fortunate, and the following is easy to see.

***Proposition 3.4*** *$Q''(-2)$ is not a (global) Vassiliev knot invariant of degree $\leq 4$.*

**Proof** Assume $v = Q''(-2)$ is a Vassiliev invariant of degree $\leq 4$. Using $Q(-2) \equiv 1$, one can correct $v$ by a multiple of $Q'(-2)^2$ to a Vassiliev invariant $\bar{v}$ that is additive under connected sum, and so primitive. By Corollary 3.3, we have that $\bar{v}$ is determined by its values on rational knots, and Kanenobu's formula [K2] shows that on rational knots $\bar{v}$ can be expressed using $V^{(n)}(1)$. Since this expression is also a Vassiliev invariant of degree $\leq 4$, it would extend to all knots. Since also $Q'(-2)$ can be expressed from $V$ using [K], we obtain that $v$ is determined by $V$ (on all knots). Then any pair of knots with equal (or conjugate) $V$ would have equal $Q''(-2)$. But the pair $5_1$ and $10_{132}$ shows that this is not the case. We quote their $V$ and $Q$ polynomials from [St3] using encoded notation:

$$V(5_1) = V(10_{132}) = \{2\}\ 1\ 0\ 1\ {-1}\ 1\ {-1},$$
$$Q(5_1) = [5]\ {-2}\ {-6}\ 2\ 2, \qquad Q(10_{132}) = [5]\ {-18}\ {-14}\ 38\ 20\ {-24}\ {-12}\ 4\ 2.$$

We thus obtain a contradiction. ■

The fact that $Q''(-2)$ is not of degree $\leq 4$ was observed by Kanenobu with similar reasoning. Of course, this argument can only work in low degree, but a more general argument for arbitrary degree and arbitrary derivative is not obvious.

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### 3.4 Braiding Sequences

The approach of braiding sequences gives another motivation for the non-triviality of the finite degree property question on the derivatives of the Brandt–Lickorish–Millett–Ho polynomial evaluated at $z = -2$. It also suggests similar phenomena for the evaluations at $z = 2$.

***Definition 3.5*** ([St]) For some odd $k \in \mathbf{Z}$, a (parallel) $k$-braiding of a crossing $p$ in a diagram $D$ is a replacement of (a neighborhood of) $p$ by the braid $\sigma_1^k$. A braiding sequence $\mathcal{B}_{D,P}$ (associated to a numbered set $P$ of crossings in a diagram $D$; all crossings by default) is a family of diagrams, parametrized by $n = |P|$ odd numbers $x_1, \ldots, x_n$, each one indicating that at the crossing numbered as $i$ an $x_i$-braiding is done.

Figure 2 shows the parallel $-3$-braiding and the antiparallel one. The theory for antiparallel braidings is almost equivalent, but for convenience the reader may assume that only parallel braidings are done.

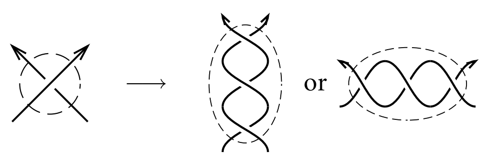


*Figure 2*: Two ways to do a $-3$-braiding at a crossing.

***Definition 3.6*** If for a knot invariant $v$ and any braiding sequence $\mathcal{B}_{D,P}$ with $|P| = n$, the map

$$\mathcal{P}_{D,P} \colon (x_1, \ldots, x_n) \mapsto v(D(x_1, \ldots, x_n))$$

is a polynomial, we call $v$ a *braiding polynomial* invariant. We call $\mathcal{P}_{D,P}$ the braiding polynomial of $v$ on $\mathcal{B}_{D,P}$.

***Theorem 3.7*** ([St]) *A knot invariant $v$ is a Vassiliev invariant of degree* $\deg v \leq d$ *if and only if it is a braiding polynomial invariant and all its braiding polynomials have degree* $\deg \mathcal{P}_{D,P} \leq d$ *for any $\mathcal{B}_{D,P}$. Herein, degree is counted in* all *variables altogether, that is, with respect to*

$$\deg \prod_{i=1}^{n} x_i^{l_i} = \sum_{i=1}^{n} l_i .$$

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***Definition 3.8*** A knot invariant is an *extended Vassiliev invariant of degree* $\leq d$, if it is a braiding polynomial invariant and for any $\mathcal{B}_{D,P}$ its braiding polynomial has degree $\deg_{x_j} \mathcal{P}_{D,P} \leq d$ in *any single* $x_j \in \{x_1, \dots, x_n\}$ with $|P| = n$, that is, with respect to

$$\deg_{x_j} \prod_{i=1}^{n} x_i^{l_i} = l_j.$$

***Example 3.9*** The determinant $\Delta(-1) = V(-1)$ is an extended Vassiliev invariant of degree 1, *if one restricts oneself to braiding sequences of antiparallel braidings only*. The squared determinant $\Delta(-1)^2 = Q(2)$ is an extended Vassiliev invariant of degree 2 (also for parallel braidings).

***Theorem 3.10*** ([St7]) *The invariants* $Q^{(k)}(-2)$ *are extended Vassiliev invariants of degree* $\leq 2k$*. The invariants* $Q^{(k)}(2)$ *are extended Vassiliev invariants of degree* $\leq 2k+2$*.*

This leads to a suggestive, but not very easy to answer, question:

***Question 3.11*** Are $Q^{(k)}(\pm 2)$, or polynomial expressions thereof, (ordinary) Vassiliev invariants?

***Definition 3.12*** A knot invariant $v$ is called *polynomially bounded of degree* $\leq d$ if there is a constant $C > 0$ such that $|v(K)| \leq C\, c(K)^d$ for any knot $K$. (Here $c(K)$ is the crossing number of §2.2.)

The following is the polynomial growth conjecture [LW], proved in [BN2, S] for knots, and in [St7] for links.

***Theorem 3.13*** *Vassiliev invariants of degree* $\leq d$ *are polynomially bounded of degree* $\leq d$*.*

Since the determinant is not a polynomially bounded invariant, it is not a Vassiliev invariant, and thus extended Vassiliev invariants are a non-trivial notion.

Now, we prove the following straightforward, but useful criterion

***Theorem 3.14*** *A knot invariant is a Vassiliev invariant (of degree* $\leq d$*) if and only if it is a polynomially bounded (of degree* $\leq d$*) and a braiding polynomial invariant.*

**Proof** The "only if" part follows from our previous results. Now assume $v$ is a braiding polynomial. We also assume that $|v(K)| < C\, c(K)^d$ for all $K$, and wish to conclude that $\deg \mathcal{P}_{D,P} \leq d$ for all $\mathcal{B}_{D,P}$. Assume that for some $\mathcal{B}_{D,P}$ we have $d_{D,P} := \deg \mathcal{P}_{D,P} > d$. Let $\mathcal{Q}_{D,P} = [\mathcal{P}_{D,P}]_{d_{D,P}} \neq 0$ be the homogeneous degree-$d_{D,P}$-part of $\mathcal{P}_{D,P}$. There are odd (in fact, positive) numbers $k_1, \dots, k_n$ with $\mathcal{Q}_{D,P}(k_1, \dots, k_n) \neq 0$. Then consider the diagrams $D_p := D(k_1 p, \dots, k_n p)$ for odd $p \to \infty$. (By proper choice of sign of $k_i$ one can achieve that $D_p$ is alternating.) The map $p \mapsto v(D_p)$ is a polynomial in $p$ of degree $d_{D,P} > d$, and the crossing number of $D_p$ is linearly bounded in $p$, so that for the knots $K_p$ represented by $D_p$ we have $|v(K_p)|$ growing faster than $c(K_p)^d$, a contradiction. ■

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***Corollary 3.15*** *The invariants* $Q^{(k)}(\pm 2)$, *or polynomial expressions thereof, are Vassiliev invariants (of some degree) if and only if they are polynomially bounded (of that degree).*

## 3.5 Vassiliev Invariants Not Obtained from the $Q$ Polynomial

### 3.5.1 An Example for Degrees 3 and 4

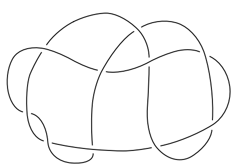
The first purpose of our investigation is to show the following statement. It explains the method of computation that is later extended to higher degrees.

***Proposition 3.16*** *Q does not contain a Vassiliev knot invariant of degree* 3 *or* 4 *that is substantial,* i.e., *not a linear combination of composite and lower degree ones.*

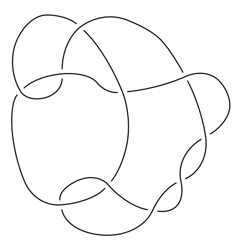
**Proof** First we recall that it does not make sense to look for a Vassiliev invariant of degree 3 (or any other odd degree), as $Q$ is a symmetric invariant [St4]. (Even non-mutually obverse examples with the same Brandt–Lickorish–Millett–Ho polynomial and different Vassiliev invariants of degree 3 are easily found, *e.g.*, $9_{12}$ and $10_{156}$.)

As is well known (see [BN, KM]), the linear space of primitive Vassiliev invariants of degree 4 (modulo degree $\leq$ 3) is 2-dimensional and generated by the projections on it of the degree 4 Vassiliev invariants $c_4$ coming from the Conway–Alexander polynomial and $v_4$ coming from the Jones polynomial.

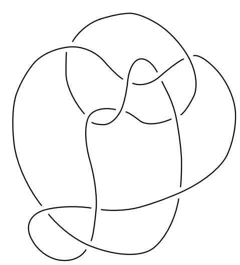
$$Q = \ [5]\ -6\ -20\ 28\ 30\ -30\ -26\ 8\ 10\ 2$$



$10_{19}$



$10_{36}$



$11_{454}$

$V = -1\ 2\ -3\ 6\ [-7]\ 8\ -8\ 7\ -5\ 3\ -1$

$\Delta = 2\ -7\ 11\ [-11]\ 11\ -7\ 2$

$V = 1\ [-2]\ 4\ -6\ 8\ -8\ 8\ -6\ 4\ -3\ 1$

$\Delta = -3\ 13\ [-19]\ 13\ -3$

$V = 1\ -3\ 5\ -7\ 8\ -9\ 8\ -5\ 4\ [-1]$

$\Delta = 1\ -5\ 12\ [-15]\ 12\ -5\ 1$

*Figure 3*: Three knots with the same $Q$ polynomial, showing that it cannot contain any interesting Vassiliev invariant of degree 4, and their $V$ and $\Delta$ polynomials (all recorded as in [St3]).

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As $Q$ contains $v_2$ and hence $v_2^2$, we may waive primitivity and adjust $c_4$ and $v_4$ in whichever way we like, only taking care that $v_4$ has no part in degree 3, *i.e.*, is symmetric. (Clearly $c_4$ is so, in whichever way we choose it, as is $\Delta$.) Thus, consider

$$c_4 := \frac{1}{24}\,\Delta^{(4)}(1) \quad \text{and} \quad v_4 := \frac{1}{12}\big(V^{(4)}(1) + 6V^{(3)}(1)\big)$$

(for which $v_4(!K) = v_4(K)$ is straightforwardly checked).

If an invariant of the kind $av_4 + bc_4$ for some $a, b \in \mathbf{R}$ is contained in $Q$, and for two knots $K_1, K_2$ we have $Q(K_1) = Q(K_2)$, then

$$b\,(c_4(K_1) - c_4(K_2)) + a\,(v_4(K_1) - v_4(K_2)) = 0. \tag{14}$$

Thus, to show that it is not the case for any $(a, b) \neq (0, 0)$, it suffices to find a triplication of $Q$, that is, knots $K_1$, $K_2$ and $K_3$ with $Q(K_1) = Q(K_2) = Q(K_3)$, such that

$$\det \begin{pmatrix} c_4(K_1) - c_4(K_2) & c_4(K_1) - c_4(K_3) \\ v_4(K_1) - v_4(K_2) & v_4(K_1) - v_4(K_3) \end{pmatrix} \neq 0. \tag{15}$$

Such an example is the triple $10_{19}$, $10_{36}$ and $11_{454}$. (This is one of the two triplications of $Q$ I found in Hoste–Thistlethwaite's tables [HT] of $\leq 11$ crossing prime knots.) We let the reader verify (15), just recording their polynomials in Figure 3. ∎

Thus, unfortunately, there seems no easy way, *e.g.*, to show via Vassiliev invariants (as it works for $V$; see [St5, corollary 7.1]) that the untwisted Whitehead doubles of a positive or almost positive knot have non-trivial $Q$ polynomial. This was my original motivation for a large part of the investigations described in [St2].

### 3.5.2 Vassiliev Invariants up to Degree 7

Now we explain how to extend our result. For degree up to 7 we can present a detailed argument and calculation.

***Theorem 3.17*** *The $Q$ polynomial determines no Vassiliev invariants up to degree 7, except those derived (as polynomials of degree at most 3) from $v_2$.*

**Proof** Let $v$ be a Vassiliev invariant of degree $\leq 7$ determined by $Q$. Since $v$ is symmetric, it has even degree [St4]. The space of symmetric invariants of degree up to 6 is generated by the primitive invariants

$$v_2;\ v_{4,1},\ v_{4,2};\ v_{6,1},\ v_{6,2},\ v_{6,3},\ v_{6,4},\ v_{6,5}\,, \tag{16}$$

and the composite invariants

$$v_2^2;\ v_2^3,\ v_3^2,\ v_2v_{4,1},\ v_2v_{4,2}. \tag{17}$$

Here so far $v_i$ (resp., $v_{i,j}$) denotes the unique (resp., $j$-th in some arbitrary ordering) primitive Vassiliev invariant of degree $i$. From now on, call all these (including composite) invariants $v_{i,j}$ (by setting $v_{i,1} := v_i$ for $i = 2, 3$ and assigning such a term for the invariants of degree $i$ in (17), with $j$ above the range in (16)). Concrete expressions for $v_{i,j}$ (with one exception, $v_{6,5}$, and up to symmetric invariants of lower degree) can be found from the Kauffman polynomial. Set $F_{i,j}$ as in (11) for $i \geq 0,\ j > 0$. The property (3) implies that there are numbers $c_{i,j}$ such that

$$\tilde{F}_{i,j} = F_{i,j} + \sum_{k=1}^{j-1} c_{i,k} F_{i,k}$$

is a symmetric invariant for $i + j$ even (and antisymmetric for $i + j$ odd). In fact, one can restrict the $k$-sum over $1 \leq k < j$ with $j - k$ odd. For instance, one can choose

$$\begin{aligned}
\tilde{F}_{d,1} &= F_{d,1}\\
\tilde{F}_{d,2} &= F_{d,2} + F_{d,1}\\
\tilde{F}_{d,3} &= F_{d,3} + 3F_{d,2}\\
\tilde{F}_{d,4} &= F_{d,4} + 6F_{d,3} - 6F_{d,1}\\
\tilde{F}_{d,5} &= F_{d,5} + 10F_{d,4} - 60F_{d,2}\\
\tilde{F}_{d,6} &= F_{d,6} + 15F_{d,5} - 300F_{d,3} + 360F_{d,1}
\end{aligned}$$

The $\tilde{F}_{i,j}$ are not primitive, but a test on a few knots (see below) shows that most of them are linearly independent. Thus one can obtain (some) primitive Vassiliev invariants $v_{i,j}$ from the $\tilde{F}_{i',j'}$ by linear combinations (possibly including products). Even more, since the $\tilde{F}_{i,j}$ exceed the dimension of the space of primitive (symmetric) invariants for $i + j \leq 6$, there are linear dependencies.

A first easy observation is that

$$\tilde{F}_{0,2} \;=\; 4\tilde{F}_{1,1},$$

which is also a multiple of $v_2 = \nabla_2$, so that we can discard $\tilde{F}_{1,1}$. Then turn to degree $\leq 4$. Consider the few thousand (including composite) knots of up to 13 crossings. (They can be generated from the tables of [HT].) A test of $\tilde{F}_{0,2}$, $\tilde{F}_{0,2}^2$, $\tilde{F}_{0,4}$, $\tilde{F}_{1,3}$, $\tilde{F}_{2,2}$, $\tilde{F}_{3,1}$ on these knots shows the linear relations

$$\begin{aligned}
31\tilde{F}_{0,2} + 5\tilde{F}_{0,4} - 16\tilde{F}_{1,3} + 16\tilde{F}_{2,2} - 4\tilde{F}_{0,2}^2 &= 0\\
3\tilde{F}_{0,2} + \tilde{F}_{0,4} - 8\tilde{F}_{1,3} + 48\tilde{F}_{2,2} - 192\tilde{F}_{3,1} &= 0.
\end{aligned}$$

Thus one can eliminate $\tilde{F}_{2,2}$ and $\tilde{F}_{3,1}$. Then a test in degree $\leq 6$ of

$$\tilde{F}_{0,2},\ \tilde{F}_{0,2}^2,\ \tilde{F}_{0,2}^3,\ \tilde{F}_{0,3}^2,\ \tilde{F}_{0,4},\ \tilde{F}_{1,3},\ \tilde{F}_{0,2}\tilde{F}_{0,4},\ \tilde{F}_{0,2}\tilde{F}_{1,3},\ \tilde{F}_{0,6},\ \tilde{F}_{1,5},\ \tilde{F}_{2,4},\ \tilde{F}_{3,3},\ \tilde{F}_{4,2},\ \tilde{F}_{5,1}$$

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shows the relations

$$-1485\tilde{F}_{0,2} - 135\tilde{F}_{0,4} + 2\tilde{F}_{0,6} + 360\tilde{F}_{1,3} - 24\tilde{F}_{1,5} + 240\tilde{F}_{2,4} - 1920\tilde{F}_{3,3} + 11,520\tilde{F}_{4,2} - 46,080\tilde{F}_{5,1} + 180\tilde{F}_{0,2}^2 = 0,$$

$$-4464\tilde{F}_{0,2} - 3564\tilde{F}_{0,4} - 45\tilde{F}_{0,6} + 4410\tilde{F}_{1,3} + 54\tilde{F}_{1,5} + 1296\tilde{F}_{2,4} - 14,688\tilde{F}_{3,3} + 97,920\tilde{F}_{4,2} - 403,200\tilde{F}_{5,1} + 64\tilde{F}_{0,3}^2 + 72\tilde{F}_{0,2}^3 - 48\tilde{F}_{0,2}\tilde{F}_{0,4} + 384\tilde{F}_{0,2}\tilde{F}_{1,3} = 0,$$

thus eliminating $\tilde{F}_{5,1}$ and $\tilde{F}_{4,2}$.

This calculation is justified by Lemma 3.2. It also confirms the well-known fact that $F$ contains both of the primitive invariants of degree 4 as well as 4 of the 5 primitive invariants of degree 6. The missing invariant $v_{6,5} = \nabla_6$ is provided (up to some correction by composite invariants) by the coefficient of $z^6$ in the Conway polynomial $\nabla(z)$, as explained in [St6] from the example of Lickorish (and recalled above in §3.3).

Now assume $Q(z) = F(1, z)$ determines $v = \sum_{i=2,4,6} \sum_j c_{i,j} v_{i,j}$. First note that if $c_{6,5} \neq 0$, then $F$ determines $\nabla_6$, a contradiction. Thus, assume $c_{6,5} = 0$, and we deal only with the Vassiliev invariants coming from the Kauffman polynomial. Then we can without loss of generality replace $v_{i,j}$ by $\tilde{F}_{i',j'}$ (for $i' + j' = i$).

Among prime knots of $\leq$ 10 crossings [Ro, appendix], $Q$ has 13 duplications. These are the pairs

$$(9_{44}, 8_2),\ (9_{45}, 8_7),\ (9_{15}, 10_{159}),\ (9_8, 10_{131}),\ (9_5, 10_{134}),\ (9_{21}, 10_{151}),\ (9_{12}, 10_{156}),$$
$$(9_{25}, 9_{26}),\ (10_{22}, 10_{35}),\ (10_{14}, 10_{31}),\ (10_{56}, 10_{33}),\ (10_{19}, 10_{36}),\ (10_{43}, 10_{72}).$$

The polynomial of (one knot of) each pair is given in Table 1.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 2 | [-7] | 0 | 22 | 2 | -20 | -4 | 6 | 2 | | |
| 8 | 7 | [-7] | 4 | 20 | -8 | -20 | 2 | 8 | 2 | | |
| 9 | 5 | [ 1] | -12 | 2 | 28 | 0 | -22 | -4 | 6 | 2 | |
| 9 | 8 | [ 1] | -8 | 8 | 22 | -12 | -22 | 2 | 8 | 2 | |
| 9 | 12 | [-3] | -6 | 10 | 14 | -12 | -16 | 4 | 8 | 2 | |
| 9 | 15 | [ 1] | 4 | -2 | -2 | -8 | -8 | 6 | 8 | 2 | |
| 9 | 21 | [-3] | -2 | 16 | 4 | -26 | -12 | 12 | 10 | 2 | |
| 9 | 25 | [-7] | 0 | 30 | 2 | -42 | -14 | 18 | 12 | 2 | |
| 10 | 14 | [ 1] | -4 | -10 | 20 | 16 | -26 | -18 | 10 | 10 | 2 |
| 10 | 19 | [ 5] | -6 | -20 | 28 | 30 | -30 | -26 | 8 | 10 | 2 |
| 10 | 22 | [ 1] | 0 | -4 | 6 | 12 | -12 | -16 | 4 | 8 | 2 |
| 10 | 33 | [ 1] | -16 | 0 | 44 | 4 | -48 | -16 | 18 | 12 | 2 |
| 10 | 43 | [-7] | -4 | 28 | 22 | -32 | -42 | 0 | 22 | 12 | 2 |

*Table 1*: The $Q$ polynomials of the $\leq$ 10 crossing prime knots occurring in duplications. (Only one knot in each pair is recorded.)

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Each such pair gives rise to a linear relation on the $c_{i,j}$ as (14) in the proof of Proposition 3.16 on $a$ and $b$. The $\tilde{F}_{i,j}$ are given in Table 2.

By imposing jointly all these 13 conditions on the $c_{i,j}$, we find that the only possible linear combinations $\sum c_{i,j} v_{i,j}$ determined by $Q$ must lie in the span of $v_2$, $v_2^2$ and $v_2^3$, as desired. ■

***Remark*** As in the proof of Proposition 3.16, we could have tried to use a single large group of knots with equal $Q$ polynomials, but different Vassiliev invariants, to find enough relations between the $c_{i,j}$. (Note that a group of $n$ knots can give up to $n-1$ independent linear relations.) However, among prime knots of up to 16 crossings, I found no group whose linear conditions on the $c_{i,j}$ eliminate anything except polynomials of $v_2$. Note that generically a considerable part of the coincidences of the $Q$ polynomial arise from mutations. But mutations preserve Vassiliev invariants up to degree 6 [CDL, CDL2, MC, Mr] and are useless for our purpose.

# 4 Vassiliev Invariants and 2-Cable Polynomials

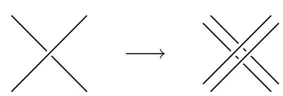
## 4.1 Calculating Invariants

If one wants to extend our result to degrees $\geq 8$, more computation is required. We will present here the outcome that suffices to cover degrees 8 and 9. A first task is to find a way to obtain all such Vassiliev invariants. Expectedly, this problem has been encountered before. In particular, a related (and still unsolved) question posed by Przytycki [Ki, Problem 1.92 (M)(c)] is

***Question 4.1*** Do all invariants of knots of degree 10 or less come from the HOMFLY and Kauffman polynomial and their 2-cables?

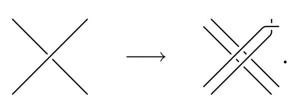
Recall that a 2-*cable* $K_p$ of a knot $K$ with framing $p$ is constructed as follows. For even $p$, (a diagram of) $K_p$ is obtained by applying

(18)



to any crossing of a diagram of $K$ of writhe $p/2$. For odd $p$ one applies (18) to a diagram of writhe $(p-1)/2$, except at one crossing, where one performs

(19)



Then $K_p$ is connected (a knot) for odd $p$ and disconnected (a 2-component link) for even $p$. We write $K_\pm$ for $K_{\pm 1}$.

In an attempt to approach Przytycki's problem, we considered the Vassiliev invariants

(20) $$\mathcal{P}_d := \{P_{i,j} \,:\, i+j \leq d\},$$

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| cr | nr | F(0,2) | F(0,4) | F(0,6) | F(1,1) | F(1,3) | F(1,5) | F(2,2) | F(2,4) | F(3,1) | F(3,3) | F(4,2) | F(5,1) | F(0,3) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 44 | 0 | -192 | 8640 | 0 | -96 | -480 | -36 | -852 | -6 | -336 | -72 | -8 | -48 |
| 8 | 2 | 0 | 576 | -2880 | 0 | 288 | 32160 | 108 | 14268 | 18 | 1920 | 136 | 12 | 48 |
| 9 | 45 | -16 | -1872 | -54720 | -4 | -768 | -65280 | -88 | -8184 | 0 | -624 | -88 | -6 | 192 |
| 8 | 7 | -16 | -336 | 14400 | -4 | -288 | -9120 | -88 | -2808 | -12 | -408 | -72 | -10 | 96 |
| 9 | 5 | -48 | -8112 | -420480 | -12 | -3600 | -522000 | -396 | -38940 | 8 | -2424 | -592 | 10 | -720 |
| 10 | 134 | -48 | -5040 | 455040 | -12 | -2640 | -210000 | -396 | -40476 | -16 | -4008 | -400 | -10 | -624 |
| 9 | 8 | 0 | -768 | 11520 | 0 | -240 | -7920 | 0 | -1728 | 6 | -384 | -40 | 2 | -96 |
| 10 | 131 | 0 | 1536 | 69120 | 0 | 624 | 80880 | 144 | 26448 | 18 | 3336 | 184 | 6 | 96 |
| 9 | 12 | -8 | -1704 | -61920 | -2 | -624 | -48240 | -60 | -7692 | 2 | -1704 | -216 | 0 | -144 |
| 10 | 156 | -8 | -168 | 7200 | -2 | -144 | -7440 | -60 | -2316 | -10 | -48 | -40 | -16 | 48 |
| 9 | 15 | -16 | -2448 | -51840 | -4 | -960 | -73920 | -100 | -8532 | 2 | -1224 | -176 | 0 | 240 |
| 10 | 159 | -16 | -912 | 17280 | -4 | -480 | -33120 | -100 | -9300 | -10 | -1104 | -128 | -16 | -144 |
| 9 | 21 | -24 | -2616 | -44640 | -6 | -1104 | -81360 | -96 | -6624 | 8 | -1128 | -188 | 8 | 288 |
| 10 | 151 | -24 | -1080 | 24480 | -6 | -624 | -25200 | -96 | -2784 | -4 | -432 | -108 | -8 | 192 |
| 9 | 25 | 0 | -960 | -72000 | 0 | -288 | -32160 | 12 | -2916 | 10 | -240 | -28 | 2 | -48 |
| 9 | 26 | 0 | 576 | -2880 | 0 | 192 | 8640 | 12 | 3996 | -2 | 936 | 84 | -2 | -48 |
| 10 | 14 | -16 | -912 | 17280 | -4 | -336 | 30000 | 44 | 21372 | 20 | 3648 | 336 | 30 | -144 |
| 10 | 31 | -16 | -144 | 5760 | -4 | -240 | -240 | -100 | -660 | -16 | -696 | -224 | -30 | 48 |
| 10 | 19 | -8 | 24 | -1440 | -2 | -144 | 240 | -120 | -1368 | -24 | -984 | -284 | -38 | 0 |
| 10 | 36 | -8 | -744 | 10080 | -2 | -240 | 7440 | 24 | 9144 | 12 | 2112 | 260 | 22 | -96 |
| 10 | 22 | 32 | 288 | -11520 | 8 | 144 | -240 | 248 | 5400 | 58 | 888 | 216 | 48 | 96 |
| 10 | 35 | 32 | 1056 | -23040 | 8 | 240 | 240 | 104 | 4104 | 22 | 672 | 120 | 24 | -96 |
| 10 | 33 | 0 | 0 | 0 | 0 | -96 | 1440 | -96 | -864 | -20 | -768 | -232 | -32 | 0 |
| 10 | 56 | 0 | 1536 | 69120 | 0 | 672 | 97440 | 192 | 37824 | 28 | 6048 | 536 | 32 | 96 |
| 10 | 43 | -16 | -336 | 14400 | -4 | -240 | -240 | -40 | -1032 | -2 | -528 | -128 | -14 | 0 |
| 10 | 72 | -16 | -1872 | -54720 | -4 | -720 | -48720 | -40 | 2808 | 10 | 2136 | 304 | 26 | -192 |

*Table 2*: This table gives the values of the $\tilde{F}_{i,j}$ (as $F(i,j)$, with $\tilde{F}_{0,3}$ accurate up to sign) on the $\leq 10$ crossing prime knots occurring in $Q$ duplications (see Table 1). *cr* and *nr* give the crossing number and order number in the tables of [Ro, appendix], that is, denote the knot $cr_{nr}$.

where $P_{i,j}$ is defined as in (13), and $i, j \geq 0$, with $i$ even. Note that $\mathcal{P}_d$ are all invariants of degree $\leq d$, as are

$$\mathcal{F}_d := \{F_{i,j} \,:\, i + j \leq d\}. \tag{21}$$

To obtain a Vassiliev invariant of degree $\leq d$, one can also use products of invariants $P_{i,j}$ and $F_{i,j}$.

The invariants in $\mathcal{P}_d$ and $\mathcal{F}_d$ were considered by Meng [Me] and Lieberum [Li], using their weight systems. Our calculation is supported by some results they obtained. However, it also shows phenomena that point to caution in some tempting conclusions concerning the structure of the algebra generated by Vassiliev invariants of the HOMFLY and Kauffman polynomials.

One can apply $\mathcal{P}_d$ and $\mathcal{F}_d$ also to 2-cables $K_p$ of $K$ with various framings. We denote by $\mathcal{P}_d(K_p)$ and $\mathcal{F}_d(K_p)$ the resulting invariants. If the framing is even, then the 2-cable is disconnected, and then the restriction to $i$ modifies to $i \geq -1$, with $i$ odd for $P_{i,j}$.

For

$$v \in (\mathcal{P}_d \setminus \mathcal{P}_{d-1})(K_*) \cup (\mathcal{F}_d \setminus \mathcal{F}_{d-1})(K_*)\,,$$

let $\widetilde{\deg}\, v := d$. Note that $\widetilde{\deg}\, v$ is not *a priori* evident to be the same as the degree of $v$ as a Vassiliev invariant (whence the notational distinction), although clearly $\deg v \leq \widetilde{\deg}\, v$. In some situations though, we have equality, and we clarify why, since the notation and arguments will be of relevance in later explanation. We formulate a statement only with $F$, letting the reader understand that most subsequent remarks on one of $P$ and $F$ also apply to the other in a similar way.

***Lemma 4.2*** *For odd $p$ and $i, j \geq 0$ with $i + j$ even, and $i$ even or $i = 1$, we have* $\deg F_{i,j}(K) = \deg F_{i,j}(K_p) = i + j$.

**Proof** Let us write for a set $M \subset \mathcal{P}_*(K_*) \cup \mathcal{F}_*(K_*)$ of invariants

$$M_d := \Big\{ \prod_{l=1}^{k} v_l : v_l \in M, \sum_{l=1}^{k} \widetilde{\deg}\, v_l \leq d \Big\},$$

and consider

$$\hat{F}(h, N) := F\big(\sqrt{-1}e^{-(N-1)h/2}, \sqrt{-1}\big(e^{h/2} - e^{-h/2}\big)\big)\;.$$

Then we have (extending the notation of coefficients to power series)

$$\begin{aligned} F_{i,j}(K) &\equiv C_{i,j}\,\big[\,\hat{F}(h,N)(K)\,\big]_{h^{i+j}N^j} \\ &\equiv C_{i,j}\,\big[\,\hat{F}(h,N)(K \cup O)\,\big]_{h^{i+j}N^{j+1}} \mod \mathcal{L}in\,(\mathcal{F}_*(K))_{i+j-1}\,, \end{aligned} \tag{22}$$

where $C_{i,j}$ are non-zero numbers, and $K \cup O$ is the split union of $K$ with an unknot. The right hand side of the congruence is a canonical Vassiliev invariant of degree

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$i + j$ by the result of Le-Murakami and Kassel [LMr, LMr2, LMr3, Ks] (compare Proposition 5 in [Li]). Now $[\hat{F}_{K\cup O}(h,N)]_{h^{i+j}N^{j+1}} \not\equiv 0$ is not hard to see. Let $K$ be the unknot, with $F(K \cup O) = (a + a^{-1})/z - 1$. The coefficient of the power series $\hat{F}(K\cup O)$ is easily found to be non-zero for the given $i, j$; for $i \neq 1$ it is, up to a factor, a Bernoulli number. (With other $K$ and a similar calculation, one can settle more $i, j$.)

Thus indeed $\deg F_{i,j}(K) = i + j$. Then the same is true for $F_{i,j}(K_p)$ if $p$ is odd. To see this, recall that connected $n$-cabling of a degree $d$ Vassiliev invariant $v$ applies a dual Adams operation $(\psi^n)^*$ of [BN] on its weight system $W_v \in \mathcal{V}_d/\mathcal{V}_{d-1} \simeq \mathcal{A}_d^*$. That $\psi^n$ is an automorphism of $\mathcal{A}_d$ was stated in [BN, Exercise 3.12]. In fact, we know that the eigenvalues of $\psi^n$ are powers of $n$ with exponents given by the number of univalent vertices of unitrivalent graphs; see [KSA, MR]. ■

For the calculation of 2-cable polynomials of $K$ it is sufficient (but also, up to algebraic transformations, necessary) to determine the polynomials of a connected cable of $K$ and $!K$. To keep the diagrams as simple as possible, we use the 2-cables with blackboard framing from the diagrams in [HT] and one negative half-twist. For the skein polynomial, this calculation was possible for all prime knots up to 13 crossings (including mirror images). The Kauffman polynomial is technically more difficult. We obtained a set $S$ of 898 prime knots up to 12 crossings (including all $\leq 10$ crossing knots, except $10_5$), where both Kauffman polynomials could be determined. We used this set $S$ for all subsequent Vassiliev invariant calculations.

## 4.2 Dimensions

Table 3 gives lower bounds for the dimension of Vassiliev invariants of bounded degree calculated for various combinations of $P_{i,j}$ and $F_{i,j}$ applied to knots and various 2-cables. With the previous designation, for example the column $d$ entry of the row $PP_+P_-$ is

$$\dim \mathcal{L}in\ \left(\mathcal{P}_*(K_+) \cup \mathcal{P}_*(K_-) \cup \mathcal{P}_*(K)\right)_d \Big|_{K\in S},$$

and $S$ is the set of knots explained above. Clearly, many linear dependencies will occur, but in degree $d \geq 7$, they are increasingly difficult to prove rigorously. Contrarily, linear independence is easy to prove if $S$ is large enough. Although some general theory behind Table 3 is known, there are many detailed aspects in the calculations it reflects that apparently were never clearly pointed out. Thus we will list below several features of the table that should be clarified, and point out phenomena and previous results it relates to.

The numbers obtained, given in the table, can only be ensured to represent lower bounds for the dimensions, since it is difficult to rigorously verify that some Vassiliev invariant is identically zero. From the fact that we evaluated enough invariants to obtain the full dimension up to degree 8, we can conclude that the set $S$ we used is $d$-deterministic, and so our numbers are exact for $d \leq 8$. However, we do not know about degrees 9 or 10. Indeed, non-trivial Vassiliev invariants of increasing degree may vanish on many low-crossings knots (for example the $\nabla_i$). All deterministic sets

| dim ≥ \ deg ≤ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| total | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 60 | 104 | 184 |
| $P/PP!$ | 1 | 1 | 2 | 3 | 6 | 9 | 16 | 24 | 40 | 60 | 95 |
| $P_+/P_-$ | 1 | 1 | 2 | 3 | 6 | 9 | 16 | 24 | 40 | 60 | 95 |
| $PP_+/PP_-$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 31 | 53 | 85 | 140 |
| $P_+P_-$ | 1 | 1 | 2 | 3 | 6 | 9 | 17 | 27 | 46 | 72 | 117 |
| $PP_+P_-$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 31 | 53 | 86 | 142 |
| $P_+P_-P_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 30 | 52 | 82 | 136 |
| $PP_+P_0/PP_-P_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 31 | 54 | 87 | 145 |
| $PP_+P_-P_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 31 | 54 | 87 | 145 |
| $PP_+P_-P_3$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 31 | 53 | 86 | 142 |
| $P_{-2}P_-P_0P_+$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 30 | 52 | 82 | 136 |
| $PP_{-2}P_-P_0P_+$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 31 | 54 | 87 | 145 |
| $PP_-P_0P_+P_2P_3$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 31 | 54 | 87 | 145 |
| $F$ | 1 | 1 | 2 | 3 | 6 | 10 | 18 | 29 | 49 | 78 | 127 |
| $F_+/F_-$ | 1 | 1 | 2 | 3 | 6 | 10 | 18 | 29 | 49 | 78 | 127 |
| $FPP_+P_-P_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 32 | 57 | 94 | 159 |
| $FF_+F_-F_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 59 | 99 | 168 |
| $FF_+F_-F_0F_{-2}$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 59 | 99 | 168 |
| $FF_+F_-F_0F_{-3}$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 59 | 99 | 168 |
| $FF_+F_-F_0F_2$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 59 | 99 | 168 |
| $FF_+PP_+P_-P_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 60 | 102 | 176 |
| $FF_0PP_+P_-P_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 60 | 102 | 176 |
| $FF_+F_-F_0PP_+P_-P_0$ | 1 | 1 | 2 | 3 | 6 | 10 | 19 | 33 | 60 | 102 | 177 |

*Table 3*: This table contains dimensions of various spaces of Vassiliev invariants for degree ≤10.

The first row gives the total dimension of Vassiliev invariants up to degree deg as calculated by Bar-Natan [BN] and Kneissler [Kn].

The second section of rows gives lower bounds for the dimension of Vassiliev invariants up to degree deg obtainable as polynomial expressions from $\mathcal{P}_{\deg}$ of the HOMFLY polynomial $P$, and its (application on) 2-cables $P_p(K) = P(K_p)$ of twist $p$. $P_\pm$ denotes $P_{\pm 1}$. The product of some of the $P$ symbols denotes that the invariants of these polynomials have been taken together. The slash separates between alternative combinations of polynomials that give, as we explain, the same dimensions (although not necessarily the same linear spaces!).

The last section gives dimensions of invariants derived via $\mathcal{F}_{\deg}$ from the Kauffman polynomial $F$ and its applications $F_p$ on 2-cables of twist $p$, with $F_\pm = F_{\pm 1}$. Some combinations of $P_*$ and $F_*$ invariants are also given. They are chosen so as to make evident that the last row's dimensions cannot be increased by adding further invariants.

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we know in degree $d > 8$, as well as the one from §3.2, are too large to allow efficient calculations. One can find smaller sets using a basis of $\mathcal{V}_d/\mathcal{V}_{d-1}$. (Its primitive part would be enough.) But such a basis is itself non-trivial to find, and was apparently never explicitly given (even if likely obtained in the course of the calculations of Bar-Natan [BN] and Kneissler [Kn]). Even if so done, the resulting reduction is still unlikely to be easily manageable. Another way to prove a set deterministic is to evaluate the remaining Vassiliev invariants, but this does not seem very efficient either. At least, the comparison of the first and last rows of the table shows that the difference between the numbers in degree 9 (resp., 10), and the actual dimension is at most 2 (resp., 7).

Once we obtain only lower bounds, it makes sense to reduce invariants modulo a large prime, which we chose to be 9091, in order to keep numbers simple. (In particular, in 2-cable Kauffman polynomials the coefficients are large enough to make $F_{i,j}$ exceed machine-size integers. Mathematica™, which bypasses this problem, could not handle well the extent of calculation needed for the upper degrees.)

Some coincidences of rows are easy to explain (even without knowing the absolute accuracy of the numbers in giving the proper dimensions), or at least known. In particular, mirroring the (set of) invariant(s) induces an involution on the space $\mathcal{V}_d/\mathcal{V}_{d-1}$. The injectivity of $\psi^2$ was mentioned in the proof of Lemma 4.2. The fact that $PP!$ contributes the same linear span of invariants as $P$ is a consequence of property (4). For that same reason, and because $(!K)_p = !(K_{-p})$, it becomes useless to consider the invariants from the various 2-cable polynomials of $!K$ for $K \in S$.

We also obtained lists of linear independent invariants (omitted here for space reasons), but we have not tried to identify a basis of the primitive part of $\mathcal{V}_d/\mathcal{V}_{d-1}$ that is obtainable. It is very difficult (see the following remarks) to determine the exact degrees of the Vassiliev invariants and their primitivity status. One should also be cautioned that the linear relations between such invariants involve up to about 30-digit coefficients, and are much more complicated than insightful.

### 4.3 "Hidden" Vassiliev Invariants

Assume for a moment that the numbers in the table are exact (rather than just lower bounds). Assume further that all the new invariants contributed by each set of $\mathcal{P}_d$ in comparison to $\mathcal{P}_{d-1}$ are invariants of degree $d$, and that all (prime) factors of all composite invariants obtained have been generated for smaller $d$. Then we find from the various rows of the table the *projected* sequences of primitive Vassiliev invariants of degree exactly $d$ that can be obtained. For example, for the $P$-row it reads $1, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5$ and for the $F$-row $1, 0, 1, 1, 2, 3, 4, 5, 6, 7, 8$. These sequences appear in [Li, Proposition 12], and seem the only case studied closely so far. However, the projected sequences may not always be correct. Consider the rows $PP_+$, where we obtain $1, 0, 1, 1, 2, 3, 5, 6, 7, 8, 9$ and $PP_-P_+$, where we obtain $1, 0, 1, 1, 2, 3, 5, 6, 7, 9, 10$. Apparently, adjoining $P_-$ seems to give a new invariant in degree 9. But it is easy to see that $\mathcal{P}_d(K_p)$ gives the same elements in $\mathcal{V}_d/\mathcal{V}_{d-1}$ for any $p$ of a given parity. Thus $\mathcal{P}_d(K_-)$ cannot increase the dimension in degree $d$. This means that a Vassiliev invariant of degree $d$ may be realizable from some $\mathcal{P}_{d'}$ with $d' > d$, but not from $\mathcal{P}_d$ (of a given set of cables). In particular, the difference

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between $PP_-P_+$ and $PP_+$ in degree $d = 9$ comes from a Vassiliev invariant $v_8$ in degree $d < 9$. We know that $v_8$ cannot be obtained from $(\mathcal{P}_*)_8$, since by the remarks in §4.2 our numbers are accurate for $d = 8$. It must have degree 7 or 8, as we have already exhausted all invariants of degrees $d \leq 6$ with $PP_+$. The additional difference in degree 10 must come from a new Vassiliev invariant of degrees 7 to 9. But these invariants are immediately lost if we work with the (degree $d$) weight systems of $\mathcal{P}_d$. This explains why the weight systems obscure sometimes essential information.

Even if we cannot explicitly observe an instance of this phenomenon, it is in principle possible that one can even obtain a composite Vassiliev invariant from some $\mathcal{P}_d(K_*)$ without being able to obtain some of its factors.

The (possible) peculiarities explained above caution the following:

(a) *The algebra of some set of Vassiliev invariants may not be isomorphic to the algebra of their weight systems.* This can occur if not all invariants are primitive and have linear independent weight systems (of the appropriate degree). To exclude such possibility, the composite and lower degree Vassiliev invariants must be proved to be generatable from previous degrees. One situation where this is needed is Theorem 3 of [Li]. It requires the result used in (22) that any $F_{i,j}(K)$ can be altered by elements in $\mathcal{L}in\,(\mathcal{F}_*(K))_{i+j-1}$ so that it becomes canonical (of degree $i+j$), and similarly $P_{i,j}(K)$. For canonical invariants, linear dependencies of the weight systems extend to linear dependencies of invariants.

(b) It is difficult to prove that some Vassiliev invariant $v$ is actually *not* obtainable from HOMFLY (or some cables of it). For $P$ we can deduce from the proof of Lemma 4.2 that if a Vassiliev invariant $v$ of degree $d = \deg v$ lies in the algebra generated by $\mathcal{P}_*(K)$, then it lies in $\mathcal{L}in\,(\mathcal{P}_*(K))_d$. On the opposite side, for cables of $P$, there is no *a priori* limit on $d'$ in terms of $d$, whose $\mathcal{P}_{d'}$ we must consider, and not only polynomials, but possibly fractions of polynomials of $\mathcal{P}_{d'}$ must be examined. There may be even other (yet unknown) ways to obtain Vassiliev invariants, not using (only) the $\mathcal{P}_*$. Thus the only approach is to find knots not distinguished by HOMFLY (or its cables) but by $v$, as in [K4, St6]. A systematic way to find such examples is unknown.

### 4.4 Connected and Disconnected Cables

It is suggestive from the skein relation of $P$ that the Vassiliev invariant $v_8$ in §4.3 can be obtained from $\mathcal{P}_8(K_0)$. This explains the difference between the $PP_+P_-$ and $PP_+P_-P_0$ rows occurring already in degree 8.

In general one can obtain the $P$-polynomial of a disconnected $n$-cable as a linear combination of polynomials of connected $n$-cables whose coefficients have a power in $m$ between 0 and $1-n$. This means that

$$\mathcal{L}in\,\mathcal{P}_d\,(\text{all } n\text{-cables}) \subset \mathcal{L}in\,\mathcal{P}_{d+n-1}\,(\text{connected } n\text{-cables})\,.$$

However, in general

$$\mathcal{L}in\,\mathcal{P}_d\,(\text{all } n\text{-cables}) \neq \mathcal{L}in\,\mathcal{P}_d\,(\text{connected } n\text{-cables})\,.$$

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That is, there is a way of obtaining new Vassiliev invariants by disconnectedly cabling invariants of the same degree, not obtainable by connected cablings. This was noticed by Dasbach [Da]. The eigenvalues of the Adams operations (mentioned in the proof of Lemma 4.2) show, as observed in [MR], that the space of invariants given by connected $n$-cablings of an invariant $v$ of degree $d$ stabilizes (modulo lower degree) for $n > d$. In contrast, Dasbach's result roughly means that, by starting from $\mathcal{P}_d$, one will obtain new invariants of degree $d$ from disconnected $n$-cables at least up to $n \leq \exp\,(C \cdot \sqrt{d})$ for some constant $C > 0$ (independent of $n$ and $d$). Thus, even though polynomials of disconnected cables are linear combinations of polynomials of connected cables, and hence the same is true for their global sets of Vassiliev invariants, the situation is quite different if one restricts oneself to their invariants of bounded degree.

On the other hand, for any connectivity, the relations between cable polynomials allow us to limit the number of cables of that connectivity which suffice to generate all possible Vassiliev invariants from all such cables. In case $n = 2$ we have

***Lemma 4.3*** *For the polynomials $P_p$ of the 2-cables of framing $p$ (connected for $p$ odd and disconnected for $p$ even), we have*

$$P_p = -l^4 P_{p-4} - (2l^2 - m^2 l^2) P_{p-2}.$$

**Proof** Consider the generating series $f(l, m, z) = \sum_{p=0}^{\infty} P_p(l, m) z^p$ (whose convergence is easy to establish). The skein relation implies $P_{p+2} = -mlP_{p+1} - l^2 P_p$, so that

$$f(l, m, z) \;=\; \frac{A(l, m, z)}{1 + mlz + l^2 z^2},$$

for some $A \in \mathbf{Q}[l, m, z]$. Taking $f(l, m, z) \pm f(l, m, -z)$, we obtain the denominator

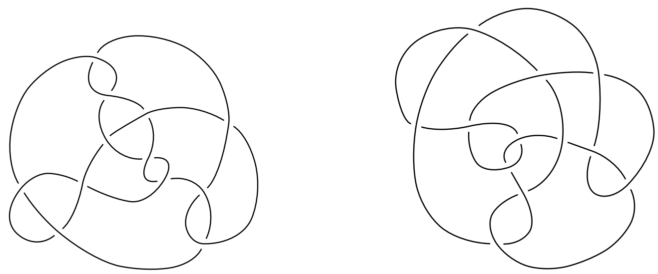
$$(1 + mlz + l^2 z^2)(1 - mlz + l^2 z^2) \;=\; 1 + l^4 z^4 + 2l^2 z^2 - m^2 l^2 z^2,$$

which leads to the stated relation. ■

This means that for connected/disconnected 2-cables, the invariants of $\mathcal{P}_d$ are exhausted if we apply them on $P_p$ for two consecutive odd (resp., even) $p$. By a similar argument for $F$, three consecutive $p$ of a given parity suffice. In practice, as the table shows, $p = \pm 1, 0$ already apparently generate all invariants from $\mathcal{P}_d$ and $\mathcal{F}_d$ for $d \leq 10$ (for both parities of $p$ taken together).

## 4.5 Mutations and Non-Mutations

Note the difference between the $P_+P_-P_0$ and the $PP_+P_-P_0$ rows. This suggests that HOMFLY may have Vassiliev invariants not contained in its 2-cables. In general, almost all knots with different HOMFLY polynomial will also have different 2-cable HOMFLY polynomial. But the Vassiliev invariant observation suggests that it may not always be so. So far, the only known examples of knots with equal 2-cable HOMFLY polynomial are mutants [LL]. They also have the same HOMFLY (and Kauffman) polynomial.



$12_{341}$ $12_{627}$

| 0 | 24 | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -8 | 10 | 5 | 27 | 69 | 99 | 75 | 6 | -44 | -44 | -20 | -4 |
| -8 | 10 | -60 | -315 | -755 | -1016 | -710 | -22 | 467 | 462 | 213 | 42 |
| -8 | 10 | 331 | 1705 | 3800 | 4752 | 3129 | 33 | -2126 | -2157 | -1040 | -206 |
| -8 | 10 | -1011 | -5283 | -11308 | -13286 | -8335 | -7 | 5603 | 5795 | 2823 | 537 |
| -8 | 10 | 1805 | 10023 | 21665 | 24516 | 14755 | -52 | -9503 | -9781 | -4556 | -794 |
| -8 | 10 | -1965 | -12201 | -27766 | -31277 | -18030 | 41 | 10913 | 10763 | 4610 | 692 |
| -8 | 10 | 1325 | 9768 | 24362 | 28057 | 15528 | -11 | -8578 | -7886 | -2968 | -355 |
| -8 | 10 | -549 | -5129 | -14689 | -17703 | -9448 | 1 | 4563 | 3811 | 1191 | 104 |
| -8 | 10 | 135 | 1728 | 5989 | 7729 | 4007 | 0 | -1583 | -1179 | -285 | -16 |
| -8 | 10 | -18 | -357 | -1602 | -2259 | -1148 | 0 | 338 | 222 | 37 | 1 |
| -8 | 8 | 1 | 41 | 267 | 418 | 210 | 0 | -40 | -23 | -2 | |
| -6 | 6 | | -2 | -25 | -44 | -22 | 0 | 2 | 1 | | |
| -4 | 0 | | | 1 | 2 | 1 | | | | | |

*Table 4*: Two knots with the same 2-cable HOMFLY polynomials ($P_+$ is displayed), which are not mutants.

However, the calculations performed while compiling the above table led to the discovery of some duplications of $P_*$ which are not mutants.

***Example 4.4*** The knots $12_{1305}$ and $!12_{1872}$ have the same $P$, $F$ and 2-cable $P$. To check the coincidence of $P_*$, comparing $P_\pm$ suffices. Still $12_{1305}$ and $!12_{1872}$ are not mutants. This is most easily shown using the result of [Ru], since their hyperbolic volumes differ: $\mathrm{vol}\,(12_{1305}) \approx 15.483$, while $\mathrm{vol}\,(!12_{1872}) \approx 15.619$. Another such group is made of the two mutants $12_{1378}$, $12_{1423}$, and the knot $!12_{1704}$. Again $P$, $F$ and 2-cable $P$ coincide, but while $\mathrm{vol}\,(12_{1378}) = \mathrm{vol}\,(12_{1423}) \approx 15.094$, we have $\mathrm{vol}\,(!12_{1704}) \approx 14.983$.

Later, after considerable calculation, we found that these pairs of knots have also different 2-cable Kauffman polynomials $F_+$, with the difference coming out as a Vassiliev invariant of degree 7. Thus there is a Vassiliev invariant of degree 7 not contained in the HOMFLY, Kauffman and 2-cable HOMFLY polynomials, but in the 2-cable Kauffman polynomial. (Note that $P_*$ exhaust all invariants up to degree 6.)

There is one further pair made up of $12_{341}$ and $12_{627}$ (see Figure 4). These knots are

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achiral, and for them, comparing $P_+$ suffices to see that $P_*$ coincide. This time they are distinguished using an invariant of degree 8 of the 2-cable Kauffman polynomials. (Note that the lowest degree of an invariant distinguishing $12_{341}$ and $12_{627}$ must be even, since by [St4] odd degree invariants can be changed by invariants of lower degree so that they vanish on achiral knots.)

There has been further work on generalizations of mutations [APR, JR, Tz, HP], but none of this seems to explain the coincidence of the 2-cable HOMFLY polynomial in these examples.

The observed coincidences of $P$ and $F$ also on non-mutants with the same 2-cable HOMFLY polynomials extend to prime $\leq 13$ crossing knots and suggest

***Question 4.5*** Does $P_p$ for some $p$ (or at least for all $p$ taken together) determine $P$ and/or $F$?

Note that this question may relate to more than mere curiosity. In [KS] we observed a (conjectural) relation between $F$ and the Whitehead double HOMFLY polynomials, and there is also Yamada's remarkable result [Y] that $F$ determines the 2-cable Jones polynomial.

***Remark*** Using Alexander Shumakovitch's database, we found that the new Khovanov polynomial $Kh$ [Kh] coincides on these examples as well, and on all other pairs of prime $\leq$ 13 crossing knots with equal $P_+$. Still, $Kh$ is known to distinguish some knots with equal $P$ and $F$ (most interestingly $9_{42}$ and its mirror image). However, I do not know of an example showing that $Kh$ can distinguish knots with equal $F$ and Murasugi-signatures.

## 4.6 Braid Index

It is known that one can estimate the braid index of a knot $K$ from its $P$ polynomial [Mo, FW]:

$$2(b(K) - 1) \geq \max \deg_l P(K) - \min \deg_l P(K). \tag{23}$$

This estimate is called the Morton–Franks–Williams inequality. Since obviously $b(K_p) \leq 2b(K)$ for any $p \in \mathbf{Z}$, we can estimate $b(K)$ also from the 2-cable $P$ polynomials of $K$, as is done in [MS]. We attempted to use this method to settle the braid index for prime knots of up to 12 crossings. This requires us to find braid representations of the strand number given as (lower) bound from the Morton–Franks–Williams inequality or its application on the 2-cable polynomials. (For a few cases of large bound, one can conclude the existence of such representations from Ohyama's inequality [Oh], and for special types of knots from Murasugi's results [Mu2].) We were able to calculate 2-cable $P$ polynomials up to 13 crossings, but were aware of the difficulties of finding braid representations. We know from [HS] of one undecidable 13 crossing knot, and in [St8] we gave a 14 crossing example of failure of the 2-cable Morton–Franks–Williams inequality. On the contrary, we indeed succeeded in finding the desired braid representations for up to 12 crossing knots, thereby showing

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9_42 | 4 | 12_1298 | 5 | 12_1499 | 5 | 12_1695 | 4 | 12_1899 | 5 |
| 9_49 | 4 | **12_1313** | **5** | 12_1503 | 5 | 12_1702 | 5 | 12_1922 | 5 |
| **10_132** | **4** | 12_1333 | 5 | 12_1506 | 5 | 12_1704 | 4 | 12_1929 | 4 |
| 10_150 | 4 | 12_1373 | 5 | 12_1541 | 4 | 12_1712 | 4 | 12_1933 | 5 |
| 10_156 | 4 | 12_1378 | 4 | 12_1542 | 4 | 12_1723 | 5 | 12_1944 | 5 |
| 11_387 | 5 | 12_1382 | 5 | 12_1548 | 5 | 12_1726 | 4 | 12_1946 | 4 |
| 11_391 | 4 | 12_1385 | 5 | 12_1553 | 5 | 12_1737 | 5 | 12_1982 | 4 |
| 11_400 | 5 | 12_1391 | 4 | 12_1598 | 5 | 12_1787 | 5 | 12_1983 | 4 |
| 11_404 | 4 | 12_1396 | 5 | 12_1600 | 5 | 12_1803 | 5 | 12_2008 | 5 |
| 11_437 | 4 | 12_1400 | 5 | 12_1610 | 5 | 12_1804 | 5 | 12_2015 | 5 |
| 11_446 | 5 | 12_1408 | 4 | 12_1628 | 4 | **12_1811** | **5** | 12_2016 | 4 |
| 11_449 | 4 | 12_1418 | 5 | 12_1650 | 4 | 12_1825 | 5 | 12_2017 | 4 |
| 11_453 | 4 | 12_1423 | 4 | 12_1652 | 5 | 12_1833 | 5 | 12_2037 | 3 |
| 11_484 | 5 | 12_1430 | 5 | 12_1653 | 5 | 12_1837 | 4 | 12_2053 | 5 |
| 11_491 | 5 | 12_1473 | 4 | 12_1657 | 4 | 12_1839 | 5 | 12_2075 | 4 |
| 11_503 | 4 | 12_1476 | 4 | 12_1672 | 5 | 12_1845 | 5 | 12_2099 | 4 |
| 11_538 | 5 | 12_1486 | 5 | 12_1679 | 5 | 12_1883 | 5 | 12_2122 | 5 |
| 11_547 | 4 | 12_1487 | 4 | 12_1683 | 5 | 12_1884 | 5 | 12_2129 | 5 |
| 11_548 | 5 | 12_1488 | 5 | 12_1684 | 5 | 12_1898 | 4 | 12_2131 | 5 |
| 12_1295 | 5 | 12_1489 | 5 | 12_1685 | 5 | | | | |

*Table 5*: Knots with unsharp Morton–Franks–Williams inequality

***Proposition 4.6*** *The* 2*-cable Morton–Franks–Williams inequality is sharp for prime knots with up to* 12 *crossings.*

To summarize the result of our computation, we assume that the calculation of $P$ is easy, so restrict ourselves to the exceptions. Table 5 gives the 98 prime knots of 12 crossings or less for which the (usual) Morton–Franks–Williams inequality is not sharp, along with their braid index. (The unsharpness of (23) is by 2, except for the knots printed in bold, where it is 4.) Note that all these knots are non-alternating, although for higher crossing numbers alternating examples are known at least for links from [Mu].

## 4.7 Main Application

With all possible framings of $P$ and $F$, we still do not obtain two invariants of degree 9, and expectedly several invariants of degree 10. Thus it seems that Question 4.1 is to be negatively answered. However, by the previous remarks, the only way to do so is to find knots not distinguished by the HOMFLY and Kauffman polynomial and their 2-cables. The only such known examples are mutants [LL], but they have the same invariants up to degree $\leq 10$ [Mr]. (In fact, this result motivated Przytycki's question.) Thus a systematic approach to answering the question negatively seems lacking.

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| | | |
|---|---|---|
| 7_2 :12_1659 | 11_415 :10_8 | 12_1728 :12_1668 |
| 8_19 :12_1727 | 11_431 :11_395 | 12_1298 :12_1295 |
| 9_44 :8_2 | 10_36 :10_19 | 12_1589 :12_1326 |
| 9_45 :8_7 | 11_370 :10_7 | 11_140 :12_1770 |
| 10_159 :9_15 | 11_473 :12_1823 | 11_210 :12_1735 |
| 10_131 :9_8 | 11_452 :10_10 | 11_110 :12_1468 |
| 10_133 :12_1670 | 11_388 :11_371 | 11_118 :11_45 |
| 11_512 :10_140 | 11_491 :10_38 | 11_294 :11_146 |
| 10_151 :9_21 | 11_374 :10_30 | 11_189 :11_30 |
| 10_156 :9_12 | 10_72 :10_43 | 11_56 :12_1608 |
| 9_26 :9_25 | 11_427 :10_46 | 11_216 :11_196 |
| 12_1750 :12_1682 | 11_546 :10_71 | 11_28 :12_1792 |
| 10_35 :10_22 | 12_1893 :12_1556 | 11_180 :12_1302 |
| 11_492 :11_435 | 12_2070 :12_1337 | 11_165 :12_1824 |
| 11_434 :12_1867 | 12_1789 :12_1576 | 11_225 :12_1630 |
| 10_31 :10_14 | 12_2105 :12_1336 | 11_279 :12_1913 |
| 11_461 :10_85 | 12_1458 :12_1394 | 11_330 :11_24 |
| 11_453 :11_385 | 12_1901 :12_1709 | 12_1150 :12_492 |
| 10_56 :10_33 | 12_1903 :12_1652 | 12_742 :12_503 |
| 11_484 :10_20 | 12_1685 :12_1600 | 12_882 :12_212 |

*Table 6*: 60 pairs of knots of $\leq$ 12 crossings with the same $Q$ polynomial, which are not mutants. The comparison of Vassiliev invariants of degree $\leq 8$ on them allows us to prove Theorem 4.7.

The calculations up to degree 8 now allow us to prove our main result.

***Theorem 4.7*** *The $Q$ polynomial determines no Vassiliev knot invariants of degree $d \leq 9$ which are not polynomials of $v_2$.*

**Proof** By the previous symmetry argument, it suffices to consider degree $d \leq 8$. Take the 60 duplications of $Q$ in Table 6. We chose them so that the knots are not mutants (which was verified using the hyperbolic volume). We already observed that the invariants of $FF_+PP_+P_-P_0$ generate all invariants up to degree 8. By evaluating these families on the 120 knots in these pairs, we can confirm this. Now consider the matrix obtained by evaluating $v(K_1) - v(K_2)$ for any Vassiliev invariant $v$ of degree $d \leq 8$ and knots $K_{1,2}$ in a pair (with rows given by a basis of invariants $v$ and columns by pairs of knots). One calculates that this matrix has rank 55, which corresponds to removing the powers $v_2^i$ for $i = 0, \ldots, 4$ from the dimension 60 of Vassiliev invariants of degree $d \leq 8$. (Thus 55 pairs would suffice, but the other 5 are used to ensure some confidence in the calculation.) ■

From Corollary 3.15 we obtain the following.

***Corollary 4.8*** *Assume that* $X \in \mathbf{Q}[x_1, x_2, x_3, \ldots, y_0, y_1, y_2, \ldots]$ *is an honest polynomial*[1]*. If* $X(Q'(-2), Q''(-2), \ldots, Q(2), Q'(2), Q''(2), \ldots)$ *is a polynomially bounded invariant of degree* $d \leq 9$*, then it is* as a knot invariant *a polynomial of degree* $d \leq 4$ *in* $Q'(-2)$.

Note that we do not know whether $X$ is a polynomial of degree $d \leq 4$ in $x_1$, since we do not know whether the $Q^{(k)}(\pm 2)$ are algebraically independent invariants. On the opposite side, one can, with just a bit of reformulation and extra argument, also incorporate the values $V^{(k)}(\pm 1)$ into $X$ in a statement of the above type.

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