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ON A PARAMETRIZED LEVI PROBLEM INVOLVING ONE COMPLEX VARIABLE

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ABSTRACT. The classical Levi problem in several complex variables characterizes domains of holomorphy in terms of a boundary condition called pseudo convexity. The purpose of this note is to give a characterization of those domains D in $\mathbb{C}\times\mathbb{R}^p$, where one can always solve the $\bar\partial$-problem with C^∞ parameters, in terms of a certain kind of convexity condition on their boundaries.

The $\bar\partial$-operator is defined on any complex manifold M. Now if X is a C^∞ manifold, then one can define a parametrized $\bar\partial$-operator on the product $M\times X$ as follows. Let $\mathscr{E}^{p,q}$ be the sheaf of germs of sections of the pull-back bundle $\pi^*(T^{*(p,q)})$, where $\pi: M\times X\to M$ is the projection and $T^{*(p,q)}$ is the bundle of covectors of type (p, q) on M. Then there is a naturally induced map $\bar\partial:\mathscr{E}^{p,q}\to\mathscr{E}^{p,q+1}$, the parametrized $\bar\partial$-operator. Let $\mathscr{A}^{p,0}:=\operatorname{Ker}(\mathscr{E}^{p,0}\xrightarrow{\bar\partial}\mathscr{E}^{p,1})$ and $\mathscr{A}:=\mathscr{A}^{0,0}$. Note that $\mathscr{A}$ is the sheaf of germs of $\mathbb{C}$-valued C^∞ functions on $M\times X$ which are "holomorphic when restricted to complex directions". Because the equation $\bar\partial u=f$ can be locally solved involving C^∞ parameters (e.g. [2, ID2]), the classical Grothendieck's Lemma extends to the parametrized $\bar\partial$-operator. Thus the sequence

$$0\to\mathscr{A}^{p,0}\xrightarrow{i}\mathscr{E}^{p,0}\xrightarrow{\bar\partial}\mathscr{E}^{p,1}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{E}^{p,n}\to 0,\qquad 0\le p\le n:=\dim_{\mathbb{C}} M,$$

is exact.

This note is concerned with characterizing domains D in $\mathbb{C}\times\mathbb{R}^p$, where one can always solve the parametrized $\bar\partial$-problem. But this just means that $\bar\partial$ must be *surjective at the section level.* Since

$$H^1(D,\mathscr{A})=\Gamma(D,\mathscr{E}^{0,1})/\bar\partial\Gamma(D,\mathscr{E}^{0,0}),$$

this is equivalent to the "cohomology vanishing" condition $H^1(D,\mathscr{A})=0$. To do this we will make use of results of Malgrange [6] and Hörmander [3]; note also the survey article of Treves [9]. In passing we mention that parametrized $\bar\partial$-problems and cohomology vanishing results, particularly involving several

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complex variables, have also been considered by Andreotti-Grauert [1], Rea [7], [8] and Jurchescu [4], [5].

DEFINITION. Suppose $X \subset \mathbb{R}^n$ is an open set and P is a differential operator defined on X with constant coefficients. The set X is said to be *P-convex with respect to supports* if for every compact set $K \subset X$ there is a compact set $K' \subset X$ such that

$$v \in \mathscr{E}'(X) \quad \text{with} \quad \operatorname{supp} P(-D)v \subset K \Rightarrow \operatorname{supp} v \subset K'.$$

THEOREM (Malgrange [6]). *Suppose X is a domain in $\mathbb{R}^n$ and P is a differential operator on X with constant coefficients. Then the map $P: C^\infty(X) \to C^\infty(X)$ is surjective if and only if X is P-convex with respect to supports.*

For $D \subset \mathbb{C} \times \mathbb{R}^p$, let $d_D(x) := d(x, \mathbb{C} \times \mathbb{R}^p \setminus D)$ and for any compact subset K of D, let $d_K := \min_{x \in K} d_D(x)$.

DEFINITION. Suppose D is a domain in $\mathbb{C} \times \mathbb{R}^p$. Then we say that $d_D(x)$ satisfies the *minimum principle in any affine space parallel to* $\mathbb{C}$ if for any $t \in \mathbb{R}^p$ and any compact subset $K \subset \pi^{-1}(t) \cap D$ one has $d_K = d_{\partial K}$, where π denotes the projection from $\mathbb{C} \times \mathbb{R}^p$ onto the second factor. By abuse of language we identify K with its image under the projection onto $\mathbb{C}$, whenever convenient.

The main tool which we shall use is the following

THEOREM (Hörmander [3]). *Suppose $P(D)$ is a differential operator in $\mathbb{R}^n$ which acts along some linear subspace $V \subset \mathbb{R}^n$ and is elliptic as an operator in V. Then an open subset $X \subset \mathbb{R}^n$ is P-convex with respect to supports if and only if $d_X(x)$ satisfies the minimum principle in any affine subspace parallel to V.*

DEFINITION. Suppose D is a domain in $\mathbb{C} \times \mathbb{R}^p$ and $c_0 = (z_0, t_0) \in \mathbb{C} \times \mathbb{R}^p \setminus D$. Then D is said to be *concave in a direction orthogonal to* $\mathbb{C}$ *relative to the point* c_0 if there exists $t_1 \in \mathbb{R}^p$ and a compact subset $K \subset \pi^{-1}(t_1) \cap D$ with the following properties:

(i) $(z_0, t_1) \in \operatorname{Int}(K)$

(ii) $\partial K \times \overline{t_0 t_1} \subset D$, where $\overline{t_0 t_1}$ is the line segment joining t_0 and t_1. Further, if the domain D is not concave orthogonal to $\mathbb{C}$ relative to any point in its complement, then we say that D satisfies the *parametrized Levi condition.*

We now prove the characterization which we seek.

THEOREM. *Suppose $D \subset \mathbb{C} \times \mathbb{R}^p$ is a domain. Then $H^1(D, \mathscr{A}) = 0$ if and only if D satisfies the parametrized Levi condition.*

Proof. Clearly it suffices to show that D is concave in a direction orthogonal to $\mathbb{C}$ relative to some point $c_0 \in \mathbb{C} \times \mathbb{R}^p \setminus D$ if and only if there exists an affine subspace parallel to $\mathbb{C}$ where d_D does not satisfy the minimum principle.

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First suppose $c_0=(z_0, t_0)\in\mathbb{C}\times\mathbb{R}^p\setminus D$, $t_1\in\mathbb{R}^p$ and $K\subset\pi^{-1}(t_1)\cap D$ such that $(z_0, t_1)\in\mathrm{Int}(K)$ and $\partial K\times\overline{t_0t_1}\subset D$. Then $\alpha:=d(K\cup(\partial K\times\overline{t_0t_1}), \mathbb{C}\times\mathbb{R}^p\setminus D)>0$. Let

$$C:=(\mathbb{C}\times\mathbb{R}^p\setminus D)\cap(K\times\overline{t_0t_1}).$$

Then C is not empty, since $(z_0, t_0)\in C$. Since both K and C are compact, $d(K, C)$ is realized by some $(z', t')\in C$ and $(z'', t_1)\in K$. Then $z'=z''$, since $(z', t_1)\in K$. Let

$$\hat{t}:=t'+\frac{\alpha}{2}\frac{t_1-t_0}{|t_1-t_0|} \quad\text{and}\quad L:=\{(z, \hat{t})\in\mathbb{C}\times\mathbb{R}^p:(z, t_1)\in K\}.$$

Then $d_L=|(z', t')-(z', \hat{t})|=|t'-\hat{t}|=\alpha/2$. But since $\partial L\subset\partial K\times\overline{t_0t_1}$, we have $d_{\partial L}\geq\alpha$ and thus there exists an appropriate compact L with $d_L<d_{\partial L}$.

Conversely suppose there exists $t_1\in\mathbb{R}^p$ and a compact subset $L\subset\pi^{-1}(t_1)\cap D$ such that $d_L<d_{\partial L}$. Choose $x=(z, t_1)\in L$ and $c_0=(z_0, t_0)\in\mathbb{C}\times\mathbb{R}^p\setminus D$ such that $d(x, c_0)=d_L$. Note that $x\in\mathrm{Int}(L)$ by force. Now $\partial L\times\overline{t_0t_1}\subset D$. For, otherwise $d_{\partial L}\leq|t_0-t_1|\leq|x-c_0|=d_L$. Finally we claim $(z_0, t_1)\in\mathrm{Int}(L)$. For, if $(z_0, t_1)\in\partial L$, then $d_{\partial L}\leq|t_0-t_1|\leq d_L$ again. And if $(z_0, t_1)\in\pi^{-1}(t_1)\setminus L$, then since $(z, t_1)\in\mathrm{Int}(L)$ there would exist a point (z', t_1) in ∂L and on the straight line segment joining (z_0, t_1) with (z, t_1) with

$$d((z', t_1), (z_0, t_0))<d((z, t_1), (z_0, t_0)).$$

This then gives the contradiction $d_{\partial L}\leq d_L$. Thus D is concave in a direction orthogonal to $\mathbb{C}$ relative to the point c_0. □

REMARK 1. If $c_j:\mathbb{R}^p\to\mathbb{C}\times\mathbb{R}^p$, $1\leq j\leq k$, are continuous functions such that $c_j(t)\in\mathbb{C}\times\{t\}$ for every $t\in\mathbb{R}^p$, then the domain $D:=\mathbb{C}\times\mathbb{R}^p\setminus\bigcup_{j=1}^k c_j(\mathbb{R}^p)$ satisfies the parametrized Levi condition. If for $p=1$, we define $c(t):=(t, t)$ for every $t\in\mathbb{R}$, then $D:=\mathbb{C}\times\mathbb{R}\setminus c(\mathbb{R})$ is *not* a regular family of domains of holomorphy in the sense of Andreotti-Grauert [1]. They show that if D is a regular family of domains of holomorphy in $\mathbb{C}^n\times\mathbb{R}^p$, then $H^q(D, \mathcal{A})=0$ for every $q>0$. This means that, while their condition is sufficient, it is certainly not necessary in order that $H^1(D, \mathcal{A})=0$.

REMARK 2. By using smooth cut-off functions and partitions of unity, the reader can easily check that the above characterization also holds for subdomains of $\mathbb{C}\times X$, where X is a smooth manifold. This follows since the parametrized Levi condition is local with respect to the real variables and the above proof need not make explicit use of the fact that $\overline{t_0t_1}$ is a line segment.

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