A NOTE ON HOMOTOPY INVARIANCE OF TANGENT BUNDLES

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Although tangent bundles of manifolds are not always homotopically invariant, but in some categories of the manifolds they can be homotopically invariant.

In this note, I show that tangent bundles of π-manifolds and almost parallelizable manifolds depend only on their homotopy types.

We denote by M^n a n-dimensional connected closed differentiable manifold, by τM^n the tangent bundle, and by $\approx$ the bundle equivalence.

1. Let M^n be a π-manifold. For a given immersion of M^n into $(n+1)$ -dimensional Euclidean space or a triviality of the stable tangent bundle $\tau M^n \oplus \varepsilon$, we can define a map of M^n into the n-dimensional sphere S^n, so called the "normal map" or the "Gauss map". This map is covered by a bundle map of τM^n into τS^n, and its degree is decisively related to the Euler characteristic $\chi(M^n)$ and the semi-Euler characteristic of M^n. (Milnor [3], Bredon-Kosinski [1])

THEOREM 1. *Let M_1^n and M_2^n be π-manifolds of dimension n, and let $f: M_1^n \to M_2^n$ be an arbitrary homotopy equivalence. Then, we have $\tau M_1^n \approx f^*(\tau M_2^n)$.*

Proof. Let F_i be a framing of the stable tangent bundle $\tau M_i^n \oplus \varepsilon$ and let ν_{F_i} be the Gauss map for $i=1,2$. Firstly we show that we can choose the framings F_1 and F_2 so that $\deg.\nu_{F_1} = \deg.\nu_{F_2}$.

If n is even the assertion is clear, since $\deg.\nu_{F_1} = \frac{1}{2}\chi(M_1^n)$, $\deg.\nu_{F_2} = \frac{1}{2}\chi(M_2^n)$ independently of the choice of the framings and M_1^n, M_2^n are of the same homotopy type.

For $n = 1, 3, 7$, since M_1^n and M_2^n are parallelizable, the theorem is trivial. Let $n = 2r+1$, $n \neq 1, 3, 7$. By Theorem 3 of [1], $\deg.\nu_{F_i} \equiv \sum_{k=0}^{r}$ rank

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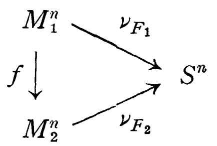
$H_k(M_i^n; Z_2)$ mod. 2 for $i = 1, 2$, independently of the choice of the framings. Since M_1^n and M_2^n are of the same homotopy type, we know that deg. $\nu_{F_1} \equiv$ deg. ν_{F_2} mod. 2 always.

Now, if we construct a new framing F_1' of $\tau M_1^n \oplus \varepsilon$ by a continuous map g: $M_1^n \to \mathrm{SO}(n+1)$, so that $F_1(x) = g(x) F_1'(x)$, there is a following relation between deg. $\nu_{F_1'}$ and deg. ν_{F_1}. ((2·2) of [1]).

deg. $\nu_{F_1'}$ = deg. $(\pi \circ g)$ + deg. ν_{F_1}, where π denotes the canonical projection of $\mathrm{SO}(n+1)$ onto S^n. Thus we have,

deg. $\nu_{F_1'}$ − deg. ν_{F_2} = deg. $(\pi \circ g)$ + (deg. ν_{F_1} − deg. ν_{F_2}). We note that deg. ν_{F_1} − deg. ν_{F_2} is even. If n is odd, since there exists a continuous map g such that deg. $(\pi \circ g)$ can be any even integer, we can choose F_1' so that deg. $\nu_{F_1'}$ = deg. ν_{F_2}.

Thus we can choose the framings F_1 and F_2 so that deg. ν_{F_1} = deg. ν_{F_2}. We may assume that deg. $f = 1$. Then, in the following diagram, deg. $\nu_{F_2} \circ f$ = deg. $f \times$ deg. ν_{F_2} = deg. ν_{F_2} = deg. ν_{F_1}. So that by Hopf's



Classification Theorem, $\nu_{F_2} \circ f$ is homotopic to ν_{F_1}, that is, the diagram is homotopy commutative. Therefore we have $\tau M_1^n \approx \nu_{F_1}^*(\tau S^n) \approx (\nu_{F_2} \circ f)^*(\tau S^n) \approx f^*(\nu_{F_2}^*(\tau S^n)) \approx f^*(\tau M_2^n)$.

This completes the proof of the theorem.

In the proof of this theorem, if $\chi(M_1^n) = \chi(M_2^n)$ or if the semi-Euler characteristic of M_1^n is equal to that of M_2^n in mod. 2, according as n is even or odd, then the map f need not be a homotopy equivalence but the condition deg. $f = 1$. So, we have

THEOREM 1'. *Let M_1^n and M_2^n be π-manifolds of dimension n. If n is even and $\chi(M_1^n) = \chi(M_2^n)$, or if n is odd and the semi-Euler characteristic of M_1^n is equal to that of M_2^n in mod. 2, then for any continuous map f: $M_1^n \to M_2^n$ of degree 1, $\tau M_1^n \approx f^*(\tau M_2^n)$.*

COROLLARY 1. *In the category of π-manifolds, whether a π-manifold of even dimension has an almost complex structure or not depends only on its homotopy type.*

Proof. This is clear.

COROLLARY 2. *Let M_1^n and M_2^n be π-manifolds of dimension n, and let f: $M_1^n \to M_2^n$ be a homotopy equivalence. Then, for any twisted spheres T_1 and T_2 of*

Γ^n, there exists a homotopy equivalence $g\colon M_1^n \# T_1 \to M_2^n \# T_2$ such that $\tau(M_1^n \# T_1) \approx g^*\tau(M_2^n \# T_2)$.

Proof. Since $M_i^n \# T_i$ $i = 1, 2$ are π-manifolds and are homeomorphic to M_i^n, this follows from Theorem 1.

2. Let us consider such a manifold M^n that $M^n -$ (a point) is an open π-manifold. If we call such a manifold to be almost π, a manifold M^n is almost π if and only if it is almost parallelizable. Because, the tangent bundle of $M^n -$ (a point) is induced from that of $M^n -$ (Interior of an imbedded n-disk), and a manifold with boundary is π if and only if it is parallelizable. (Kervaire-Milnor [5]).

THEOREM 2. *Let M_1^n and M_2^n be almost parallelizable manifolds, and let $f\colon M_1^n \to M_2^n$ be a homotopy equivalence. Then, we have that $\tau M_1^n \approx f^*(\tau M_2^n)$. In other words, tangent bundles of $(n-1)$-parallelizable manifolds are homotopically invariant.*

Proof. Let $O_i \in H^n(M_i^n; \pi_{n-1}(SO_{n+1}))$ $i = 1, 2$. be the obstruction class for extending the triviality of $\tau M_i^n \oplus \varepsilon$ on the $(n-1)$-skeleton over the whole. If $n \equiv 1, 2, 3, 5, 6, 7 \mod. 8$, by the analogous argument of Kervaire and Milnor [4], [5], we know that M_1^n and M_2^n are π-manifolds. So, the theorem is valid by Theorem 1. If $n = 4k$, since the k-th Pontrjagin classes $P_k(M_i^n) = m\ O_i$ (m: an integer) $i = 1, 2$ and the indexes of M_1^n and M_2^n are equal (we may assume that deg. $f = 1$.), we know that $O_1 = f^*O_2$. So that, the obstruction class $f^*O_2 - O_1$ for extending the isomorphism of $\tau M_1^n \oplus \varepsilon$ onto $f^*(\tau M_2^n) \oplus \varepsilon$ on the $(n-1)$-skeleton over the whole vanish. Thus, $f^*(\tau M_2^n)$ is stably equivalent to τM_1^n. But, in this case, we can show that $f^*(\tau M_2^n)$ is equivalent to τM_1^n; If we denote by $\alpha_i \in H^n(M_i^n; \pi_{n-1}(SO_n))$ $i = 1, 2$ the obstruction classes for extending the triviality of τM_i^n on the $(n-1)$-skeleton over the whole, then the obstruction class for extending the isomorphism of τM_1^n onto $f^*(\tau M_2^n)$ on the $(n-1)$-skeleton is given by $f^*\alpha_2 - \alpha_1$. we can show that $f^*\alpha_2 - \alpha_1 = 0$. The proof is included in that of K. Shiraiwa [7].

Finally, note that if M^n is a $(n-1)$-parallelizable manifold, then M^n is almost parallelizable. Because, choose a point of M^n and tie to an interior point of every n-simplex with an imbedded arc. Then, there exists a n-cell which contains the tree.

This completes the proof.

COROLLARY 3. *For* $(n-1)$*-connected* $2n$*-manifolds,* $n \equiv 3, 5, 6, 7$ *mod.* 8, *their tangent bundles are homotopically invariant. For* $(n-1)$*-connected* $(2n+1)$*-manifolds,* $n \equiv 5, 6$ *mod.* 8, *their tangent bundles are homotopically invariant.*

Proof. In this case, these manifolds are almost parallelizable or stably parallelizable.

COROLLARY 4. *The matters corresponding to the corollaries* 1, 2 *are also valid for almost parallelizable manifolds.*

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