Can. J. Math., Vol. XXVI, No. 6, 1974, pp. 1455–1465

TOPOLOGICAL PROPERTIES OF LIMITS OF INVERSE SYSTEMS OF MEASURES

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Introduction. It has been shown (Mallory and Sion [**6**]) that the problem of finding "limit" measures for inverse systems of measure spaces $(X_i, \mu_i)_{i \in I}$ can be successfully attacked by establishing the existence of a "limit" measure $\tilde{\mu}$ on the product space $\prod_{i \in I} X_i$, then considering the restriction to the inverse limit set $L \subset \prod_{i \in I} X_i$.

In this paper we use a similar point of view to establish conditions under which a system of Radón measures has a "limit" measure which is also Radón.

In Section 1 we establish that a Radón limit measure exists in $\prod_{i \in I} X_i$ under quite weak conditions for the product topology and a related topology (Theorems 4, 8). In Section 2 we examine the restriction of the resulting measures to the inverse limit set L and show that previously known results can be obtained and extended by these means (Theorems 2, 4). We conclude with an example (5, Section 2) which illustrates the difficulty of obtaining Radón limit measures on L with weaker conditions.

Preliminaries. Almost all of the set theory and topology used is quite standard. However, if for each $i \in I, \mathscr{H}_i$ is a family of subsets of a space X_i we define

$$\text{Rect}\,(\mathscr{H}) = \Big\{\alpha : \alpha = \prod_{i \in I} A_i, A_j = X_j \text{ for all but a finite number of } j \in I, \text{ and } A_j \in \mathscr{H}_j \text{ otherwise}\Big\}.$$

Also, for any $\alpha \subset \prod_{i \in I} X_i$ we define:

$$J_\alpha = \{j \in I : \pi_j(\alpha) \neq X_j\}.$$

(If α is a "rectangle" J_α is the set of indices of its base.)

We will use Carathéodory measure theory concepts and notation throughout (see, e.g., Sion [**10**: **11**]). If Y is a space with topology $\mathscr{G}$, ν is a *Radón outer measure* on Y if and only if ν is an outer measure on Y, $\mathscr{G} \subset \mathscr{M}_\nu$, closed compact sets have finite measure and for every $A \subset Y$,

$$\nu(A) = \inf \{\nu(G) : A \subset G, G \in \mathscr{G}\},$$

and for every $G \in \mathscr{G}$,

$$\nu(G) = \sup \{\nu(K) : K \subset G, K \text{ closed and compact}\}.$$

Received May 29, 1973 and in revised form, August 10, 1973.

https://doi.org/10.4153/CJM-1974-140-7 Published online by Cambridge University Press

We will denote by (X, p, μ, I) an *inverse system of outer measures*, where I is the index set (directed by $\prec$), $X = \{X_i\}_{i \in I}$ is an indexed family of spaces, $\mu = \{\mu_i\}_{i \in I}$ is an indexed family of outer measures μ_i on the spaces X_i and for $i \prec j$ $p_{ij} : X_j \to X_i$ is a measureable function. We denote by $\mathscr{M}_i$ the μ_i-measurable subsets of X_i and require that for $i \prec j \prec k$ and $A \in \mathscr{M}_i$, $\mu_j(p_{ij}^{-1}[A]) = \mu_i(A)$ and

$$\mu_k(p_{ik}^{-1}[A] \Delta (p_{ij} \circ p_{jk})^{-1}[A]) = 0$$

(instead of the usual $p_{ij} \circ p_{jk} = p_{ik}$). We define ν to be a *π-limit outer measure* for (X, p, μ, I) if and only if ν is an outer measure on $\prod_{i \in I} X_i$ and whenever $i \prec j$ and $A \in \mathscr{M}_i$,

(i) $\pi_i^{-1}[A] \in \mathscr{M}_\nu$

(ii) $\nu(\pi_i^{-1}[A]) = \mu_i(A)$

(iii) $\nu(\pi_i^{-1}[A] \Delta (p_{ij} \circ \pi_j)^{-1}[A]) = 0$.

We will say ν is an *inverse limit outer measure* for (X, p, μ, I) if and only if ν is a π-limit outer measure and ν is carried by the inverse limit set

$$L = \left\{ x \in \prod_{i \in I} X_i : p_{ij}(x_j) = x_i \quad \text{whenever} \quad i \prec j, i, j \in I \right\}.$$

For any system (X, p, μ, I) we define g to be the set function on Rect $(\mathscr{M})$ given by

$$g(\alpha) = \mu_k\Big(\bigcap_{j \in J_\alpha} p_{jk}^{-1}[\pi_j[\alpha]] \Big)$$

for all $\alpha \in$ Rect $(\mathscr{M})$, where $k \succ j$ for all $j \in J_\alpha$, and $\tilde{\mu}$ to be the Carathéodory measure on $\prod_{i \in I} X_i$ generated by g and Rect $(\mathscr{M})$.

A detailed discussion of the concepts of inverse limits etc. defined above is given by Mallory and Sion [**6**], cf. also Bourbaki [**1**], Choksi [**2**] and Metivier [**7**].

1. Topological properties of π-limit measures. This section is concerned with Radón π-limit measures on the product space $\prod_{i \in I} X_i$.

We obtain such measures by "regularizing" the measure $\tilde{\mu}$. In order to do this we examine approximation properties of $\tilde{\mu}$ in 2.1, 2.2 and prove a Radónization lemma (3) designed to utilize these properties. We are then able to obtain Radón π-limit measures for two topologies on $\prod_{i \in I} X_i$: the product topology (Theorem 8), and a coarser topology (Theorem 4) which arises from the approximating sets used in 2.1, 2.2.

1. *General assumptions and notation.* Throughout this section we suppose:

1.1. (X, p, μ, I) is a system.

1.2. For every $i \in I$:

1.2.1. X_i is a topological space and $\mathscr{G}_i, \mathscr{T}_i$ and $\mathscr{K}_i$ represent respectively the families of open sets, closed sets, and closed compact sets in the topology of X_i.

1.2.2 μ_i is a σ-finite Radón outer measure on X_i.

1.3. $\mathscr{G}^\sim, \mathscr{F}^\sim, \mathscr{K}^\sim$ represent respectively the families of open sets, closed sets and closed compact sets in the product topology on $\prod_{i\in I} X_i$.

1.4. $\mathscr{C}' = \{A : A = \bigcup_{j=1}^n K_j,\ n \in \omega, K_1 \ldots K_n \in \text{Rect}(\mathscr{K})\}$. (Rect $(\mathscr{K})$ is the family of rectangles with compact bases.)

1.5. $\mathscr{G}'$ is the topology on $\prod_{i\in I} X_i$ generated by the complements of elements of Rect $(\mathscr{K})$.

The above systems satisfy the hypotheses of [**6**, Theorem 2.7] so that $\tilde{\mu}$ is a π-limit outer measure.

Note that the complements of elements of Rect $(\mathscr{K})$ are open in the product topology $\mathscr{G}^\sim$, and form a base for $\mathscr{G}^\sim$ if the spaces X_i are compact. Notice also that Rect $(\mathscr{K})$, hence $\mathscr{C}'$ consists of sets which are closed and compact in the topology $\mathscr{G}'$.

We now note some approximation properties of the measure $\tilde{\mu}$.

2.1 Theorem. *Let* $A \in \mathscr{M}_{\tilde{\mu}}$ *and* $\tilde{\mu}(A) < \infty$. *Then*

$$(*) \quad \tilde{\mu}(A) = \sup\{\tilde{\mu}(C) : C \in (\mathscr{C}')_\delta, C \subset A\}.$$

Proof. See Mallory and Sion [**6**].

2.2 Proposition. *Every* $A \in$ Rect $(\mathscr{M})$ *satisfies* (*).

Proof. See Mallory and Sion [**6**].

The properties of $\tilde{\mu}$ given above allow us to produce Radón π-limit measures without requiring particular topological properties (e.g. regularity) of the spaces $\{X_i\}_{i\in I}$. Lemma 3 below will be used to construct Radón π-limit measures for the topologies $\mathscr{G}'$ and $\mathscr{G}^\sim$.

3. Lemma. *Let Y be a topological space with topology* $\mathscr{G}^-$, *and let* $\mathscr{K}^-$ *be the closed compact sets.*

Let $\mathscr{B}$ *be a base for* $\mathscr{G}^-$ *closed under finite unions and intersections.*

Let ν *be an outer measure on Y such that* ν *is finitely additive on* $\mathscr{K}^-$, *finite on* $\mathscr{K}^- \cup \mathscr{B}$, *and such that for every* $B \in \mathscr{B}$,

$$\nu(B) = \sup\{\nu(F) : F \subset B,\ F \text{ closed},\ F \in \mathscr{M}_\nu\}.$$

If we let

$$h(G) = \sup\{\nu(K) : K \in \mathscr{K}^-, K \subset G\}$$

for every $G \in \mathscr{G}^-$ *and*

$$\mu(A) = \{\inf h(G) : G \in \mathscr{G}^-, A \subset G\}$$

for every $A \subset Y$, *then:*

3.1. μ *is a Radón outer measure for* $\mathscr{G}^-$,

3.2. $\mu(G) = h(G)$ *for every* $G \in \mathscr{G}^-$,

3.3. $\mu(K) \geqq \nu(K)$ *for every* $K \in \mathscr{K}^-$.

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Proof. Let $K \in \mathscr{K}^-$, $B_1, B_2, \ldots, B_n \in \mathscr{B}$ and $K \subset \bigcup_{m=1}^n B_n$. Let $\epsilon \in 0$ and choose, for each $j \leqq n$, $F_j \subset B_j$, F_j closed such that

$$\nu(B_j) < \nu(F_j) + \epsilon/n.$$

Then

$$\nu(K) \leqq \sum_{j=1}^n \nu(K \cap B_j) \leqq \sum_{j=1}^n \nu(K \cap F_j) + \epsilon \leqq \sum_{j=1}^n h(B_j) + \epsilon$$

hence

$$(\dagger) \quad \nu(K) \leqq \sum_{j=1}^n h(B_j).$$

From this and the fact that h is clearly monotone it follows that h is countably subadditive on $\mathscr{G}^-$.

Since $\mathscr{G}^-$ is closed under unions.

$$\inf\left\{\sum_{n\in\omega} h(G_n) : A \subset \bigcup_{n\in\omega} G_n, G_n \in \mathscr{G}^-, n \in \omega\right\}$$
$$= \inf\{h(G) : A \subset G, G \in \mathscr{G}^-\}$$

so that μ is the Carathéodory measure generated by h and $\mathscr{G}^-$. It thus follows from the standard Carathéodory Extension Theorem (see, e.g., Mallory and Sion [**6**]) that μ is a Carathéodory measure and $\mu(G) = h(G)$ for all $G \in \mathscr{G}^-$ (3.2).

That $\mathscr{G}^- \subset \mathscr{M}_\mu$ follows from straightforward use of approximation properties and the fact that ν is finitely additive on $\mathscr{K}^-$.

That $\mu(K) < \infty$ for every $K \in \mathscr{K}^-$ follows immediately from (†). Thus μ is Radón for $\mathscr{G}^-$. 3.3 is immediate from the definitions.

4. THEOREM. *If for each $i \in I$, μ_i is bounded then there exists a π-limit outer measure which is Radón for the topology $\mathscr{G}'$* (Definition 1.5).

Proof. We begin by checking that $\tilde{\mu}$ satisfies the conditions on ν in Lemma 3. Let

$$\mathscr{B} = \left\{G : G = \prod_{i\in I} X_i \sim C \quad \text{for some} \quad C \in \mathscr{C}'\right\},$$

and $\mathscr{K}'$ be the closed compact subsets of $\prod_{i\in I} X_i$ for the topology $\mathscr{G}'$. Then since $\mathscr{C}'$, hence $\mathscr{B}$, is contained in the σ-field generated by Rect $(\mathscr{M})$, $\mathscr{B} \subset \mathscr{M}_\mu$, and since $\mathscr{C}_\delta' \subset \mathscr{K}'$ we have from Theorem 2.1, for every $B \in \mathscr{B}$,

$$\tilde{\mu}(B) = \sup\{\tilde{\mu}(C) : C \in \mathscr{C}_\delta', C \subset B\}$$
$$= \sup\{\tilde{\mu}(K) : K \in \mathscr{K}', K \subset B\}.$$

It also follows from the fact that $\mathscr{B}$ is a base and $\mathscr{B} \subset \mathscr{M}_\mu$ that $\tilde{\mu}$ is finitely additive on $\mathscr{K}'$. Since for each $i \in I$, μ_i is bounded, $\tilde{\mu}$ satisfies the conditions on ν.

https://doi.org/10.4153/CJM-1974-140-7 Published online by Cambridge University Press

Let ψ be the Radón measure generated by $\tilde{\mu}$. It remains to show that ψ is a π-limit outer measure.

For any $M \in \text{Rect}(\mathscr{M})$, from Theorem 2.1

$$\begin{aligned}\psi(M) &\geqq \sup\{\psi(C) : C \in \mathscr{C}_\delta', C \subset M\} \\ &\geqq \sup\{\tilde{\mu}(C) : C \in \mathscr{C}_\delta', C \subset M\} \geqq \tilde{\mu}(M).\end{aligned}$$

Since Rect $(\mathscr{M})$ is a semi-ring and $\mathscr{C}_\delta'$ is closed under finite unions, a similar result holds for $\prod_{i\in I} X_i \sim M$. Thus $\psi(M) = \tilde{\mu}(M)$ and $M \in \mathscr{M}_\psi$ (since $\mathscr{C}_\delta' \subset \mathscr{M}_\psi$).

Note that as a result of working on $\prod_{i\in I} X_i$ instead of the inverse limit set L we need no conditions on the functions p_{ij} beyond those necessary to form a system, also that no special topological properties of the spaces X_i are needed.

We next seek conditions under which we can find a π-limit outer measure which is Radón for the product topology $\mathscr{G}^\sim$.

First we define a candidate for such a measure by "regularizing" $\tilde{\mu}$ with respect to $\mathscr{G}^\sim$.

5. *Definition.* $\tilde{\mu}^*$ is the set function on the subsets of $\prod_{i\in I} X_i$ defined by

$$\tilde{\mu}^*(A) = \{\inf h(G) : A \subset G, G \in \mathscr{G}^\sim\}$$

for every $A \subset \prod_{i\in I} X_i$, where

$$h(G) = \sup\{\tilde{\mu}(K) : K \subset G, K \in \mathscr{K}^\sim\}$$

for every open set G.

6. THEOREM.

6.1. *$\tilde{\mu}^*$ is a Radón outer measure for the product topology $\mathscr{G}^\sim$.*

6.2. *$\tilde{\mu}^*(G) = h(G)$ for every $G \in \mathscr{G}^\sim$ and $\tilde{\mu}^*(K) \geqq \tilde{\mu}(K)$ for every $K \in \mathscr{K}^\sim$.*

6.3. Rect $(\mathscr{M}) \subset \mathscr{M}_{\tilde{\mu}^*}$.

6.4. *If $\tilde{\mu}^*$ is a π-limit outer measure then for every $G \in$* Rect $(\mathscr{G})$

$$\tilde{\mu}^*(G) = \tilde{\mu}(G) = \sup\{\tilde{\mu}(K) : K \subset G, K \in \mathscr{K}^\sim\}.$$

6.5. *If ψ is any π-limit outer measure which is Radón for $\mathscr{G}^\sim$, then $\psi = \tilde{\mu}^*$.*

Proof of 6. We first check that $\tilde{\mu}$ satisfies the hypotheses of Lemma 3 for the product topology $\mathscr{G}^\sim$.

The proof that $\tilde{\mu}$ is additive on $\mathscr{K}^\sim$ is the same as in the proof of Theorem 4.

If we now let

$$\begin{aligned}\mathscr{T} &= \{G : G \in \text{Rect}(\mathscr{G}) \text{ and } \tilde{\mu}(G) < \infty\}, \text{ and} \\ \mathscr{E} &= \{\text{finite unions of elements of } \mathscr{T}\},\end{aligned}$$

then since $\mathscr{C}_\delta'$ consists of sets which are closed in the product topology and $\mathscr{C}_\delta' \subset \mathscr{M}_{\tilde{\mu}}$, we see from Theorem 2.1 that the hypotheses of Lemma 3 are satisfied if we replace ν by $\tilde{\mu}$, $\mathscr{B}$ by $\mathscr{E}$ and $\mathscr{K}^-$ by $\mathscr{K}^\sim$. Thus $\tilde{\mu}^*$ is a Radón outer measure for the product topology $\mathscr{G}^\sim$, and 6.2 holds.

To show that 6.3 holds we first note that for $\epsilon > 0$ and $\alpha \in \text{Rect}(\mathscr{M})$ there exists $G \in \text{Rect}(\mathscr{G})$ with $\alpha \subset G$ and $\tilde{\mu}(G \sim \alpha) < \epsilon$. Since $\text{Rect}(\mathscr{G}) \subset \text{Rect}(\mathscr{M})$ and $\text{Rect}(\mathscr{M})$ is a semi-ring we can write $G \sim \alpha$ as a disjoint union of a finite number of elements of $\text{Rect}(\mathscr{M})$. Each of these can be approximated by elements of $\text{Rect}(\mathscr{G})$. Thus there exists $\bar{G} \in \mathscr{G}$ with $G \sim \alpha \subset \bar{G}$ and $\tilde{\mu}^*(\bar{G}) \leqq \tilde{\mu}(\bar{G}) \leqq 2\epsilon$. Hence $\alpha \in \mathscr{M}_{\tilde{\mu}^*}$.

6.4 follows from 6.2 and the fact that $\text{Rect}(\mathscr{G}) \subset \text{Rect}(\mathscr{M})$.

6.5 follows from the fact that $\mathscr{E}$ is closed under finite unions and is a base for $\mathscr{G}^\sim$, hence determines the values of a Radón measure.

We next establish a condition under which $\tilde{\mu}^*$ is a π-limit measure (essentially when the result in 6.4 holds).

7. Lemma. *If for every* $G \in \text{Rect}(\mathscr{G})$

(a) $\tilde{\mu}(G) = \sup\{\tilde{\mu}(K) : K \subset G, K \in \mathscr{K}^\sim\}$,

then $\tilde{\mu}^*$ *will be a* π*-limit outer measure.*

Proof. In view of Theorem 6 it remains only to show that $\tilde{\mu}^*(\alpha) = \tilde{\mu}(\alpha)$ for every $\alpha \in \text{Rect}(\mathscr{M})$.

First we note that (a) assures $\tilde{\mu}^*(G) = h(G) = \tilde{\mu}(G)$ for $G \in \text{Rect}(\mathscr{G})$. For $\alpha \in \text{Rect}(\mathscr{M})$ and $\epsilon > 0$ choose $G \in \text{Rect}(\mathscr{G})$ such that $\alpha \subset G$ and $\tilde{\mu}(G \sim \alpha) \leqq \epsilon$. Let $G \sim \alpha = \bigcup_{j=1}^n \alpha_j, \alpha_1, \ldots, \alpha_n$ Rect $(\mathscr{M})$ and $\alpha_i \cap \alpha_j = \emptyset, i \neq j$. Then clearly there exists $B_1, \ldots, B_n \in \text{Rect}(\mathscr{G})$ with $\sum_{j=1}^n \tilde{\mu}(B_j) < 2\epsilon$ so that

$$\tilde{\mu}(\alpha) - 2\epsilon \leqq \tilde{\mu}(G) - 2\epsilon = \tilde{\mu}^*(G) - 2\epsilon \leqq \tilde{\mu}^*(\alpha) \leqq \tilde{\mu}^*(G) \leqq \tilde{\mu}(\alpha) + \epsilon,$$

hence $\tilde{\mu}^*(\alpha) = \tilde{\mu}(\alpha)$ and $\tilde{\mu}^*$ is a π-limit outer measure.

We are now able to give conditions on (X, p, μ, I) which will guarantee that $\tilde{\mu}^*$ is a Radón π-limit outer measure for the product topology $\mathscr{G}^\sim$.

8. Theorem. $\tilde{\mu}^*$ *is a* π*-limit outer measure whenever any one of the following conditions holds:*

8.1 I *is countable,*

8.2. *there exists a countable cofinal set* $I_0 \subset I$ *and* $p_{ij}[A] \in \mathscr{K}_i$ *for every* $A \in \mathscr{K}_j$ *whenever* $i < j$,

8.3. $p_{ij}[A] \in \mathscr{K}_i$ *for every* $A \in \mathscr{K}_j$ *and* $p_{ij}^{-1}[B] \in \mathscr{K}_j$ *for every* $B \in \mathscr{K}_i$ *whenever* $i < j$.

Proof. $\tilde{\mu}^*$ is Radón by Theorem 6. Hence it is only necessary to show that (a) in Lemma 7 holds.

First assume that 8.1 holds. Let $I = \{i_0, i_1, i_2, \ldots\}$, $G \in \text{Rect}(\mathscr{G})$ and $t < \tilde{\mu}(G)$. Then choose by recursion $K_n \in \mathscr{K}_{i_n}$ such that $K_n \subset \pi_{i_n}[G]$ and

$$\left(G \cap \bigcap_{m=0}^{n} \pi_{i_m}^{-1} K_m\right) > t$$

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(this can be done by Lemma B in Theorem 2.7 of [**6**]). Now let

$$C = \bigcap_{n \in \omega} \pi_{i_n}^{-1}[K_n].$$

Then $C \subset G$, $C \in \mathscr{K}^{\sim}$ and $\tilde{\mu}(C) \geqq t$. Hence (a) holds.

Now assume 8.2 holds. Let $G \in$ Rect $(\mathscr{G})$ and choose a cofinal subset $\{i_0, i_1, \ldots\}$ of I with $j \prec i_0$ for every $j \in J_G$ and $i_n \prec i_{n+1}$ for every $n \in \omega$. Let $t < \tilde{\mu}(G)$ and for each $n \in \omega$ let

$$A_n = \bigcap_{j \in J_G} p_{j i_n}^{-1}[\pi_j[G]]$$

so that

$$\mu_{i_n}(A_n) = \tilde{\mu}(G) > t$$

and

$$\mu_{i_{n+1}}(A_{n+1} \Delta p_{i_n}^{-1}{}_{i_{n+1}}[A_n]) = 0.$$

By recursion choose $C_n \in \mathscr{K}_{i_n}$ so that $C_0 \subset A_0$,

$$C_{n+1} \subset A_{n+1} \cap \bigcap_{m=0}^{n} p_{i_m}^{-1}{}_{i_{n+1}}[C_m],$$

and $\mu_{i_n}(C_n) > t$.

For any $j \in I$ let n be the smallest integer with $j \prec i_n$ and set $K_j = p_{j i_n}[C_n]$ and $K = \prod_{j \in I} K_j$. Then $K \in \mathscr{K}^{\sim}$ and $K \subset G$.

To check that $\tilde{\mu}(K) > t$ note first that if $m < n$

$$\mu_{i_n}(C_n \sim p_{i_n}^{-1}{}_{i_m}[C_m]) = 0$$

so that if $j \prec i_n$

$$\mu_{i_n}(K_{i_n} \sim p_{j i_n}^{-1}[K_j]) = 0.$$

(Note that $K_{i_n} = C_n$.) Hence for any finite $J \subset I$, and $i_n \succ j$ for all $j \in J$,

$$\tilde{\mu}\Big(\bigcap_{j \in J} \pi_j^{-1}[K_j]\Big) = \mu_{i_n}\Big(\bigcap_{j \in J} p_{j i_n}^{-1}[K_j]\Big) \geqq \mu_{i_n}(K_{i_n}) > t.$$

Let $\epsilon > 0$ and $\mathscr{H} = \{H_0, H_1, \ldots\} \subset$ Rect $(\mathscr{M})$ be a cover of K such that for each $n \in \omega$, $\mu(H_n) < \infty$ and $\sum_{n \in \omega} \tilde{\mu}(H_n) \leqq \tilde{\mu}(K) + \epsilon$.

We can choose for each $n \in \omega$, $G_n \in$ Rect $(\mathscr{G})$ such that $H_n \subset G_n$ and $\tilde{\mu}(G_n) < \tilde{\mu}(H_n) + \epsilon/2^{n+1}$. Then $K \subset \bigcup_{n \in \omega} G_n$, thus for some $m \in \omega$, $K \subset \bigcup_{k=1}^{m} G_k$.

Let $J' = \bigcup_{k=0}^{m} J_{G_k}$; then J' is finite and for any $x \in \bigcap_{j \in J'} \pi_j^{-1}[K_j]$ there exists $y \in K$ with $y_j = x_j$ for all $j \in J'$.

Since $y \in \bigcup_{k=0}^{m} G_k$,

$$x \in \bigcap_{j \in J'} \pi_j^{-1}[\{y_j\}] \subset G_k$$

for some $k \leqq m$. Thus

$$\bigcap_{j \in J'} \pi_j^{-1}[K_j] \subset \bigcup_{k=0}^{m} G_k$$

so that

$$\sum_{k \in \omega} \tilde{\mu}(G_k) \geqq \tilde{\mu}\left(\bigcup_{l=0}^{m} G_l\right) \geqq \mu\left(\bigcap_{j \in J'} \pi_j^{-1}[K_j]\right) > t$$

and thus

$$\tilde{\mu}(K) \geqq \sum_{n \in \omega} \tilde{\mu}(H_n) - \epsilon \geqq \sum_{n \in \omega} \tilde{\mu}(G_n) - 2\epsilon > t - 2\epsilon.$$

Hence $\tilde{\mu}(K) \geqq t$, and (a) holds.

Finally let 8.3 hold. Let $G \in \text{Rect}(\mathscr{G})$ and $k \succ j$ for every $j \in J_G$. Let $t < \tilde{\mu}(G)$ and choose $C \in \mathscr{K}_k$ such that

$$C \subset \bigcap_{j \in J_G} p_{jk}^{-1}[\pi_j[G]]$$

and $\mu_k(C) > t$. For every $i \in I$ let

$$K_i = \begin{cases} p_{ik}[C], \text{ if } i \prec k \\ p_{ij}[p_{kj}^{-1}[K]], \text{ for some } j \succ i, j \succ k \text{ otherwise.} \end{cases}$$

As in the previous case we see that if we let $K = \prod_{i \in I} K_i$ then $K \in \mathscr{K}^{\sim}$, $K \subset G$ and $\tilde{\mu}(K) > t$ so that (a) holds.

2. Topological properties of inverse limit measures. In this section we establish conditions under which a system (X, p, μ, I) of Radón outer measures has an inverse limit which is also a Radón outer measure. Our approach is to restrict $\tilde{\mu}^*$ (Definition 5, Section 1) to the inverse limit set L.

We will use the general assumptions and notation of Section 1 and assume also that X, p and I form an inverse system of spaces, i.e. for $i \prec j \prec k$

$$p_{ik} = p_{ij} \circ p_{jk},$$

and that simple maximality holds, i.e. for every $i \in I$ and $x \in X_i$,

$$\pi_i^{-1}[\{x\}] \cap L \neq \emptyset.$$

1. Theorem. *If L is a closed set in the product topology, then $\tilde{\mu}^*$ is a Radón measure which is supported by L.*

Proof. First we note the following lemma [**6**, Lemma 4.2].

Lemma A. *Let $\alpha \in$ Rect $(\mathscr{M})$, $\alpha \subset \pi X_i \sim L$. Then $\tilde{\mu}(\alpha) = 0$.*

Next we note that Rect $(\mathscr{G})$ forms a base for the product topology, and, since each μ_i, $i \in I$, is Radon, Rect $(\mathscr{G}) \subset$ Rect $(\mathscr{M})$. Now let $\mathscr{B} \subset$ Rect $(\mathscr{G})$

be such that

$$\prod_{i\in I} X_i \sim L = \bigcup_{B\in\mathscr{B}} B.$$

For any $K \in \mathscr{K}^{\sim}$, $K \subset \prod_{i\in I} X \sim L$ there exists $B_1, \ldots, B_n \in \mathscr{B}$ such that $K \subset \bigcup_{m=1}^{n} B_m$. Thus

$$\tilde{\mu}(K) \leqq \tilde{\mu}\left(\bigcup_{m=1}^{n} B_m\right) \leqq \sum_{m=1}^{n} \tilde{\mu}(B_m) = 0$$

by Lemma A. From this we see that

$$\tilde{\mu}^*\left(\prod_{i\in I} X_i \sim L\right) = h\left(\prod_{i\in I} X_i \sim L\right)$$
$$= \sup\left\{\tilde{\mu}(K) : K \in \mathscr{K}^{\sim}, K \subset \prod_{i\in I} X_i \sim L\right\} = 0$$

(Theorem 6.2, Section 1). Hence $\tilde{\mu}^*$ is supported by L.

It is clear that L is closed whenever the spaces X_i are Hausdorff and the functions p_{ij} are continuous. Thus we can combine Theorems 1 and 8, Section 1 to obtain the following theorem. Similar results have been obtained by Choksi [**2**] and Metivier [**7**] using other approaches.

2. THEOREM. *Suppose that for each $i \in I$, X_i is a Hausdorff space, and that p_{ij} is continuous whenever $i \prec j$. Then μ^* is an inverse limit outer measure which is Radón for the product topology whenever any one of the following conditions holds:*

2.1. *There exists a countable cofinal set $I_0 \subset I$.*

2.2. *$p_{ij}^{-1}[K] \in \mathscr{K}_j$ whenever $i \prec j$ and $K \in \mathscr{K}_i$.*

Proof. By Theorem 8, Section 1, $\tilde{\mu}^*$ is a π-limit outer measure which is Radón for the product topology. By Theorem 1, $\tilde{\mu}^*$ is supported by L. Hence $\tilde{\mu}^*$ is an inverse limit outer measure which is Radón for the product topology.

We next turn to a case in which we require weaker conditions on the functions but stronger conditions on the measures.

3. *Definition.* An outer measure φ on a space S is almost separable if and only if there exists a countable family $\mathscr{B} \subset \mathscr{M}_\varphi$, and a set $T \subset S$ with $\varphi(T) = 0$ such that for every $x, y \in S \sim T$ with $x \neq y$ there exists $B \in \mathscr{B}$ with $c \in B$ and $y \notin B$.

4. THEOREM. *Suppose that I is countable and that for each $i \in I$,*

4.1. *X_i is a topological space,*

4.2. *μ_i is almost separable.*

Then $\tilde{\mu}^*$ is an inverse limit outer measure which is Radón for the product topology.

Proof. By Lemma 2.31 of Section 3 in [**6**], $\tilde{\mu}$ is carried by L, i.e.,

$\tilde{\mu}(\prod_{i\in I} X_i \sim L) = 0$. Hence for $\epsilon > 0$ there exists $\alpha_1, \alpha_2 \ldots \in$ Rect $(\mathscr{M})$ such that

$$\prod_{i\in I} X_i \sim L \subset \bigcup_{n\in\omega} \alpha_n \quad \text{and} \quad \sum_{n\in\omega} \mu(\alpha_n) < \epsilon.$$

We may choose for every $n \in \omega$, $G_n \in$ Rect $(\mathscr{G})$ with $\alpha_n \subset G_n$ and $\tilde{\mu}(G_n) \leqq \tilde{\mu}(\alpha) + \epsilon/2^{n+1}$. Then since $\tilde{\mu}^*(G_n) \leqq \tilde{\mu}(G_n)$ we have

$$\tilde{\mu}^*\Big(\prod_{i\in I} X_i \sim L\Big) \leqq \mu^*\Big(\bigcup_{n\in\omega} G_n\Big) \leqq \sum_{n\in\omega} \mu^*(G_n)$$
$$\leqq \epsilon + \sum_{n\in\omega} \mu^*(\alpha) < 2\epsilon.$$

Hence $\mu^*(\prod_{i\in I} X_i \sim L) = 0$ and $\tilde{\mu}^*$ is carried by L. By Theorem 8, Section 1, $\tilde{\mu}^*$ is a π-limit outer measure which is Radón for the product topology. Hence $\tilde{\mu}^*$ is an inverse limit measure which is Radón for the product topology.

Note that the above theorem requires no conditions on the functions p_{ij} beyond those necessary to make (X, p, μ, I) a system. An examination of the proof shows that it is not necessary that X, p, and I form an inverse system of spaces.

The following example illustrates some of the difficulties of obtaining inverse limit measures which are Radón for the product topology. In this example the spaces X_i are locally compact and Hausdorff, the measures μ_i are bounded, Radón and atomless, and the functions p_{ij} are continuous except at one point. The system also has a "natural" inverse limit measure, but the topology induced on the inverse limit set L is such that only discrete measures could be Radón.

However, there exists a π-limit outer measure Radón for $\mathscr{G}'$ (Definition 1.5 Section 1) and $\tilde{\mu}|L$ is Radón for the topology induced on L by $\mathscr{G}$.

5. *Example.* Let $S = [0, 1)$ and λ be Lebesque outer measures on S. Define (X, p, μ, I) as follows: Let $I = (0, 1]$ with the usual ordering and for each $j \in I$ let

$X_j = S$
$\mu_j = \lambda$
$\mathscr{G}_j$ be the usual topology on X_j
$\mathscr{G}_j$ be the closed compact subsets of X_j
p_{ij} be the function defined by $p_{ij}(x) = x + j - i$ (modulo 1), whenever $i \leqq j$.

Note that p_{ij} is continuous except at $1 - (j - i)$.

It is clear that the inverse limit set L can be identified with S by the mapping $\varphi : L \to X_1$ defined by $\varphi(x) = \pi_1(x) = x_1$ for every $x \in L$. Furthermore $\tilde{\mu}|L$ is just λ and is an inverse limit outer measure.

However the topology induced on L by the product topology can be identified with the "half open" interval topology on X_1 $(= S)$, since for $s \in S$, $0 < h < s$

$$\varphi^{-1}[p_{s1}^{-1}[[0, h)]] = \varphi^{-1}[[s, s + h)]$$

and $[0, h)$ is open in X_s.

This topology has no uncountable compact sets. Hence only discrete measures could be Radón for this topology.

Note also that L is closed in the product topology hence any π-limit outer measure Radón for the product topology would be supported by L (Theorem 1.0). Since the above argument shows this to be impossible, no π-limit outer measure can be Radón for the product topology.

Theorem 4 Section 1 shows that there exists a π-limit outer measure Radón for $\mathscr{G}'$. Furthermore the topology induced on L by $\mathscr{G}'$ is the usual topology except at 0. At 0 neighbourhoods contain sets of the form $[0, a) \cup (b, 1)$, $0 < a \leqq b < 1$. This topology is compact and Hausdorff and λ is Radón for this topology.

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